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On L_2-Convergence of the Quasi-periodic Interpolation

(Submitted by academician A. B. Nersessian 8 /IV 2013)

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1. In this paper we consider the problem of function reconstruction by its values on the equidistant grid.

It is well known (see [1]) that for a rather smooth on $[-1,1]$ function f with smooth 2-periodic extension on the real line the classical trigonometric interpolation

$$I_N(f,x) = \sum_{n=-N}^{N} \hat{f}_n e^{i\pi n x},$$

$$\hat{f}_n = \frac{1}{2N+1}\sum_{k=-N}^{N} f\left(\frac{2k}{2N+1}\right) e^{-\frac{2i\pi nk}{2N+1}} \qquad (1)$$

effectively solves the problem. Otherwise, if 2-periodic extension of a smooth on $[-1,1]$ f function on the real line after extension is discontinuous then the quality of interpolation near the endpoints is degraded by the Gibbs phenomenon (see[2]).

An approach which is not eliminating the Gibbs phenomenon but mitigates its effect is suggested in [3] where the "quasi-periodic" interpolation $I_{N,m}(f,x)$, $m \in Z$, $m \geq 0$ is introduced. This interpolation is exact on the segment $[-1,1]$ for quasi-periodic functions

$$\left\{e^{i\pi n a x}\right\}_{n=-N}^{N}, \ a = \frac{2N}{2N+m+1}, \qquad (2)$$

with the period $2/\alpha$. Therefore, when $N \to \infty$ then $\alpha \to 1$. Interesting feature of such interpolations is the possibility to interpolate functions on the grid

$$x_k = \frac{k}{N}, \ \ k = -N, \ldots, N \qquad (3)$$

which includes also the endpoints $x = \pm 1$ of the interval. Such interpolations are known as the "full-interpolations" ([3]).

We introduce explicit formula for the quasi-periodic interpolation for all $m \geq 0$ and investigate its convergence properties on the entire interval $[-1,1]$ of interpolation in the L_2 norm. Exact constants of the asymptotic errors are obtained. The results of numerical experiments confirm theoretical estimates. Some results of this research were presented in [4] and [5].

2. In this section we derive an explicit form of the quasi-periodic interpolation for all $m \geq 0$. The cases $m = 0$ and $m = 1$ are presented in [3].

Consider the following formula

$$I_{N,m}(f,x) = \sum_{k=-N}^{N} f\left(\frac{k}{N}\right) a_k(x), \ x \in [-1,1] \tag{4}$$

with unknowns $a_k(x)$. We get the following system of equations for determination of the unknowns as we assumed that (4) is exact for the system (2)

$$e^{\frac{2i\pi lNx}{2N+m+1}} = \sum_{k=-N}^{N} e^{\frac{2i\pi lk}{2N+m+1}} a_k(x), \ |l| \leq N \ . \tag{5}$$

For solution of (5) we add some new unknowns and equations getting the new system

$$e^{\frac{2i\pi lNx}{2N+m+1}} = \sum_{k=-N}^{N+m} e^{\frac{2i\pi lk}{2N+m+1}} a_k^*(x) + \varepsilon_l(x), \ l = -N, \ldots, N+m, \tag{6}$$

where

$$a_k^*(x) = a_k(x) \ \text{ if } \ |k| \leq N, \text{ and } \ a_k^*(x) = 0 \ \text{ if } \ k = N+1, \ldots, N+m \tag{7}$$

and

$$\varepsilon_l(x) = 0, \quad |l| \leq N.$$

We multiply the both sides of equation (6) by $e^{-\frac{2i\pi ls}{2N+m+1}}$ and sum over l

$$\sum_{l=-N}^{N+m} e^{\frac{2i\pi l(Nx-s)}{2N+m+1}} = \sum_{l=-N}^{N+m} \sum_{k=-N}^{N+m} e^{\frac{2i\pi l(k-s)}{2N+m+1}} a_k^*(x) + \sum_{l=N+1}^{N+m} e^{-\frac{2i\pi ls}{2N+m+1}} \varepsilon_l(x).$$

By application of the DFT we get

$$a_k^*(x) = \frac{1}{2N+m+1} \left(\sum_{l=-N}^{N+m} e^{\frac{2i\pi l(Nx-s)}{2N+m+1}} - \sum_{l=N+1}^{N+m} e^{-\frac{2i\pi ls}{2N+m+1}} \varepsilon_l(x) \right) \tag{8}$$

Using conditions (7) we derive the following system with the Vandermonde matrix for determination of $\varepsilon_l(x)$

$$\sum_{l=N+1}^{N+m} e^{-\frac{2i\pi ls}{2N+m+1}} \left(\varepsilon_l(x) - e^{\frac{2i\pi lx}{2N+m+1}} \right) = \sum_{t=-N}^{N} e^{\frac{2i\pi t(Nx-s)}{2N+m+1}}, \ s = N+1, \ldots, N+m \tag{9}$$

After some transformations we obtain

$$\sum_{l=1}^{m} v_{s+1,l}\tilde{\varepsilon}_l(x) = \sum_{t=-N}^{N} e^{\frac{2i\pi tNx}{2N+m+1}} e^{\frac{2i\pi t(s-N-m)}{2N+m+1}}, \quad s = 0,\ldots,m-1$$

where

$$\tilde{\varepsilon}_l(x) = e^{-\frac{2i\pi(l+N)(N+m)}{2N+m+1}} \left(\varepsilon_{l+N}(x) - e^{\frac{2i\pi(l+N)Nx}{2N+m+1}} \right),$$

and

$$v_{s,l} = a_l^{s-1}, \quad a_l = e^{\frac{2i\pi(l+N)}{2N+m+1}}.$$

Following [6] (see also [7]), where the explicit form of the inverse of the Vandermonde matrix was constructed, we derive

$$v_{l,s}^{-1} = -\frac{1}{\alpha_l^s \prod_{j=1,i\neq l}^{m}(\alpha_l - \alpha_i)} = \sum_{i=0}^{s-1} \gamma_j \alpha_l^j, \quad l,s = 1,\ldots,m,$$

where γ_j are the coefficients of the polynomial

$$\prod_{i=1}^{m}(x - a_i) = \sum_{i=0}^{m} \gamma_i x^i.$$

Hence the solution of (9) can be written explicitly

$$\varepsilon_l(x) = e^{\frac{2i\pi lNx}{2N+m+1}} + e^{\frac{2i\pi l(N+m)}{2N+m+1}} \sum_{s=0}^{m-1} v_{l-N,s+1}^{-1} \sum_{t=-N}^{N} e^{\frac{2i\pi tNx}{2N+m+1}} e^{\frac{2i\pi t(s-N-m)}{2N+m+1}},$$

$$l = N+1,\ldots,N+m.$$

Substituting $\varepsilon_l(x)$ into (8) we get

$$a_k(x) = \frac{1}{2N+m+1}\left(\sum_{l=-N}^{N} e^{\frac{2i\pi lNx}{2N+m+1}} e^{-\frac{2i\pi lk}{2N+m+1}} - \sum_{l=N+1}^{N+m} e^{\frac{2i\pi l(N+m)}{2N+m+1}} e^{-\frac{2i\pi lk}{2N+m+1}} \right.$$

$$\left. \times \sum_{s=0}^{m-1} v_{l-N,s+1}^{-1} \sum_{t=-N}^{N} e^{\frac{2i\pi tNx}{2N+m+1}} e^{\frac{2i\pi t(s-N-m)}{2N+m+1}} \right), \quad k = -N,\ldots,N$$

Substituting this into (4) we get the explicit form of the quasi-periodic interpolation

$$I_{N,m}(f,x) = \sum_{n=-N}^{N} F_{n,m} e^{\frac{2i\pi nNx}{2N+m+1}}, \tag{10}$$

$$F_{n,m}(f,x) = \breve{f}_{n,m} - \sum_{l=1}^{m} \theta_{n,l}(m) \breve{f}_{l+N,m}, \tag{11}$$

$$\breve{f}_{n,m} = \frac{1}{2N+m+1} \sum_{k=-N}^{N} f\left(\frac{k}{N}\right) e^{-\frac{2i\pi nk}{2N+m+1}}$$

and

$$\theta_{n,l}(m) = e^{\frac{2i\pi(l+N)(N+m)}{2N+m+1}} \sum_{s=0}^{m-1} v_{l,s+1}^{-1} e^{\frac{2i\pi n(s-N-m)}{2N+m+1}}$$

By $R_{N,m}(f,x)$ we denote the error of interpolation

$$R_{N,m}(f,x) = f(x) - I_{N,m}(f,x). \tag{12}$$

3. In this section we investigate the convergence of the quasi-periodic interpolation on the entire interval in the L_2 norm.

First we need some lemmas. Let $f \in C^{(q+p)}[-1,1]$ and

$$f^*(x) = \begin{cases} l(x) & x \in \left[-1, -\frac{2N}{2N+m+1}\right] \\ f\left(\frac{2N+m+1}{2N}x\right), & x \in \left[-\frac{2N}{2N+m+1}, \frac{2N}{2N+m+1}\right] \\ r(x), & x \in \left[\frac{2N}{2N+m+1}, 1\right] \end{cases} \tag{13}$$

where

$$l(x) = \sum_{j=0}^{q+p} \frac{f^{(j)}(-1)}{j!} \left(\frac{2N+m+1}{2N}x + 1\right)^j,$$

$$r(x) = \sum_{j=0}^{q+p} \frac{f^{(j)}(-1)}{j!} \left(\frac{2N+m+1}{2N}x - 1\right)^j.$$

By f_n we denote the n-th Fourier coefficient of f

$$f_n = \frac{1}{2}\int_{-1}^{1} f(x) e^{-i\pi n x} dx.$$

Also we denote

$$A_{kl}(f) = f^{(l)}(1) - (-1)^{l-k} f^{(l)}(-1), \; k,l = 0,\ldots,q$$

Lemma 1. *Let* $f^{(q)} \in AC[-1,1]$ *for some* $q \ge 0$ *and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \; k = 0,\ldots,q-1. \tag{14}$$

Then the following estimate holds as $N \to \infty$ *and* $|n| > N$

$$f_n^* = \frac{(-1)^{n+1}}{(2N+m+1)N^q} \mu_{q,m}\left(\frac{2n}{2N+m+1}\right) + o\left(n^{-q-1}\right), \tag{15}$$

where

$$\mu_{q,m}(x) = \sum_{k=0}^{q} \frac{A_{kq}(f)(m+1)^{q-k}}{2^{q-k}(i\pi)^{k+1}(q-k)!x^{k+1}}. \tag{16}$$

Lemma 2. *Let* $f^{(q+m)} \in AC[-1,1]$ *for some* $q,m \ge 0$, $q+m \ne 0$ *and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \; k = 0,\ldots,q-1. \tag{17}$$

Then the following estimate holds as $|n| \le N$ *and* $N \to \infty$

$$F_{n,m} - f_n^* = \frac{(-1)^{n+1}}{(2N+m+1)N^q} v_{q,m}\left(\frac{2n}{2N+m+1}\right) + o\left(N^{-q-1}\right), \tag{18}$$

where

$$v_{q,m}(x)=\sum_{k=0}^{q}\frac{A_{kq}(f)(m+1)^{q-k}}{2^{q-k}(i\pi)^{k+1}(q-k)!}\left(\sum_{r\neq 0}\frac{(-1)r^{(m+1)}}{(2r+x)^{k+1}}-\right.$$
$$\left.-e^{-i\pi\frac{m-1}{2}x}\sum_{j=0}^{m-1}\frac{1}{j!}\Phi_{k,m}^{(j)}(-1)\left(e^{i\pi x}+1\right)^{j}\right) \tag{19}$$

and

$$\Phi_{k,m}\left(e^{i\pi x}\right)=e^{\frac{i\pi}{2}(m-1)x}\sum_{r=-\infty}^{\infty}\frac{(-1)^{r(m+1)}}{(2r+x)^{k+1}}. \tag{20}$$

Now we prove the main theorem of this paper.

Theorem 1. *Let* $f^{(q+m)}\in AC[-1,1]$ *for some* $q,m\geq 0$, $q+m\neq 0$ *and*

$$f^{(k)}=f^{(k)}(1)=0,\ k=0,\ldots,q-1$$

Then the following estimate holds

$$\lim_{N\to\infty}N^{2q+1}\left\|R_{N,m}(f,x)\right\|_{L_2(-1,1)}^{2}=\frac{1}{2}\int_{-1}^{1}\left|v_{q,m}(x)\right|^{2}dx+\frac{1}{2}\int_{|x|>1}\left|\mu_{q,m}(x)\right|^{2}dx-$$
$$\frac{m+1}{2}\int_{-1}^{1}\left|\int_{-1}^{1}v_{q,m}(h)e^{\frac{i\pi(m+1)xh}{2}}dh-\int_{|h|>1}\mu_{q,m}(h)e^{\frac{i\pi(m+1)xh}{2}}dh\right|^{2}dx$$

where functions $v_{q,m}(x)$ *and* $\mu_{q,m}(x)$ *are defined in Lemmas 1 and 2.*

Proof. We divide $R_{N,m}(f,x)$ into three parts

$$\left\|R_{N,m}(f,x)\right\|_{L_2(-1,1)}^{2}=\int_{-1}^{1}\left|R_{N,m}(f,x)\right|^{2}dx=$$
$$=\frac{2N+m+1}{2N}\int_{-\frac{2N}{2N+m+1}}^{\frac{2N}{2N+m+1}}\left|R_{N,m}\left(f,\frac{2N+m+1}{2N}x\right)\right|^{2}dx=$$
$$=\frac{2N+m+1}{2N}\int_{-1}^{1}\left|R_{N,m}\left(f,\frac{2N+m+1}{2N}x\right)\right|^{2}dx-$$
$$-\frac{2N+m+1}{2N}\int_{\frac{2N}{2N+m+1}}^{1}\left|R_{N,m}\left(f,\frac{2N+m+1}{2N}x\right)\right|^{2}dx=$$
$$-\frac{2N+m+1}{2N}\int_{-1}^{-\frac{2N}{2N+m+1}}\left|R_{N,m}\left(f,\frac{2N+m+1}{2N}x\right)\right|^{2}dx=I_1-I_2-I_3.$$

where

$$I_1=\frac{2N+m+1}{2N}\int_{-1}^{1}\left|R_{N,m}\left(f,\frac{2N+m+1}{2N}x\right)\right|^{2}dx,$$
$$I_2=\frac{m+1}{2N}\int_{0}^{1}\left|R_{N,m}\left(f,\frac{2N+m+1}{2N}-\frac{m+1}{2N}x\right)\right|^{2}dx$$

and

$$I_3 = \frac{m+1}{2N}\int_0^1 \left| R_{N,m}\left(f, \frac{m+1}{2N}x - \frac{2N+m+1}{2N} \right) \right|^2 dx .$$

First we estimate I_1. We have

$$R_{N,m}(f,x) = \sum_{n=-N}^{N} \left(f_n^* - F_{n,m} \right) e^{i\pi n \frac{2N}{2N+m+1}x} + \sum_{|n|>N} f_n^* e^{i\pi n \frac{2N}{2N+m+1}x} . \tag{21}$$

Therefore

$$I_1 = \frac{2N+m+1}{N}\sum_{n=-N}^{N} \left| f_n^* - F_{n,m} \right|^2 + \frac{2N+m+1}{N}\sum_{|n|>N} \left| f_n^* \right|^2$$

In view of Lemmas 1 and 2 we obtain

$$I_1 = \frac{1}{(2N+m+1)N^{2q+1}} \sum_{n=-N}^{N} \left| v_{q,m}\left(\frac{2n}{2N+m+1} \right) \right|^2 +$$

$$+ \frac{1}{(2N+m+1)N^{2q+1}} \sum_{|n|>N} \left| \mu_{q,m}\left(\frac{2n}{2N+m+1} \right) \right|^2 + o\left(N^{2q-1} \right), \ N \to \infty$$

Tending N to infinity and replacing the sums by the corresponding integrals we get

$$\lim_{N\to\infty} N^{2q+1} I_1 = \frac{1}{2}\int_{-1}^{1} \left| v_{q,m}(x) \right|^2 dx + \frac{1}{2}\int_{|x|>1} \left| \mu_{q,m}(x) \right|^2 dx$$

Now we estimate L_2. From (21) we get

$$R_{N,m}\left(f, \frac{2N+m+1}{2N} - \frac{m+1}{2N}x \right) =$$

$$\sum_{n=-N}^{N} (-1)^n \left(f_n^* - F_{n,m} \right) e^{-i\pi x \frac{(m+1)n}{2N+m+1}} + \sum_{|n|>N} (-1)^n f_n^* e^{-i\pi x \frac{(m+1)n}{2N+m+1}}$$

According to Lemmas 1 and 2 we derive

$$R_{N,m}\left(f, \frac{2N+m+1}{2N} - \frac{m+1}{2N}x \right) =$$

$$= \frac{1}{(2N+m+1)N^q} \sum_{n=-N}^{N} v_{q,m}\left(\frac{2n}{2N+m+1} \right) e^{-i\pi x \frac{(m+1)n}{2N+m+1}}$$

$$- \frac{1}{(2N+m+1)N^q} \sum_{|n|>N} \mu_{q,m}\left(\frac{2n}{2N+m+1} \right) e^{-i\pi x \frac{(m+1)n}{2N+m+1}} + o\left(N^{-q} \right)$$

Tending N to infinity and replacing the sums by the corresponding integrals we derive

$$\lim_{N\to\infty} N^q R_{N,m}\left(f, \frac{2N+m+1}{2N} - \frac{m+1}{2N}x \right) =$$

$$= \frac{1}{2}\int_{-1}^{1} v_{q,m}(h) e^{-\frac{i\pi(m+1)xh}{2}} - \frac{1}{2}\int_{|h|>1} \mu_{q,m}(h) e^{-\frac{i\pi(m+1)xh}{2}} dh$$

Hence

$$\lim_{N\to\infty} N^{2q+1} I_2 = \frac{m+1}{8}\int_0^1 \left| \int_{-1}^{1} v_{q,m}(h) e^{\frac{-i\pi(m+1)xh}{2}} dh - \int_{|h|>1} \mu_{q,m}(h) e^{\frac{-i\pi(m+1)xh}{2}} dh \right|^2 dx .$$

Similarly

$$\lim_{N\to\infty} N^{2q+1} I_3 = \frac{m+1}{8}\int_0^1 \left| \int_{-1}^{1} v_{q,m}(h) e^{\frac{i\pi(m+1)xh}{2}} dh - \int_{|h|>1} \mu_{q,m}(h) e^{\frac{i\pi(m+1)xh}{2}} dh \right|^2 dx ,$$

which completes the proof.

Remark 1. Lemma 2 and Theorem 1 are valid also in the case $q+m=0$ with more strict condition $f' \in L_2[-1,1]$.

4. Consider the following function

$$f(x) = \sin(x-1)$$

Let's denote

$$c_{N,m}(f) = N^{1/2} \left\| R_{N,m}(f,x) \right\|_{L_2(-1,1)}$$

and

$$c_m(f) = \lim_{N\to\infty} N^{1/2} \left\| R_{N,m}(f,x) \right\|_{L_2(-1,1)} .$$

Table 1 presents the values of $c_{N,m}(f)$ for different m and moderate values of N. For comparison the last column presents the values of $c_m(f)$. We see that numerical results (however for this specific example) confirm theoretical estimates of Theorem 1.

Table1

Numerical values of $c_{N,m}(f)$ and $c_m(f)$ for different values of N and m while interpolating the function $f(x) = \sin(x-1)$ by the quasi-periodic interpolation

	$N=16$	$N=32$	$N=64$	$N=128$	$c_m(f)$
$m=0$	0.174552	0.172947	0.172117	0.171695	0.171268
$m=1$	0.034082	0.03315	0.03275	0.032568	0.032399
$m=2$	0.01044	0.010017	0.009836	0.009754	0.009678
$m=3$	0.003763	0.003545	0.003455	0.003415	0.003379
$m=4$	0.00148	0.001364	0.001318	0.001298	0.00128
$m=5$	0.000617	0.000554	0.000529	0.000519	0.00051
$m=6$	0.000268	0.000233	0.00022	0.000215	0.000211
$m=7$	0.00012	0.000101	0.000094	0.000091	0.000089

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L. D. Poghosyan
On L_2 - Convergence of the Quasi-Periodic Interpolation

We discuss the problem of function reconstruction via pointwise values on uniform grid and in that context consider the quasi-periodic interpolation. The latest is exact for quasi-exponential functions which allow inclusion the endpoints of the interval into the grid. We derive explicit formulae for the corresponding interpolations and investigate their convergence. Exact asymptotic estimate of the L_2 error is obtained for smooth functions. Comparison with the classical trigonometric interpolation is performed.

Լ. Դ. Պողոսյան

Քվազի-պարբերական ինտերպոլիացիայի L_2 - զուգամիտության մասին

Քննարկվում է ֆունկցիայի վերականգնման խնդիրը հավասարաչափ ցանցի վրա հայտնի արժեքներով, և այդ համատեքստում դիտարկվում է քվազի-պարբերական ինտերպոլիացիան: Հնարավորություն է առաջանում ցանցում ներառելու նաև հատվածի ծայրակետերը, քանի որ ինտերպոլիացիան ճշգրիտ է քվազի-պարբերական էքսպոնենցիալ ֆունկցիաների համար: Ստացվել են համապատասխան ինտերպո-լիացիաների բացահայտ բանաձևեր, ուսումնասիրվել է նրանց զուգամիտությունը: Ողորկ ֆունկցիաների համար ներկայացվել է L_2 սխալի ասիմպտոտական ճշգրիտ գնահատական: Կատարվել է համեմատություն դասական եռանկյունաչափական ինտերպոլիացիայի հետ:

Л. Д. Погосян
Об L_2-сходимости квазипериодической интерполяции

Обсуждается проблема восстановления функции по известным значениям на равномерной сети и в этом контексте рассматривается квазипериодическая интерполяция. Точность интерполяции для квазипериодических экспоненциальных функций позволяет включение в сеть также концов отрезка. Получены явные формулы для соответсвующих интерполяций и изучена их сходимость. Получены точные асимптотические оценки L_2 ошибки для гладких функций. Проведен сравнительный анализ с классической тригонометрической интерполяцией.

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