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ОБ ОПЕРАТОРЕ ДРОБНОГО ИНТЕГРОДИФФЕРЕНЦИРОВАНИЯ В $\mathbb{R}^n$

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Аннотация. В статье введена новая форма дробной производной для функций, заданных в единичном шаре из $\mathbb{R}^n$. В качестве приложения получено интегральное представление для гармонических в шаре функций с конечной смешанной нормой.

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Ключевые слова: дробный интеграл; дробная производная; гармоническая функция; единичный шар в $\mathbb{R}^n$; пространство со смешанной нормой; ядро Пуассона–Бергмана.

1. Введение

Возникновение понятий дробного интегрирования и дифференцирования в XIX веке поначалу было связано с теорией интегральных и дифференциальных уравнений. Дробными интегралами и производными занимались Абель, Лиувилль, Риман, Адамар, Вейль, Харди, Литтлвуд, М. Рисс и другие, см. монографию Самко, Килбаса, Маричева [1], где можно найти подробные справки. Позднее, дробное интегродифференцирование нашло новые применения в теории функций. Оно оказалось весьма эффективным средством, в частности, в теории классических функциональных пространств, см. [2] – [9].

Одним из недостатков дробных интегралов и производных является их разнообразие и часто несовместимость друг с другом, в том числе в различных приложениях. Широкий выбор различных типов дробных интегралов и производных обращается в преимущество, когда нужно их приспособить к тому или иному применению.

В настоящей работе мы развиваем и применяем классическое интегродифференцирование Римана–Лиувилля в задачах из теории гармонических пространств в $\mathbb{R}^n$ $(n \geq 2)$. Введем необходимые обозначения. Пусть $B = B_n$ — открытый единичный шар в $\mathbb{R}^n$, $S = \partial B$ — его граница, единичная сфера. Интегральные средние порядка p гармонической функции $u(x) = u(r\zeta)$ на сфере $|x| = r$ обозначены, как обычно, через

$$M_p(u;r) = \big\|u(r\cdot)\big\|_{L^p(S;d\sigma)}, \qquad 0 \leq r < 1,\ 0 < p \leq \infty,$$

где $d\sigma$ — $(n-1)$-мерная поверхностная лебегова мера на S, нормированная условием $\sigma(S) = 1$. Множество всех вещественных гармонических функций в шаре B обозначим через $h(B)$.

Определим пространство $h(p,q,\alpha)\big(0 < p,q \leq \infty, \alpha \in \mathbb{R}\big)$ со смешанной нормой как пространство тех гармонических функций $u \in h(B)$, для которых конечна квазинорма

$$\|u\|_{h(p,q,\alpha)} = \begin{cases} \left(\displaystyle\int_0^1 (1-r)^{\alpha q-1} M_p^q(u;r)dr\right)^{1/q}, & 0 < q < \infty, \\ \displaystyle\sup_{0<r<1} \ (1-r)^{\alpha} M_p(u;r), & q = \infty. \end{cases}$$

При $p = q < \infty$ пространства $h(p,q,\alpha)$ совпадают с весовыми классами Бергмана, а при $q = \infty$ их обычно называют весовыми пространствами Харди.

Много работ посвящено пространствам $h(p,q,\alpha)$ со смешанной нормой или их аналогам, состоящим из голоморфных, плюригармонических или гармонических функций в круге или шаре из $\mathbb{C}^n$ или $\mathbb{R}^n$.

Пространства со смешанной нормой для голоморфных в единичном круге функций были введены Харди и Литтлвудом [2],[3]. Позднее, Флетт [6],[7] значительно развил теорию, а в дальнейшем были получены многомерные обобщения результатов Харди, Литтлвуда, Флетта в областях из $\mathbb{C}^n$ и $\mathbb{R}^n$. Пространства $h(p,p,\alpha), h(p,q,\alpha)$ в единичном шаре из $\mathbb{R}^n$ изучались в работах [10]–[21], а пространства $h(p,q,\alpha)$, состоящие из n-гармонических функций в полидиске из $\mathbb{C}^n$, изучались в статьях [22] – [24].

Основным нашим приложением дробных операторов в теории гармонических пространств является следующее интегральное представление в пространствах $h(p,q,\alpha)$ со смешанной нормой.

Теорема 1.1. *Пусть* $0 < p,q \leq \infty$, $\alpha > 0$, $u \in h(p,q,\alpha)$ *— произвольная функция. Пусть выполнено* $0 < p,q \leq \infty$, $\beta > \max\{\alpha + (n-1)(1/p-1), \alpha\}$, *либо*

$1 \le p \le \infty$, $0 < q \le 1$, $\beta \ge \alpha$. *Тогда*

$$u(x) = \frac{2}{n\,\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1} P_\beta(x,y)\, u(y)\, dV(y), \qquad x \in B, \tag{1.1}$$

где $P_\beta(x,y)$ — *ядро Пуассона–Бергмана, определенное в параграфе 5.*

Замечание 1.1. *В частном случае* $p = q \ge 1$ *и* $\beta = \alpha$ *или для еще более узких классов аналогичное интегральное представление получено в* [10],[11],[13],[14], [18], *а с более общими весами — в* [17],[19]. *Для голоморфных функций в единичном шаре из* $\mathbb{C}^n$ *Теорему 1.1 при* $1 \le p, q \le \infty$ *можно найти в* [25],[26], *а для* n*-гармонических функций в полидиске — в* [22].

2. Дробное интегродифференцирование в единичном круге на плоскости

В этом параграфе мы приводим определение известного оператора D^α дробного интегродифференцирования Римана–Лиувилля и одну его модификацию $\mathcal{D}^\alpha$. Всегда будем считать, что при положительной степени α операторы D^α и $\mathcal{D}^\alpha$ представляют дробное дифференцирование порядка $\alpha > 0$, а при отрицательной степени α операторы D^α и $\mathcal{D}^\alpha$ выражают дробное интегрирование порядка $|\alpha| > 0$. В случае $\alpha = 0$ полагаем, что D^0 и $\mathcal{D}^0$ — тождественные операторы, т.е. $D^0 f = f$ и $\mathcal{D}^0 f = f$.

Определение 2.1. *(Дробное интегродифференцирование Римана-Лиувилля на* $\mathbb{R}^2$*) Пусть задана функция* $f(r)$ *одной переменной* $r \in [0,1)$. *Тогда*

$$D_2^{-\alpha} f(r) := \frac{1}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} f(t)\, dt = \frac{r^\alpha}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tr)\, dt, \tag{2.1}$$

$$D_2^m f(r) := \left(\frac{d}{dr}\right)^m f(r), \qquad D_2^\alpha f(r) := D_2^{-(m-\alpha)} D_2^m f(r), \tag{2.2}$$

$$\mathcal{D}_2^{-\alpha} f(r) := r^{-\alpha} D_2^{-\alpha} f(r) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tr)\, dt, \tag{2.3}$$

$$\mathcal{D}_2^\alpha f(r) := D_2^\alpha \Big\{ r^\alpha f(r) \Big\}, \tag{2.4}$$

где $0 \le r < 1$, $\alpha > 0$, $m \in \mathbb{Z}$, $m \ge 0$, $m - 1 < \alpha \le m$.

Нижний индекс 2 при операторах ставится, поскольку далее мы применяем их по отношению к функциям, заданным на плоскости, и в частности на единичном круге $\mathbb{D}$. В последующих разделах мы будем рассматривать операторы $\mathcal{D}^\alpha$ также по отношению к функциям, заданным в $\mathbb{R}^n$. Введение модификации $\mathcal{D}_2^\alpha$

обусловлено тем, что оператор D_2^α не сохраняет голоморфные или гармонические функции, а оператор $\mathcal{D}_2^\alpha$ — сохраняет, т.е. $\mathcal{D}_2^\alpha : H(\mathbb{D}) \longrightarrow H(\mathbb{D})$ и $\mathcal{D}_2^\alpha : h(\mathbb{D}) \longrightarrow h(\mathbb{D})$. Заметим, что оператор дифференцирования $\mathcal{D}_2^1$ первого порядка не совпадает с обычной частной или комплексной производной, именно $\mathcal{D}_2^1 f = \frac{\partial}{\partial r}(rf)$, которую называют радиальной производной функции f. Иногда будем писать $D_{2,r}^\alpha$ и $\mathcal{D}_{2,r}^\alpha$, чтобы указать переменную r, по которой выполняется интегрирование или дифференцирование.

Операторы D_2^α и $\mathcal{D}_2^\alpha$ хорошо приспособлены к применению к степенным функциям. Следующая лемма проверяется простыми вычислениями.

Лемма 2.1. *При $\alpha > 0$, $k > -1$, $m \in \mathbb{Z}$, $m \geq 0$, $D^\alpha \equiv D_2^\alpha$, $\mathcal{D}^\alpha \equiv \mathcal{D}_2^\alpha$ имеют место следующие тождества:*

$$D^{-\alpha}\{r^k\} = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}\, r^{k+\alpha}, \tag{2.5}$$

$$D^m\{r^k\} = \frac{\Gamma(k+1)}{\Gamma(k-m+1)}\, r^{k-m}, \qquad k-m \neq -1,-2,\dots, \tag{2.6}$$

$$D^\alpha\{r^{k+\alpha}\} = \frac{\Gamma(\alpha+k+1)}{\Gamma(k+1)}\, r^k, \tag{2.7}$$

$$\mathcal{D}^{-\alpha}\{r^k\} = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}\, r^k, \tag{2.8}$$

$$\mathcal{D}^\alpha\{r^k\} = \frac{\Gamma(\alpha+k+1)}{\Gamma(k+1)}\, r^k. \tag{2.9}$$

Помимо определения 2.1 посредством дробных интегралов имеются другие способы определения дробного интегродифференцирования через разложения в степенные ряды, см. [1], [3], [5] – [7]. Подобные разложения удобны для распространения дробного интегродифференцирования на функции, заданные в $\mathbb{R}^n$.

3. Дробное интегрирование Римана–Лиувилля для гармонических и негармонических функций в $\mathbb{R}^n$

Вначале определим дробное интегродифференцирование для гармонических в $\mathbb{R}^n$ функций. Положим, что гармоническая функция $f(x) \in h(B)$ разложена в ряд по однородным сферическим гармоникам Y_{kj}

$$f(x) = \sum_{k=0}^{\infty}\sum_{j=1}^{d_k} a_{kj}\, r^k\, Y_{kj}(\zeta) \equiv \sum_{k,j} a_{kj}\, r^k\, Y_{kj}(\zeta), \qquad x = r\zeta \in B, \tag{3.1}$$

где $d_k = \frac{(2k+n-2)(n+k-3)!}{(n-2)!\,k!}$ — размерность пространства $\mathcal{H}_k(S)$ сферических гармоник степени k. Теорию пространств $\mathcal{H}_k(S)$ можно найти, например, в монографиях [27],[16]. Хорошо известно понятие дробного интегродифференцирования (см. [1],[8], [10]–[20]) для гармонических функций (3.1) в $\mathbb{R}^n$, выраженное в виде ряда из сферических гармоник.

Определение 3.1.

$$\mathcal{D}^{-\alpha}_{n(ser)} f(x) := \sum_{k=0}^{\infty}\sum_{j=1}^{d_k} \frac{\Gamma(k+n/2)}{\Gamma(\alpha+k+n/2)}\, a_{kj}\, r^k\, Y_{kj}(\zeta), \qquad \alpha \geq 0, \tag{3.2}$$

$$\mathcal{D}^{\alpha}_{n(ser)} f(x) := \sum_{k=0}^{\infty}\sum_{j=1}^{d_k} \frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\, a_{kj}\, r^k\, Y_{kj}(\zeta), \qquad \alpha \geq 0. \tag{3.3}$$

Другой, более общий способ определения дробного интегрирования порядка $\alpha > 0$ выражается через дробные интегралы Римана–Лиувилля подобно случаю $n = 2$.

Определение 3.2.

$$D_n^{-\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} f(t\zeta)\, t^{n/2-1}\, dt =$$
$$= \frac{r^{\alpha+n/2-1}}{\Gamma(\alpha)} \int_0^1 (1-\xi)^{\alpha-1} f(\xi x)\, \xi^{n/2-1}\, d\xi = D_2^{-\alpha}\Big\{r^{n/2-1} f(x)\Big\}. \tag{3.4}$$

Это прямое обобщение оператора $D_2^{-\alpha}$ (см. (2.1)) на n-мерный случай. Как и при $n = 2$ определяем его модификацию $\mathcal{D}_n^{-\alpha}$ (сравни с [12], [8]).

Определение 3.3.

$$\mathcal{D}^{-\alpha}_{n(int)} f(x) := r^{-(\alpha+n/2-1)} D_n^{-\alpha} f(x) = r^{-(\alpha+n/2-1)} D_2^{-\alpha}\Big\{r^{n/2-1} f(x)\Big\} =$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\xi)^{\alpha-1} f(\xi x)\, \xi^{n/2-1}\, d\xi. \tag{3.5}$$

Преимущество модификации $\mathcal{D}^{-\alpha}_{n(int)}$ перед оператором $D_n^{-\alpha}$ состоит в том, что оператор $\mathcal{D}^{-\alpha}_{n(int)}$ сохраняет гармонические функции, т.е. $\mathcal{D}^{-\alpha}_{n(int)} : h(B) \longrightarrow h(B)$. Этим же преимуществом обладают операторы $\mathcal{D}^{\pm\alpha}_{n(ser)}$. Поскольку оба оператора $\mathcal{D}^{-\alpha}_{n(ser)}$ и $\mathcal{D}^{-\alpha}_{n(int)}$ выражают первообразную порядка $\alpha > 0$ функции f, то возникает естественный вопрос об их связи. На самом деле эти операторы совпадают для гармонических функций, что выражается следующей леммой.

Лемма 3.1. *Два определения* (3.2) *и* (3.5) *для первообразной* $\mathcal{D}_n^{-\alpha}$ ($\alpha > 0$) *совпадают для гармонических функций в* B, *т.е.*

$$\mathcal{D}_{n(ser)}^{-\alpha} f(x) = \mathcal{D}_{n(int)}^{-\alpha} f(x) =: \mathcal{D}_n^{-\alpha} f(x), \quad x \in B, \quad f \in h(B).$$

Доказательство. Пусть гармоническая функция $f \in h(B)$ разложена в ряд по сферическим гармоникам по формуле (3.1). Тогда по определениям (3.5) и (3.3)

$$\begin{aligned}\mathcal{D}_{n(int)}^{-\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tx)\, t^{n/2-1}\, dt = \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} \left(\sum_{k,j} a_{kj}\, t^k r^k\, Y_{kj}(\zeta) \right) t^{n/2-1}\, dt = \\ &= \frac{1}{\Gamma(\alpha)} \sum_{k,j} a_{kj}\, r^k\, Y_{kj}(\zeta) \left(\int_0^1 (1-t)^{\alpha-1}\, t^{k+n/2-1}\, dt \right) = \\ &= \sum_{k,j} \frac{\Gamma(k+n/2)}{\Gamma(\alpha+k+n/2)}\, a_{kj}\, r^k\, Y_{kj}(\zeta) = \mathcal{D}_{n(ser)}^{-\alpha} f(x),\end{aligned}$$

что и требовалось показать. □

В частности, действие оператора $\mathcal{D}_n^{-\alpha}$ на мономы типа r^γ ($\alpha > 0$, $\gamma > -n/2$) приводит к полезной формуле

$$\mathcal{D}_n^{-\alpha} \{r^\gamma\} = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} (tr)^\gamma\, t^{n/2-1}\, dt = \frac{\Gamma(\gamma+n/2)}{\Gamma(\alpha+\gamma+n/2)}\, r^\gamma.$$

4. Определение модификации дробной производной Римана–Лиувилля в $\mathbb{R}^n$

Если определение 3.3 оператора дробного интегрирования довольно естественно и удобно, то определение соответствующего оператора дробного дифференцирования вызывает некоторые проблемы. Мы уже определили дробное дифференцирование $\mathcal{D}_{n(ser)}^{\alpha}$ ($\alpha > 0$) для гармонической функции через разложение в ряд по сферическим гармоникам (3.2). Такое определение вполне естественно и немедленно влечет обратимость оператора, выраженное формулой, весьма полезной в приложениях,

$$\mathcal{D}_{n(ser)}^{\alpha} \mathcal{D}_n^{-\alpha} f(x) = \mathcal{D}_n^{-\alpha} \mathcal{D}_{n(ser)}^{\alpha} f(x) = f(x), \qquad f \in h(B).$$

Тем не менее, хотелось бы отстраниться от гармонических функций и их разложений в ряды по сферическим гармоникам и определить дробные производные как обратные к дробным первообразным для более общих функций. Поэтому будем решать следующую задачу: **найти явную и удобную форму для дробной производной $\mathfrak{D}_n^\alpha$ как обратного оператора к дробному интегралу $\mathfrak{D}_n^{-\alpha}$**. Естественное полугрупповое свойство $\mathfrak{D}^\alpha \mathfrak{D}^\beta f = \mathfrak{D}^{\alpha+\beta} f$ желательно, но не обязательно.

Одна форма для дробной производной $\mathfrak{D}_n^\alpha$ в неявном виде содержится в работах [10], [13] – [15], [19], [20]. Выразим ее в терминах оператора D_2^α, см. (2.2).

Определение 4.1. *(Дробная производная порядка $\alpha > 0$)*

$$\mathfrak{D}_{n(int)}^\alpha f(x) := D_{2,t}^\alpha \Big[t^{\alpha+n/2-1} f(tx) \Big]_{t=1}, \qquad \alpha > 0, \tag{4.1}$$

где производная $D_{2,t}^\alpha$ берется по переменной t.

Все же, здесь из-за введения дополнительной переменной t такая форма для дробной производной создает некоторые неудобства в применении.

Следующее наше определение новой формы дробной производной более удобно и приспособлено к дальнейшим применениям.

Определение 4.2. *(Дробная производная порядка $\alpha > 0$)*

$$\mathfrak{D}_n^\alpha f(x) := r^{-(n/2-1)} D_2^\alpha \Big[r^{\alpha+n/2-1} f(x) \Big], \tag{4.2}$$

где производная $D_2^\alpha = D_{2,r}^\alpha$, см. (2.2), *берется по переменной $r = |x|$, как обычно.*

При $n = 2$, очевидно, оператор $\mathfrak{D}_2^\alpha$ сводится к $\mathfrak{D}_2^\alpha$, см. (2.4).

Итак, на данный момент в $\mathbb{R}^n$ мы имеем три определения (3.3), (4.1) и (4.2) для дробной производной. На самом деле эти три определения совпадают для пригодных функций.

Теорема 4.1. *Три определения* (3.3), (4.1) *и* (4.2) *дробных производных совпадают для гармонических функций $f \in h(B)$, т.е.*

$$\mathfrak{D}_{n(ser)}^\alpha f(x) = \mathfrak{D}_{n(int)}^\alpha f(x) = \mathfrak{D}_n^\alpha f(x), \qquad \alpha > 0. \tag{4.3}$$

Для негармонических, но достаточно гладких функций f совпадают определения (4.1) *и* (4.2), *т.е.*

$$\mathfrak{D}_{n(int)}^\alpha f(x) = \mathfrak{D}_n^\alpha f(x), \qquad \alpha > 0. \tag{4.4}$$

Доказательство. Вначале применим операторы к мономам типа r^k $(k \geq 0)$. Согласно (4.1), (2.7), (3.3),

$$\mathcal{D}^{\alpha}_{n(int)}\{r^k\} = D^{\alpha}_{2,t}\left[t^{\alpha+n/2-1}(tr)^k\right]_{t=1} = r^k D^{\alpha}_{2,t}\left[t^{\alpha+k+n/2-1}\right]_{t=1} =$$

$$= r^k \frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\left[t^{k+n/2-1}\right]_{t=1} = \frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\, r^k,$$

Согласно (4.2), (2.7), (3.3),

$$\mathfrak{D}^{\alpha}_{n}\{r^k\} = r^{-(n/2-1)} D^{\alpha}_{2}\left[r^{\alpha+n/2-1}\, r^k\right] = r^{-(n/2-1)} D^{\alpha}_{2}\left[r^{\alpha+k+n/2-1}\right] =$$

$$= r^{-(n/2-1)} \frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\left[r^{k+n/2-1}\right] = \frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\, r^k.$$

Таким образом,

$$\mathcal{D}^{\alpha}_{n(int)}\{r^k\} = \mathfrak{D}^{\alpha}_{n}\{r^k\} = \mathcal{D}^{\alpha}_{n(ser)}\{r^k\} = \frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\, r^k. \tag{4.5}$$

Теперь пусть гармоническая в B функция f разложена в ряд (3.1). Ввиду (4.5) и линейности операторов получаем

$$\mathcal{D}^{\alpha}_{n(int)} f(x) = \mathcal{D}^{\alpha}_{n(int)}\left[\sum_{k,j} a_{kj}\, r^k\, Y_{kj}(\zeta)\right] = \sum_{k,j} a_{kj}\left[\mathcal{D}^{\alpha}_{n(int)}\{r^k\}\right] Y_{kj}(\zeta) =$$

$$= \sum_{k,j} a_{kj}\left[\frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\, r^k\right] Y_{kj}(\zeta) = \mathcal{D}^{\alpha}_{n(ser)} f(x),$$

$$\mathfrak{D}^{\alpha}_{n} f(x) = \mathfrak{D}^{\alpha}_{n}\left[\sum_{k,j} a_{kj}\, r^k\, Y_{kj}(\zeta)\right] = \sum_{k,j} a_{kj}\left[\mathfrak{D}^{\alpha}_{n}\{r^k\}\right] Y_{kj}(\zeta) =$$

$$= \sum_{k,j} a_{kj}\left[\frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\, r^k\right] Y_{kj}(\zeta) = \mathcal{D}^{\alpha}_{n(ser)} f(x),$$

что доказывает равенства (4.3).

Для негармонических функций f нам нужно доказать равенство (4.4), $\mathcal{D}^{\alpha}_{n(int)} f(x) = \mathfrak{D}^{\alpha}_{n} f(x)$, или что то же

$$D^{\alpha}_{2,t}\left[t^{\alpha+n/2-1} f(tx)\right]_{t=1} = r^{-(n/2-1)} D^{\alpha}_{2,r}\left[r^{\alpha+n/2-1} f(x)\right], \quad x = r\zeta \in B.$$

Вначале заметим, что для целых $m \geq 1$

$$D^{m}_{2,t}\left[t^{\alpha+\beta} f(tx)\right]_{t=1} = r^{-\beta} D^{m}_{2,r}\left[r^{\alpha+\beta} f(x)\right], \qquad \alpha > 0,\ \beta \geq 0, \tag{4.6}$$

что проверяется непосредственным дифференцированием. Это влечет равенство (4.4), $\mathcal{D}^{\alpha}_{n(int)} f(x) = \mathfrak{D}^{\alpha}_{n} f(x)$ для целых $\alpha = m \geq 1$. Для нецелых $\alpha > 0$ обозначим $m \in \mathbb{Z}$, $m \geq 1$, $m-1 < \alpha < m$, т.е. $[\alpha] = m-1$. Тогда (4.4) равносильно равенству

$$D^{-(m-\alpha)}_{2,t} D^{m}_{2,t}\left[t^{\alpha+n/2-1} f(tx)\right]_{t=1} = r^{-(n/2-1)} D^{-(m-\alpha)}_{2,r} D^{m}_{2,r}\left[r^{\alpha+n/2-1} f(x)\right],$$

а это следует из (4.6). Этим завершается доказательство (4.4). □

Таким образом, для всех типов операторов дробного дифференцирования будем пользоваться единым обозначением $\mathcal{D}_n^\alpha$:

$$\mathcal{D}^\alpha_{n(ser)} f(x) = \mathcal{D}^\alpha_{n(int)} f(x) = \mathfrak{D}^\alpha_n f(x) =: \mathcal{D}^\alpha_n f(x), \qquad \alpha > 0. \tag{4.7}$$

Лемма 4.1. *Для достаточно гладких функций f в шаре B и, в частности, для гармонических функций $f \in h(B)$ имеет место формула обратимости*

$$\mathcal{D}^\alpha_n \mathcal{D}^{-\alpha}_n f(x) = \mathcal{D}^{-\alpha}_n \mathcal{D}^\alpha_n f(x) = f(x), \qquad x \in B, \quad \alpha > 0. \tag{4.8}$$

Доказательство. Для гармонических функций формула (4.8) очевидна ввиду их разложимости в ряд сферических гармоник и определений (3.2), (3.3).

Для более общих функций, дифференцируемых достаточное число раз, будем доказывать (4.8) посредством определений (3.4), (3.5), (4.2). Ввиду этих определений и тождества $D_2^\alpha D_2^{-\alpha} g(x) = g(x)$ для интегрируемых функций g, получим

$$\begin{aligned}
\mathcal{D}^\alpha_n \mathcal{D}^{-\alpha}_n f(x) &= \mathcal{D}^\alpha_n \Big[r^{-(\alpha+n/2-1)} D_2^{-\alpha} \big\{ r^{n/2-1} f(x) \big\} \Big] = \\
&= r^{-(n/2-1)} D_2^\alpha \Big[r^{\alpha+n/2-1}\, r^{-(\alpha+n/2-1)} D_2^{-\alpha} \big\{ r^{n/2-1} f(x) \big\} \Big] = \\
&= r^{-(n/2-1)} D_2^\alpha D_2^{-\alpha} \big\{ r^{n/2-1} f(x) \big\} = r^{-(n/2-1)} r^{n/2-1} f(x) = f(x).
\end{aligned}$$

Аналогично доказываем

$$\begin{aligned}
\mathcal{D}^{-\alpha}_n \mathcal{D}^\alpha_n f(x) &= \mathcal{D}^{-\alpha}_n \Big[r^{-(n/2-1)} D_2^\alpha \big\{ r^{\alpha+n/2-1} f(x) \big\} \Big] = \\
&= r^{-(\alpha+n/2-1)} D_2^{-\alpha} \Big[r^{n/2-1}\, r^{-(n/2-1)} D_2^\alpha \big\{ r^{\alpha+n/2-1} f(x) \big\} \Big] = \\
&= r^{-(\alpha+n/2-1)} D_2^{-\alpha} D_2^\alpha \big\{ r^{\alpha+n/2-1} f(x) \big\} = r^{-(\alpha+n/2-1)} r^{\alpha+n/2-1} f(x) = f(x).
\end{aligned}$$

Здесь мы воспользовались тождеством $D_2^{-\alpha} D_2^\alpha \big\{ r^\beta f(x) \big\} = r^\beta f(x)$, $\beta \geq \alpha$. □

5. Интегральное представление для гармонических функций со смешанной нормой

Перейдем к применению операторов дробного интегродифференцирования для изучения весовых классов гармонических функций. Везде будем пользоваться стандартными обозначениями: $x = r\zeta$, $y = \rho\eta$, $0 \leq r, \rho < 1$, $\zeta, \eta \in S$, dV — нормированная n-мерная объемная мера на шаре B, $dV(x) = n\, r^{n-1} dr d\sigma(\zeta)$, $V(B) = 1$. Каждая гармоническая функция $u(x) \in h(B)$ разложима в ряд однородных сферических гармоник

$$u(x) = \sum_{k=0}^\infty \sum_{j=1}^{d_k} a_{kj}\, r^k\, Y_{kj}(\zeta) \equiv \sum_{k,j} a_{kj}\, r^k\, Y_{kj}(\zeta), \qquad x = r\zeta \in B, \tag{5.1}$$

Напомним определение расширенного ядра Пуассона в шаре B, см. [16, глава 6].

Определение 5.1. *(Расширенное ядро Пуассона в шаре B)*

$$P(x,y) \equiv P_0(x,y) := \sum_{k=0}^{\infty} Z_k(x,y) = \frac{1-|x|^2|y|^2}{(1-2x\cdot y+|x|^2|y|^2)^{n/2}}, \qquad x\in B,\ y\in \overline{B},$$

где Z_k — зональные гармоники, см. [27],[16].

Расширенное ядро Пуассона гармонично в B по каждой из переменных x и y, а также гармонично по x при $y \in \overline{B}$. Оно обладает полезными свойствами $P(x,y) = P(y,x)$, $P(x,y) = P(ry,\zeta)$, $x = r\zeta$, разлагается в ряд сферических гармоник

$$P(x,y) = \sum_{k=0}^{\infty} Z_k(x,y) = \sum_{k=0}^{\infty}\sum_{j=1}^{d_k} \overline{Y_{k_j}(y)}\, r^k\, Y_{k_j}(\zeta).$$

Если $\eta \in S$, то $P(x,\eta)$ — обычное (нерасширенное) ядро Пуассона в B.

Определим теперь ядро Пуассона–Бергмана.

Определение 5.2. *(Ядро Пуассона–Бергмана в шаре B)*

$$(5.2)\qquad P_\alpha(x,y) := \mathcal{D}_n^\alpha P(x,y) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\, Z_k(x,y), \quad x,y\in B, \quad \alpha \geq 0.$$

Подобные ядра определены в работах [8], [10] (для целых α), [11] – [15], [18] – [20]. При $\alpha = 0$, очевидно, ядро (5.2) сводится к расширенному ядру Пуассона $P \equiv P_0$ в шаре B. Ядро Пуассона–Бергмана (5.2) вместе с техникой дробного интегродифференцирования позволяет вывести интегральную формулу из Теоремы 1.1 типа Пуассона–Бергмана для функций из $h(p,q,\alpha)$.

Доказательство Теоремы 1.1. Обозначим интеграл в правой части формулы (1.1) как оператор

$$(T_\alpha u)(x) := \frac{2}{n\,\Gamma(\alpha)} \int_B (1-|y|^2)^{\alpha-1}\, P_\alpha(x,y)\, u(y)\, dV(y), \qquad x\in B, \qquad \alpha>0.$$

Сначала докажем теорему для функций $u(x) \in h(1,1,\beta)$. Для этого вначале предположим, что $u(x) \in h(B)\cap C(\overline{B})$. Полагая, что функция $u(x)$ разложена в ряд сферических гармоник (5.1), преобразуем интеграл $(T_\beta u)(x)$,

$$(T_\beta u)(x) = \frac{2}{n\,\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1}\, P_\beta(x,y)\, u(y)\, dV(y)$$

$$= \frac{2}{n\,\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1} \left[\sum_{k=0}^{\infty} \frac{\Gamma(\beta+k+n/2)}{\Gamma(k+n/2)}\, Z_k(x,y)\right] \left(\sum_{m,j} a_{mj}\, Y_{mj}(y)\right) dV(y)$$

$$= \frac{2}{n\,\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\beta+k+n/2)}{\Gamma(k+n/2)} \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} a_{mj} \int_B (1-|y|^2)^{\beta-1}\, Z_k(x,y)\, Y_{mj}(y)\, dV(y).$$

Учитывая, что $Z_k(x,y) = r^k \rho^k Z_k(\zeta,\eta)$, $\quad Z_k(\zeta,\eta) = \sum_{i=1}^{d_k} Y_{ki}(\zeta)\,\overline{Y_{ki}(\eta)}$, отдельно упростим последний интеграл, считая $k \geq 0$ фиксированным,

$$\sum_{m=0}^{\infty} \sum_{j=1}^{d_m} a_{mj} \int_B (1-|y|^2)^{\beta-1}\, Z_k(x,y)\, \rho^m\, Y_{mj}(\eta)\, dV(y)$$

$$= r^k \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} a_{mj} \int_B (1-|y|^2)^{\beta-1}\, \rho^k \rho^m\, Z_k(\zeta,\eta)\, Y_{mj}(\eta)\, dV(y)$$

$$= n\, r^k \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} a_{mj} \int_0^1 (1-\rho^2)^{\beta-1}\, \rho^k \rho^m\, \rho^{n-1} \left(\int_S Z_k(\zeta,\eta)\, Y_{mj}(\eta)\, d\sigma(\eta)\right) d\rho$$

$$= n\, r^k \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} a_{mj} \int_0^1 (1-\rho^2)^{\beta-1}\, \rho^{k+m+n-1} \sum_{i=1}^{d_k} Y_{ki}(\zeta) \left[\int_S \overline{Y_{ki}(\eta)}\, Y_{mj}(\eta)\, d\sigma(\eta)\right] d\rho$$

$$= n\, r^k \sum_{j=1}^{d_k} a_{kj}\, Y_{kj}(\zeta) \int_0^1 (1-\rho^2)^{\beta-1}\, \rho^{2k+n-1}\, d\rho$$

$$= \frac{n}{2}\, r^k\, \frac{\Gamma(\beta)\,\Gamma(k+n/2)}{\Gamma(\beta+k+n/2)} \sum_{j=1}^{d_k} a_{kj}\, Y_{kj}(\zeta).$$

Здесь мы воспользовались ортогональностью в $L^2(S)$ сферических гармоник разных степеней $(k \neq m)$. Далее, подставляя, получаем

$$(T_\beta u)(x) = \frac{2}{n\,\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\beta+k+n/2)}{\Gamma(k+n/2)} \left(\frac{n}{2}\, r^k\, \frac{\Gamma(\beta)\,\Gamma(k+n/2)}{\Gamma(\beta+k+n/2)} \sum_{j=1}^{d_k} a_{kj}\, Y_{kj}(\zeta)\right) =$$

$$= \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} a_{kj}\, r^k\, Y_{kj}(\zeta) = u(x).$$

Искомая функция $u(x) = (T_\beta u)(x)$, (1.1) получена. Для произвольной функции $u(x) \in h(1,1,\beta)$ применим доказанную часть теоремы по отношению к растянутой функции $u_\delta(x) := u(\delta x)$, $0 < \delta < 1$, что приводит к $u_\delta(x) = (T_\beta u_\delta)(x)$. Теперь остается перейти к пределу при $\delta \to 1^-$ (см., например, [17],[20])

$$\|u - u_\delta\|_{h(1,1,\beta)} \longrightarrow 0, \qquad \delta \to 1^-, \tag{5.3}$$

и в итоге получаем искомую формулу (1.1) для произвольной функции $u \in h(1,1,\beta)$. Теперь докажем теорему для общей функции $u \in h(p,q,\alpha)$.

Если $1 \le p \le \infty$, $0 < q \le 1$, $\beta \ge \alpha$, то согласно вложениям [21, Теор.1(i)-(iii)] имеем $h(p,q,\alpha) \subset h(1,1,\beta)$.

Если $1 \le p \le \infty$, $0 < q \le \infty$, $\beta > \alpha$, то согласно вложениям [21, Теор.1(ii),(vi)] имеем $h(p,q,\alpha) \subset h(1,q,\alpha) \subset h(1,1,\beta)$.

Если $0 < p < 1$, $0 < q \le \infty$, $\beta > \alpha + (n-1)(1/p - 1)$, то согласно вложениям [21, Теор.1(iv),(vi)] имеем $h(p,q,\alpha) \subset h\big(1,q,\alpha + (n-1)(1/p-1)\big) \subset h(1,1,\beta)$.

Во всех случаях получаем $u(x) \in h(1,1,\beta)$, что сводит доказательство к уже доказанному случаю. $\square$

Замечание 5.1. *Легко заметить, что в приведенном доказательстве мы лишь проверили истинность* (1.1) *без конструктивного вывода этой формулы. Такой метод доказательства в специальных случаях применялся, например, в* [11], [12],[8],[19],[20]. *Было бы полезно найти конструктивный вывод формулы* (1.1). *Наше определение дробной производной* (4.2), *Теорема 4.1 и Лемма 4.1 позволяют найти конструктивное и очень короткое доказательство Теоремы 1.1.*

Второе доказательство Теоремы 1.1. Пусть $u(x) \in h(1,1,\beta)$, $\beta > 0$ — произвольная функция. В силу формулы обратимости (4.8) и определения 4.2, можем преобразовать

$$\begin{aligned} u(x) &= \mathcal{D}_n^{-\beta}\mathcal{D}_n^{\beta}u(x) = \frac{1}{\Gamma(\beta)}\int_0^1 (1-t)^{\beta-1}\mathcal{D}_n^{\beta}u(tx)\, t^{n/2-1}\, dt = \\ &= \frac{1}{\Gamma(\beta)}\int_0^1 (1-\rho^2)^{\beta-1}\mathcal{D}_n^{\beta}u(\rho^2 x)\, \rho^{2(n/2-1)}\, 2\rho\, d\rho = \\ &= \frac{2}{\Gamma(\beta)}\int_0^1 (1-\rho^2)^{\beta-1}\mathcal{D}_n^{\beta}\left[\int_S P(x,\rho\eta)\, u(\rho\eta)\, d\sigma(\eta)\right] \rho^{n-1}\, d\rho = \\ &= \frac{2}{n\,\Gamma(\beta)}\int_0^1\int_S (1-\rho^2)^{\beta-1}\mathcal{D}_n^{\beta}P(x,\rho\eta)\, u(\rho\eta)\, n\, \rho^{n-1}\, d\rho\, d\sigma(\eta) = \\ &= \frac{2}{n\,\Gamma(\beta)}\int_B (1-|y|^2)^{\beta-1}\, P_\beta(x,y)\, u(y)\, dV(y), \qquad x \in B. \end{aligned}$$

Сходимость последнего интеграла внутри шара B следует из оценки ядра Пуассона–Бергмана (см., например, [13],[14],[18])

$$|P_\beta(x,y)| \le C(\beta,n)\frac{1}{(1-|x||y|)^{\beta+n-1}}, \qquad x,y \in B.$$

Abstract. In the paper, a new form of fractional derivative is introduced for functions defined in a unit ball in $\mathbb{R}^n$. As an application, an integral representation for harmonic functions with finite mixed-norm in the ball is obtained.

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ЗАВИСЯЩЕЕ ОТ ОРИЕНТАЦИИ РАСПРЕДЕЛЕНИЕ ДЛИНЫ ХОРДЫ КАК ФУНКЦИЯ МАКСИМАЛЬНОЙ ХОРДЫ

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Аннотация. В [3] и [4] авторы доказали, что ковариограмма и зависящая от ориентации функция распределения длины хорды для треугольника и эллипса являются функциями максимальной хорды $t_{max}(u)$ в направлении u. В статье получено необходимое условие того, что зависящая от ориентации функция распределения длины хорды была бы функцией зависящей от максимальной хорды. Также приведен класс параллелограммов для которых необходимое условие не выполняется.

MSC2010 numbers: 60D05; 52A22; 53C65.

Ключевые слова: ковариограмма; максимальная хорда; зависящее от ориентации распределение.

1. Введение

Пусть $R^n (n \geq 2)$ n-мерное евклидово пространство, $\mathbf{D} \subset R^n$ - ограниченная выпуклая область с внутренними точками, и V_n - n-мерная мера Лебега в R^n. Пусть S^{n-1} - $(n-1)$-мерная единичная сфера с центром в начале координат. Случайная прямая, параллельная направлению $u \in S^{n-1}$ и пересекающая $\mathbf{D}$ имеет точку пересечения (обозначим ее через x) с $\Pi r_{u^\perp} \mathbf{D}$, где $\Pi r_{u^\perp} \mathbf{D}$ ортогональная проекция $\mathbf{D}$ на гиперплоскость $u^\perp (u^\perp$ - гиперплоскость проходящая через начало координат с нормальным вектором u). Можно отождествлять точки $\Pi r_{u^\perp} \mathbf{D}$ с прямыми, которые пересекают $\mathbf{D}$ и параллельны направлению u. Предположив, что точка пересечения x равномерно распределена в $\Pi r_{u^\perp} \mathbf{D}$, мы можем определить следующую функцию распределения:

Определение 1.1. *Функция*

$$F(u,t) = \frac{V_{n-1}\{x \in \Pi r_{u^\perp} \mathbf{D} : V_1(g(u,x) \cap \mathbf{D}) < t)\}}{b_{\mathbf{D}}(u)}$$

[1]Работа первого автора выполнена при частичной поддержке государственного комитета по науке Республики Армения, Грант 14А-1а09

называется зависящей от ориентации функцией распределения длины хорды $\mathbf{D}$ *в направлении* u *в точке* $t \in R^1$, *где* $g(u,x)$ - *прямая параллельная* $u \in S^{n-1}$ *и пересекающая* $\Pi r_{u^\perp}\mathbf{D}$ *в точке* x, *а* $b_{\mathbf{D}}(u) = V_{n-1}(\Pi r_{u^\perp}\mathbf{D})$.

Определение 1.2. *(см. [1]). Функция*

$$C(\mathbf{D}, h) = V_n(\mathbf{D} \cap (\mathbf{D} + h)), \quad h \in R^n,$$

называется ковариограммой $\mathbf{D}$. *Здесь* $\mathbf{D} + h = \{x + h, x \in \mathbf{D}\}$.

Мы можем представить каждый вектор $h \in R^n$ как $h = tu$, где u - направление h, а t его длина.

Лемма 1.1. *(см. [1]). Пусть* $u \in S^{n-1}$ *и* $t > 0$ *такое, что* $\mathbf{D} \cap (\mathbf{D} + tu)$ *содержит внутренние точки. Тогда* $C(\mathbf{D}, u, t)$ *дифференцируема по* t *и*

$$-\frac{\partial C(\mathbf{D}, u, t)}{\partial t} = (1 - F(u,t)) \cdot b_{\mathbf{D}}(u). \tag{1.1}$$

В точке $t = 0$ *существует правосторонняя производная левой части, и равенство (1.3) остается в силе.*

Ж. Матерон сформулировал гипотезу, что ковариограмма выпуклого тела $\mathbf{D}$ определяет $\mathbf{D}$ в классе всех выпуклых тел, с точностью до параллельных переносов и отражений (см. [1]). Г. Бианчи и Г. Аверков доказали, что каждая плоская выпуклая область определяется в классе всех плоских выпуклых областей по ковариограмме, с точностью до параллельных переносов и отражений (см. [2]).

Пусть $L(\omega)$ - случайный отрезок длины $l > 0$, параллельный фиксированному направлению u и пересекающий $\mathbf{D}$. Рассмотрим случайную величину $|L|(\omega) = V_1(L(\omega) \cap \mathbf{D})$, где $L(\omega)$ принадлежит следующему множеству:

$$\Omega(u) = \{\text{отрезки длины } l, \text{ пересекающие } \mathbf{D} \text{ и имеющие направление } u\}.$$

Случайный отрезок $L(\omega)$, лежащий на прямой $g(u,x)$, можно задавать координатами $(g(u,x), y)$, где $g(u,x)$ – прямая с направлением $u \in S^{n-1}$, проходящая через точку $x \in \Pi r_{u^\perp}\mathbf{D}$, а y одномерная координата центра отрезка $L(\omega)$ на прямой $g(u,x)$. Началом координат на прямой $g(u,x)$ берется одна из точек пересечений $g(u,x)$ с $\mathbf{D}$. Используя вышеупомянутые обозначения можно отождествлять $\Omega(u)$ с множеством:

$$\Omega(u) = \left\{(x,y) : x \in \Pi r_{u^\perp}\mathbf{D}, \quad y \in \left[-\frac{l}{2}, \chi(u,x) + \frac{l}{2}\right]\right\},$$

где $\chi(u,x) = V_1(g(u,x) \cap \mathbf{D})$.

Определение 1.3. *Функция*

$$F_{|L|}(u,t)=\frac{V_n(\{(x,y)\in\Omega(u):\ |L|(x,y)<t\})}{V_n(\Omega(u))}$$

называется зависящей от ориентации функцией распределения длины случайного отрезка в направлении $u\in S^{n-1}$.

В [6] получена связь между функцией распределения случайной величины $|L|(\omega)$ и зависящей от ориентации функцией распределения длины хорды в R^n:

$$(1.2)\quad F_{|L|}(u,t)=\begin{cases}0 & \text{для}\quad t\le 0,\\ \dfrac{b_{\mathbf{D}}(u)\left[2t+F(u,t)(l-t)-\int_0^t F(u,z)dz\right]}{V_n(\mathbf{D})+lb_{\mathbf{D}}(u)} & \text{для}\quad 0\le t\le l,\\ 1 & \text{для}\quad t>l.\end{cases}$$

2. Необходимое условие на $F(u,t)$

Зафиксируем $u\in S^{n-1}$. Пусть $L(\omega)$ случайный отрезок длины $t_{max}(u)$ ($t_{max}(u)$ - хорда максимальной длины $\mathbf{D}$ в направлении $u\in S^{n-1}$), параллельный фиксированному направлению u и пересекающий $\mathbf{D}$. Подставляя в (1.2) $|L|=l=t_{max}(u)$ получаем:

(2.1)

$$F_{t_{max}(u)}(u,t)=\begin{cases}0 & \text{для}\quad t\le 0,\\ C\cdot\left[2t+F(u,t)(t_{max}(u)-t)-\displaystyle\int_0^t F(u,z)dz\right] & \text{для}\quad 0\le t\le t_{max}(u),\\ 1 & \text{для}\quad t>t_{max}(u).\end{cases}$$

где $C=\frac{b_{\mathbf{D}}(u)}{V_n(\mathbf{D})+t_{max}(u)b_{\mathbf{D}}(u)}$.

Левая часть (2.1) в точке t есть вероятность, что длина пересечения $\mathbf{D}$ и случайного отрезка $L(\omega)$ меньше чем t, следовательно $F_{t_{max}(u)}(u,t_{max}(u))=1$. Таким образом, подставляя в (2.1) $t=t_{max}(u)$, получаем

$$(2.2)\qquad 1=\frac{b_{\mathbf{D}}(u)}{V_n(\mathbf{D})+t_{max}(u)b_{\mathbf{D}}(u)}\left[2t_{max}(u)-\int_0^{t_{max}(u)}F(u,z)dz\right]$$

После элементарных преобразований (2.2) получаем следующее утверждение.

Теорема 2.1. *Для любого выпуклого тела* $\mathbf{D}$ *и любого направления* $u\in S^{n-1}$ *интеграл зависящей от ориентации функции распределения длины хорды на интервале* $[0,t_{max}(u)]$ *дается следующей формулой:*

$$(2.3)\qquad \int_0^{t_{max}(u)}F(u,z)dz=t_{max}(u)-\frac{V_n(\mathbf{D})}{b_{\mathbf{D}}(u)}$$

Так как левая часть (2.3) неотрицательна, получаем следующее следствие теоремы 2.1:

Следствие 2.1. *Для каждого выпуклого тела* $\mathbf{D} \subset R^n$ *и направления* $u \in S^{n-1}$ *имеет место неравенство:*

$$V_n(\mathbf{D}) \leq b_{\mathbf{D}}(u) \cdot t_{max}(u).$$

Из (2.3) также имеем:

Следствие 2.2. *Для любых двух направлений* $u_1, u_2 \in S^{n-1}$ *таких, что* $t_{max}(u_1) = t_{max}(u_2)$ *и*

$$\int_0^{t_{max}(u_1)} F(u_1, z)dz = \int_0^{t_{max}(u_2)} F(u_2, z)dz$$

следует

$$b_{\mathbf{D}}(u_1) = b_{\mathbf{D}}(u_2).$$

Допустим, что существует функция $G(x, y)$ от двух переменных такая, что $F(u, t) = G(t_{max}(u), t)$. В этом случае, из равенства $t_{max}(u_1) = t_{max}(u_2)$ следует:

$$\int_0^{t_{max}(u_1)} F(u_1, z)dz = \int_0^{t_{max}(u_2)} F(u_2, z)dz$$

Следовательно, из следствия 2.2 получаем следующий результат:

Следствие 2.3. *Если зависящая от ориентации функция распределения длины хорды зависит от направления только через* $t_{max}(u)$*, тогда для любых двух направлений* $u_1, u_2 \in S^{n-1}$ *таких, что* $t_{max}(u_1) = t_{max}(u_2)$ *следует*

$$b_{\mathbf{D}}(u_1) = b_{\mathbf{D}}(u_2).$$

Следствие 2.3 является необходимым условием того, что зависящая от ориентации функция распределения длины хорды была бы функцией, зависящей от направления через $t_{max}(u)$. Зависящая от ориентации функция распределения длины хорды для треугольников и эллипсов зависит от направления через $t_{max}(u)$ (см. [3] и [4]), следовательно необходимое условие в этих случаях удовлетворяется. Проверим это для треугольника (для эллипса это можно сделать аналогично). Из [3] имеем, что ковариограмма треугольника Δ с площадью S дается следующей формулой:

$$C(\Delta, u, t) = S\left(1 - \frac{t}{t_{\max}(u)}\right)^2 \tag{2.4}$$

где $t \in [0, t_{\max}(u)]$. Поэтому из (2.4) получаем

$$-\frac{\partial C(\Delta, u, t)}{\partial t}(0) = \frac{2S}{t_{\max}(u)}. \tag{2.5}$$

С другой стороны из (1.3) имеем

$$-\frac{\partial C(\Delta, u, t)}{\partial t}(0) = b_\Delta(u). \tag{2.6}$$

Из (2.5) и (2.6) получаем

$$S = \frac{1}{2} t_{\max}(u) b_\Delta(u). \tag{2.7}$$

Используя (2.7), легко заметить, что для любых двух направлений $u_1, u_2 \in S^{n-1}$ таких, что $t_{max}(u_1) = t_{max}(u_2)$, следует $b_\Delta(u_1) = b_\Delta(u_2)$.

3. Необходимое условие для параллелограмма

Пусть $ABCD$ - параллелограмм. Обозначим $|AB| = a_1$, $|AD| = a_2$, $\angle BAD = \alpha$. Рассмотрим направление луча AD как нулевое, а направление против часовой стрелки в качестве положительного. В [5] получены явные выражения для максимальной хорды и функции ширины для параллелограмма $ABCD$:

$$t_{\max}(u) = \begin{cases} \frac{a_2 \sin\alpha}{\sin(\alpha-u)}, & u \in \left(0, \operatorname{arcctg}\left(\frac{a_2}{a_1 \sin\alpha} + \cot\alpha\right)\right] \\ \frac{a_1 \sin\alpha}{\sin u}, & u \in \left[\operatorname{arcctg}\left(\frac{a_2}{a_1 \sin\alpha} + \cot\alpha\right), \operatorname{arcctg}\left(\cot\alpha - \frac{a_2}{a_1 \sin\alpha}\right)\right], \ u \neq \alpha \\ \frac{a_2 \sin\alpha}{\sin(u-\alpha)}, & u \in \left[\operatorname{arcctg}\left(\cot\alpha - \frac{a_2}{a_1 \sin\alpha}\right), \pi\right) \\ a_2, & u = 0 \\ a_1, & u = \alpha \end{cases} \tag{3.1}$$

$$b_{ABCD}(u) = \begin{cases} a_2 \sin u + a_1 \sin(\alpha - u), & u \in [0, \alpha] \\ a_2 \sin u + a_1 \sin(u - \alpha), & u \in [\alpha, \pi] \end{cases} \tag{3.2}$$

Рассмотрим ромб со стороной a и $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$. В этом случае, из (3.1) и (3.2), для максимальной хорды и функции ширины получаем:

$$t_{\max}(u) = \begin{cases} \frac{a \sin\alpha}{\sin(\alpha-u)}, & u \in \left(0, \frac{\alpha}{2}\right] \\ \frac{a \sin\alpha}{\sin u}, & u \in \left[\frac{\alpha}{2}, \frac{\pi}{2} + \frac{\alpha}{2}\right], \ u \neq \alpha \\ \frac{a \sin\alpha}{\sin(u-\alpha)}, & u \in \left[\frac{\pi}{2} + \frac{\alpha}{2}, \pi\right) \\ a, & u = 0, \alpha \end{cases} \tag{3.3}$$

$$b_{ABCD}(u) = \begin{cases} a(\sin u + \sin(\alpha - u)), & u \in [0, \alpha] \\ a(\sin u + \sin(u - \alpha)), & u \in [\alpha, \pi] \end{cases} \tag{3.4}$$

Обозначим $u_1 = \frac{\alpha}{n}$ и $u_2 = 2\alpha - \frac{\alpha}{n}$, где n натуральное число. Очевидно, что $u_1 \in (0, \frac{\alpha}{2}]$ для любого $n \geq 2$. Поскольку $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$, то $\frac{\alpha}{2} + \frac{\pi}{2} < 2\alpha < \pi$, следовательно, $u_2 \in [\frac{\pi}{2} + \frac{\alpha}{2}, \pi)$ для достаточно большого натурального числа n. Из (3.3) имеем что равенство $t_{\max}(u_1) = t_{\max}(u_2)$ выполняется тогда и только тогда, когда

$$\sin(\alpha - u_1) = \sin(u_2 - \alpha) \tag{3.5}$$

Подставляя в (3.5) значения u_1 и u_2 легко проверить, что (3.5) выполнено. Теперь покажем, что необходимое условие из следствия 2.3 не выполняется для направлений u_1 и u_2. Допустим необходимое условие выполняется. В этом случае из (3.4) имеем, что равенство $b_{ABCD}(u_1) = b_{ABCD}(u_2)$ эквивалентно следующему

$$a(\sin u_2 - \sin u_1) = 0 \tag{3.6}$$

Подставляя в (3.6) значения u_1 и u_2 получаем эквивалентное уравнение:

$$\sin\frac{\alpha}{n} - \sin\left(2\alpha - \frac{\alpha}{n}\right) = 0 \tag{3.7}$$

После элементарных преобразований из (3.7) получаем:

$$\sin\left(\frac{1-n}{n} \cdot \alpha\right) \cdot \cos\alpha = 0 \tag{3.8}$$

Уравнение (3.8) имеет место, если $\alpha = \frac{\pi}{2}$ или $\frac{1-n}{n} \cdot \alpha = \pi k$ где k целое число, следовательно для достаточно большого натурального числа n (3.8) не имеет места (так как $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$). Следовательно $b_{ABCD}(u_1) \neq b_{ABCD}(u_2)$.

Abstract. In [3] and [4] have been proved that covariogram and orientation-dependent chord length distribution function of a triangle and an ellipse depends on maximal chord $t_{max}(u)$ in direction u. In this paper a necessary condition for orientation-dependent chord length distribution function as a function of maximal chord is obtained. A class of parallelograms is constructed for which this necessary condition is violated.

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SPECIFIC PROPERTIES OF SOLUTIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS WITH SEVERAL DELAY ARGUMENTS

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Abstract. The paper considers the following differential equation $x'(t) + \sum_{i=1}^{m} p_i(t)\, x(\tau_i(t)) = 0$, $t \geq 0$, where $p_i \in L_{loc}(R_+; R_+)$, $\tau_i \in C(R_+; R)$, $\tau_i(t) \leq t$ and $\lim_{t\to+\infty} \tau_i(t) = +\infty$, $i = 1, \ldots, m$. New oscillation criteria of all solutions for this equation are established.

MSC2010 numbers:* 34K11
Keywords: Oscillation; proper solution; differential equation; delay argument.

1. Introduction

It is a trivial consequence of the uniqueness of solutions of initial value problems that first order linear ordinary differential equations cannot have oscillatory solutions. As for the equation

$$x'(t) + p(t)\, x(\tau(t)) = 0, \tag{1.1}$$

the presence of a delay leads to the fact that oscillatory solutions do appear. Moreover, if p is nonnegative and the delay is sufficiently large, all proper solutions (see Definition 2.1 below) turn out to be oscillatory. Specific criteria for the oscillation of proper solutions of linear delay equations were for the first time proposed by A. D. Myshkis [1]. It follows from the results of [2, 3] that if the functions $p : R_+ \to R_+$ ($R_+ = [0, +\infty)$) and $\tau : R_+ \to R$ are continuous, $\tau(t) \leq t$ for $t \in R_+$, $\lim_{t\to+\infty} \tau(t) = +\infty$,

$$p^* = \limsup_{t\to+\infty} \int_{\tau(t)}^{t} p(s)ds, \quad p_* = \liminf_{t\to+\infty} \int_{\tau(t)}^{t} p(s)ds$$

and

$$\text{either} \quad p^* > 1 \quad \text{or} \quad p_* > \frac{1}{e}, \tag{1.2}$$

then the equation (1.1) is oscillatory. Note that if $p_* \leq \frac{1}{e}$, the condition $p^* > 1$ can be improved.

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For $\tau(t) = t - \tau$ ($\tau = \text{const} > 0$) such an improvement was carried out successively in [4–6], where the condition $p^* > 1$ was replaced, respectively, by $p^* > 1 - \frac{p_*^2}{4}$, $p^* > 1 - \frac{p_*^2}{2(1-p_*)}$ and

$$p^* > 1 - \frac{1 - p_* - \sqrt{1 - 2p_* - p_*^2}}{2}.$$

On the other hand, in [7, 8] sufficient conditions for oscillation of all proper solutions of equation (1.1) were given, which involve classes of inequalities not satisfying condition (1.2). In the present paper, using the ideas of [9], we establish some new criteria for the equation

$$x'(t) + \sum_{i=1}^{m} p_i(t)\, x(\tau_i(t)) = 0, \tag{1.3}$$

where

$$\begin{gathered} p_i \in L_{\mathrm{loc}}(R_+; R_+), \quad \tau_i \in C(R_+; R), \quad \tau_i(t) \le t \quad \text{for} \quad t \in R_+ \\ \text{and} \quad \lim_{t \to +\infty} \tau_i(t) = +\infty \quad (i = 1, \dots, m), \end{gathered} \tag{1.4}$$

to be oscillatory.

2. The main results

Throughout the paper we assume that $\tau_*(t) = \min\{\tau_i(t) : i = 1, \dots, m\}$. Put

$$\begin{gathered} \eta^{\tau_*}(t) = \max\{s : \tau_*(s) \le t\} \quad \text{for} \quad t \in R_+, \\ \eta_1^{\tau_*}(t) = \eta^{\tau_*}(t), \quad \eta_i^{\tau_*} = \eta^{\tau_*} \circ \eta_{i-1}^{\tau_*} \quad (i = 2, 3, \dots). \end{gathered} \tag{2.1}$$

Definition 2.1. *Let $t_0 \in R_+$. A continuous function $x : [t_0, +\infty) \to R$ is said to be a proper solution of equation* (1.3) *if it is locally absolutely continuous on $[\eta^{\tau_*}(t_0), +\infty)$, satisfies* (1.3) *almost everywhere on $[\eta^{\tau_*}(t_0), +\infty)$, and*

$$\sup\{|x(s)| : t \le s < +\infty\} > 0 \quad \textit{for} \quad t \ge t_0.$$

Definition 2.2. *A proper solution of equation* (1.3) *is said to be oscillatory if the set of its zeros is unbounded from above. Otherwise it is said to be nonoscillatory.*

Definition 2.3. *The equation* (1.3) *is said to be oscillatory, if any of its proper solutions is oscillatory. Define*

$$\psi_1(t) = 0, \quad \psi_i(t) = \exp\left\{\sum_{j=1}^{m} \int_{\tau_j(t)}^{t} \Big(\prod_{\ell=1}^{m} p_\ell(s)\Big)^{\frac{1}{m}} \psi_{i-1}(s)\, ds\right\}, \quad i = 2, 3, \dots. \tag{2.2}$$

Theorem 2.1. *Let there exist $k \in N$ and non-decreasing functions $\sigma_i \in C(R_+;R)$ such that*

$$\tau_i(t) \le \sigma_i(t) \le t, \quad i = 1,\dots,m \tag{2.3}$$

and

$$\limsup_{t\to+\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left\{ m \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} \psi_k(\xi)\, d\xi \right\} ds \right]^{\frac{1}{m}} >$$

$$> \frac{1}{m^m} - \prod_{\ell=1}^{m} c_\ell(p_{*\ell}). \tag{2.4}$$

Then the equation (1.3) *is oscillatory, where*

$$p_{*\ell} = \liminf_{t\to+\infty} \int_{\sigma_\ell(t)}^{t} p_\ell(s)\, ds,$$

$$c_\ell(p_{*\ell}) = \frac{1 - p_{*\ell} - \sqrt{1 - 2p_{*\ell} - p_{*\ell}^2}}{2}, \quad \ell = 1,\dots,m. \tag{2.5}$$

Theorem 2.1′. *Let $\overline{p}_* \le \frac{1}{e}$ and there exist nondecreasing functions $\sigma_i \in C(R_+;R)$ such that the conditions* (2.3) *are fulfilled and for some $\varepsilon \in (0, \lambda^*(\overline{p}_*))$*

$$\limsup_{t\to+\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left\{ (\lambda^*(\overline{p}_*) - \varepsilon) \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} d\xi \right\} ds \right]^{\frac{1}{m}} >$$

$$> \frac{1}{m^m} - \prod_{\ell=1}^{m} c_\ell(p_{*\ell}). \tag{2.6}$$

Then equation (1.3) *is oscillatory, where $p_{*\ell}$ and $c_\ell(p_{*\ell})$ are given by* (2.5) *and $\lambda^*(\overline{p}_*)$ is the smallest root of the equation*

$$e^{\overline{p}_*\lambda} = \lambda, \tag{2.7}$$

$$\overline{p}_* = \liminf_{t\to+\infty} \sum_{i=1}^{m} \int_{\tau_i(t)}^{t} \Big(\prod_{\ell=1}^{m} p_\ell(s) \Big)^{\frac{1}{m}} ds > 0. \tag{2.8}$$

Corollary 2.1. *Let τ_i be nondecreasing functions, $p_i(t) \ge p(t)$ $(i = 1,\dots,m)$, $p \in L_{\mathrm{loc}}(R_+ : R_+)$ and for some $\varepsilon \in (0, \lambda^*(\overline{p}_*))$*

$$\limsup_{t\to+\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\tau_j(t)}^{t} p(s) \exp\left\{ m(\lambda^*(\overline{p}_*) - \varepsilon) \int_{\tau_i(s)}^{\tau_i(t)} p(\xi)\, d\xi \right\} ds \right]^{\frac{1}{m}} >$$

$$> \frac{1}{m^m} - \prod_{i=1}^{m} c_\ell(p_{*\ell}). \tag{2.9}$$

Then equation (1.3) *is oscillatory, where $p_{*\ell}$ and c_ℓ are given by* (2.5) *and $\lambda^*(\overline{p}_*)$ is the smallest root of equation* (2.7).

Theorem 2.2. *Let there exist nondecreasing functions $\sigma_i \in C(R_+; R)$ such that conditions* (2.3) *are fulfilled,*

$$\limsup_{t\to+\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} d\xi\, ds \right]^{\frac{1}{m}} > 0, \tag{2.10}$$

and let

$$\liminf_{t\to+\infty} \sum_{i=1}^{m} \int_{\tau_i(t)}^{t} \Big(\prod_{\ell=1}^{m} p_\ell(s) \Big)^{\frac{1}{m}} ds > \frac{1}{\ell}. \tag{2.11}$$

Then equation (1.3) *is oscillatory.*

Corollary 2.2. *Let τ_i be nondecreasing functions,*

$$\liminf_{t\to+\infty} \int_{\tau_j(t)}^{t} p_i(s) \int_{\tau_i(s)}^{\tau_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} d\xi\, ds > 0, \quad i,j = 1,\dots,m, \tag{2.12}$$

and let condition (2.11) *be fulfilled. Then equation* (1.3) *is oscillatory.*

Theorem 2.3. *Let τ_i be nondecreasing functions,*

$$\liminf_{t\to+\infty} \int_{\tau_i(t)}^{t} p_i(s)\, ds > 0, \quad i = 1,\dots,m, \tag{2.13}$$

and let condition (2.11) *be fulfilled. Then the equation* (1.3) *is oscillatory.*

3. Some auxiliary statements

In this section we establish estimates of the quotient

$$\frac{\Big(\prod\limits_{i=1}^{m} x(\tau_i(t)) \Big)^{\frac{1}{m}}}{x(t)},$$

where x is a nonoscillatory solution of equation (1.3).

Lemma 3.1. *Let $t_0 \in R_+$ and $x : [t_0, +\infty) \to (0, +\infty)$ be a solution of equation* (1.3). *Then for any $i \in \{1, 2, \dots\}$*

$$\Big(\prod_{\ell=1}^{m} x(\tau_\ell(t)) \Big)^{\frac{1}{m}} \ge \psi_i(t)\, x(t) \quad \text{for} \quad t \ge \eta_i^{\tau_*}(t_0), \tag{3.1}$$

where the functions $\eta_i^{\tau_*}$ and ψ_i are defined by (2.1) *and* (2.2), *respectively.*

Proof. From (1.3) for $t \ge \eta_1^{\tau_*}(t_0)$ we have

$$\frac{x(\tau_j(t))}{x(t)} \ge \exp\left\{ \int_{\tau_j(t)}^{t} \sum_{i=1}^{m} p_i(s) \frac{x(\tau_i(s))}{x(s)}\, ds \right\} \quad (j = 1,\dots,m).$$

Using the arithmetic mean-geometric mean inequality, from the last inequality we get

$$\frac{x(\tau_j(t))}{x(t)} \geq \exp\left\{m\int_{\tau_j(t)}^{t}\Big(\prod_{\ell=1}^{m}p_\ell(s)\Big)^{\frac{1}{m}}\frac{\Big(\prod\limits_{\ell=1}^{m}x(\tau_\ell(s))\Big)^{\frac{1}{m}}}{x(s)}\,ds\right\} \quad (j=1,\dots,m).$$

Therefore for $t \geq \eta_1^{\tau_*}(t_0)$,

$$\frac{\Big(\prod\limits_{j=1}^{m}x(\tau_j(t))\Big)^{\frac{1}{m}}}{x(t)} \geq \exp\left\{\sum_{j=1}^{m}\int_{\tau_j(t)}^{t}\Big(\prod_{\ell=1}^{m}p_\ell(s)\Big)^{\frac{1}{m}}\frac{\Big(\prod\limits_{\ell=1}^{m}x(\tau_\ell(s))\Big)^{\frac{1}{m}}}{x(s)}\,ds\right\}. \tag{3.2}$$

Inequality (3.1) is obviously fulfilled for $i=1$. Assuming its validity for some $i \in \{1,2,\dots\}$, by (3.2) for $t \geq \eta_{i+1}^{\tau_*}(t_0)$, we obtain

$$\Big(\prod_{\ell=1}^{m}x(\tau_\ell(t))\Big)^{\frac{1}{m}} \geq \exp\left\{\sum_{j=1}^{m}\int_{\tau_j(t)}^{t}\Big(\prod_{\ell=1}^{m}p_\ell(s)\Big)^{\frac{1}{m}}\psi_i(s)\,ds\right\}x(t) = \psi_{i+1}(t)\,x(t),$$

and the result follows. □

Lemma 3.2. *Let $\overline{p}_* \leq \frac{1}{\ell}$, where $\overline{p}_*$ is defined by (2.8). Then*

$$\lim_{i\to+\infty}\big(\liminf_{t\to+\infty}\psi_i(t)\big) \geq \lambda^*(\overline{p}_*), \tag{3.3}$$

where functions ψ_i are given by (2.2) and $\lambda^*(\overline{p}_*)$ is the smallest root of equation (2.7).

Proof. Suppose on the contrary, that (3.3) is not true. Let

$$\lim_{i\to+\infty}\big(\liminf_{t\to+\infty}\psi_i(t)\big) = \gamma < \lambda^*(\overline{p}_*). \tag{3.4}$$

Then there exists $\varepsilon_0 > 0$ such that

$$\frac{e^{\gamma\overline{p}_*}}{\gamma} \geq 1+\varepsilon_0. \tag{3.5}$$

On the other hand, for any $\varepsilon > 0$, by (2.8) and (3.4) there exist $k_\varepsilon \in N$ and $t_\varepsilon \in R_+$ such that

$$\sum_{j=1}^{m}\int_{\tau_j(t)}^{t}\Big(\prod_{\ell=1}^{m}p_\ell(s)\Big)^{\frac{1}{m}}ds \geq \overline{p}_*-\varepsilon, \quad \psi_k(t) \geq \gamma-\varepsilon \quad \text{for} \quad t \geq t_\varepsilon \quad \text{and} \quad k \geq k_\varepsilon. \tag{3.6}$$

According to (3.6) from (2.2) we get

$$\psi_{k+1}(t) \geq e^{(\gamma-\varepsilon)(\overline{p}_*-\varepsilon)} \quad \text{for} \quad t \geq t_\varepsilon \quad \text{and} \quad k \geq k_\varepsilon\,.$$

Therefore

$$\lim_{k\to+\infty}\big(\liminf_{t\to+\infty}\psi_k(t)\big) \geq e^{(\gamma-\varepsilon)(\overline{p}_*-\varepsilon)},$$

which implies $\gamma \geq e^{\gamma\cdot\overline{p}_*}$. In view (3.5), this is a contradiction, and the proof of the lemma 3.2 is complete.

Quite similarly one can prove the next result.

Lemma 3.3. *Let*

$$\bar{p}_* = \liminf_{t\to+\infty} \sum_{j=1}^{m} \int_{\tau_j(t)}^{t} \Big(\prod_{\ell=1}^{m} p_\ell(s)\Big)^{\frac{1}{m}} ds > \frac{1}{e}, \tag{3.7}$$

then

$$\lim_{k\to+\infty} \big(\liminf_{t\to+\infty} \psi_k(t)\big) = +\infty, \tag{3.8}$$

where ψ_k are given by (2.2).

The proof of the next lemma can be found in [9].

Lemma 3.4. *Let there exist nondecreasing functions $\sigma_i \in C(R_+; R)$ such that condition* (2.3) *is fulfilled and equation* (1.3) *has an eventually positive solution $x : [t_0, +\infty) \to (0, +\infty)$. Then*

$$\liminf_{t\to+\infty} \frac{x(t)}{x(\sigma_i(t))} \geq c_i(p_{*i}) \quad (i = 1, \dots, m), \tag{3.9}$$

where p_{*i} and $c_i(p_{*i})$ are defined by (2.5)

4. Proofs of the theorems

Proof of Theorem 2.1. Suppose on the contrary that the equation (1.3) has a nonoscillatory proper solution $x : [t_0, +\infty) \to R$. Since $-x(t)$ is also a solution for (1.3), we confine ourselves only to the case where $x(t)$ is an eventually positive solution of equation (1.3). Then there exists $t_1 > t_0$ such that $x(\tau_i(t)) > 0$ for $t \geq t_1$ $(i = 1, \dots, m)$. As we have seen, while proving Lemma 3.1

$$\Big(\prod_{\ell=1}^{m} x(\tau_\ell(t))\Big)^{\frac{1}{m}} \geq \psi_i(t)\, x(t) \quad \text{for} \quad t \geq \eta_i^{\tau^*}(t_1) \quad (i = 1, 2, \dots), \tag{4.1}$$

where the functions $\eta_i^{\tau^*}$ and ψ_i are defined by (2.1) and (2.2), respectively. From (1.3) we have

$$x(s) = x(t) \exp\left\{\int_s^t \sum_{i=1}^{m} p_i(\xi) \frac{x(\tau_i(\xi))}{x(\xi)}\, d\xi\right\} \quad \text{for} \quad t \geq s \geq t_1. \tag{4.2}$$

Integrating (1.3) from $\sigma_j(t)$ to t, for sufficiently large t, we get

$$x(\sigma_j(t)) = \sum_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s)\, x(\tau_i(s))\, ds + x(t). \tag{4.3}$$

On the other hand, taking into account (2.3), from (4.2) for sufficiently large t, we obtain

$$\frac{x(\tau_j(t))}{x(t)} = \exp\left\{\int_{\tau_j(t)}^{t} \sum_{i=1}^{m} p_i(s) \frac{x(\tau_i(s))}{x(s)}\, ds\right\} \quad (j = 1, \dots, m) \tag{4.4}$$

and

$$x(\tau_j(s)) = x(\sigma_j(t)) \exp\left\{ \int_{\tau_j(s)}^{\sigma_j(t)} \sum_{i=1}^{m} p_i(\xi) \frac{x(\tau_i(\xi))}{x(\xi)}\, d\xi \right\} \tag{4.5}$$

$$\text{for} \quad t \geq s > \eta_1^{\tau_*}(t_1) \quad (j = 1, \dots, m).$$

Combining (4.1), (4.3) – (4.5) and using the arithmetic mean-geometric mean inequality for $j = 1, 2, \dots, m$, we can write

$$x(\sigma_j(t)) \geq \sum_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s)\, x(\sigma_i(t)) \times$$

$$\times \exp\left\{ m \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} \frac{\Big(\prod_{\ell=1}^{m} x(\tau_\ell(\xi)) \Big)^{\frac{1}{m}}}{x(\xi)}\, d\xi \right\} + x(t) \geq$$

$$\geq \sum_{i=1}^{m} x(\sigma_i(t)) \int_{\sigma_j(t)}^{t} p_i(s) \exp\left\{ m \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} \psi_k(\xi)\, d\xi \right\} ds + x(t) \geq$$

$$\geq m \Big[\prod_{i=1}^{m} x(\sigma_i(t)) \Big]^{\frac{1}{m}} \Big[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \cdot$$

$$\cdot \exp\left\{ m \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} \psi_k(\xi)\, d\xi \right\} ds \Big]^{\frac{1}{m}} + x(t).$$

Therefore

$$w(t) \geq m^m w(t) \prod_{j=1}^{m} \Big[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \times$$

$$\times \exp\left\{ m \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} \psi_k(\xi)\, d\xi \right\} ds \Big]^{\frac{1}{m}} + x^m(t),$$

where $w(t) = \prod_{j=1}^{m} x(\sigma_j(t))$. Hence

$$\limsup_{t \to +\infty} \Bigg(\prod_{j=1}^{m} \Big[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left\{ m \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} \psi_k(\xi)\, d\xi \right\} ds \Big]^{\frac{1}{m}} +$$

$$+ \frac{x^m(t)}{\prod_{\ell=1}^{m} x(\sigma_\ell(t))} \Bigg) \leq \frac{1}{m^m}. \tag{4.6}$$

On the other hand, by Lemma 3.4 we have

$$\liminf_{t \to +\infty} \frac{x(t)}{x(\sigma_\ell(t))} \geq c_\ell(p_{*\ell}), \quad \ell = 1, \dots, m, \tag{4.7}$$

where $p_{\ell *}$ and c_ℓ are given by (2.5). Therefore, according to (4.6) and (4.7), we get

$$\limsup_{t\to+\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\cdot\exp\left\{m\int_{\tau_i(s)}^{\sigma_i(t)}\Big(\prod_{\ell=1}^{m}p_\ell(\xi)\Big)^{\frac{1}{m}}\psi_k(\xi)\,d\xi\right\}ds\right]^{\frac{1}{m}}+$$
$$+\prod_{\ell=1}^{m}c_\ell(p_{*\ell})\le$$
$$\le\limsup_{t\to+\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left\{m\int_{\tau_i(s)}^{\sigma_i(t)}\Big(\prod_{\ell=1}^{m}p_\ell(\xi)\Big)^{\frac{1}{m}}\psi_k(\xi)\,d\xi\right\}ds\right]^{\frac{1}{m}}+$$
$$+\liminf_{t\to+\infty}\frac{x^m(t)}{\prod\limits_{\ell=1}^{m}x(\sigma_\ell(t))}\le$$
$$\le\limsup_{t\to+\infty}\left(\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left\{m\int_{\tau_i(s)}^{\sigma_i(t)}\Big(\prod_{\ell=1}^{m}p_\ell(\xi)\Big)^{\frac{1}{m}}\psi_k(\xi)\,d\xi\right\}ds\right]^{\frac{1}{m}}+\right.$$
$$\left.+\frac{x^m(t)}{\prod\limits_{\ell=1}^{m}x(\sigma_\ell(t))}\right)\le\frac{1}{m^m},$$

which contradicts (2.4). □

Proof of Theorem 2.1′. Suppose on the contrary that equation (1.3) has a nonoscillatory solution $x:[t_0,+\infty)\to(0,+\infty)$. Then by Lemma 3.2, the inequality (3.3) is fulfilled. Therefore, for any $\varepsilon>0$, there exist $t_\varepsilon\in R_+$ and $k_0\in N$ such that

$$\psi_{k_0}(t)\ge\lambda^*(\overline{p}_*)-\varepsilon\quad\text{for}\quad t\ge t_\varepsilon, \tag{4.8}$$

where $\overline{p}_*$ is defined by (2.8) and $\lambda^*(\overline{p}_*)$ is the smallest root of equation (2.7). Taking into account that (2.6) and (2.8) imply (2.4), it is easy to see that Theorem 2.1′ is a straightforward consequence of Theorem 2.1. □

Proof of Corollary 2.1 immediately follows from Theorem 2.1′, if we take $\tau_i(t)\equiv\sigma_i(t)$. □

Proof of Theorem 2.2. Suppose on the contrary that equation (1.3) has a nonoscillatory solution $x:[t_0,+\infty)\to(0,+\infty)$. Then by Lemma 3.3, condition (3.8) holds. In view of (2.10), we can choose $M>0$ to satisfy

$$(e\,M)^{\frac{1}{m}}\cdot\limsup_{t\to+\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t}p_i(s)\int_{\tau_i(s)}^{\sigma_i(t)}\Big(\prod_{\ell=1}^{m}p_\ell(\xi)\Big)^{\frac{1}{m}}d\xi\,ds\right]^{\frac{1}{m}}>$$
$$>\frac{1}{m^m}-\prod_{\ell=1}^{m}c_\ell(p_{*\ell}), \tag{4.9}$$

where $p_{*\ell}$ and c_ℓ are given by (2.5).

On the other hand, according to (3.8), there exist $k_0 \in N$ and $t_0 \in R_+$ such that $\psi_{k_0}(t) \geq M$ for $t \geq t_0$. Since $e^x \geq ex$, by (4.9) the condition (2.4) is fulfilled for $k = k_0$, and the result follows. □

Proof of Corollary 2.2 is similar to that of Theorem 2.2, and so is omitted.

Proof of Theorem 2.3. Suppose on the contrary that equation (1.3) has an eventually positive solution $x(t)$. Then by (2.13) and Lemma 3.4 we have

$$\limsup_{t\to+\infty} \frac{x(\tau_i(t))}{x(t)} < +\infty, \quad i = 1, \dots, m. \tag{4.10}$$

On the other hand, by Lemmas 3.3 and 3.4 conditions (3.1) and (3.8) hold. But this contradicts condition (4.10), and the result follows. □

Remark 4.1. Condition (2.11) for any $\varepsilon > 0$, cannot be replaced by the condition

$$\liminf_{t\to+\infty} \sum_{i=1}^{m} \int_{\tau_i(t)}^{t} \Big(\prod_{\ell=1}^{m} p_\ell(s)\Big)^{\frac{1}{m}} ds \geq \frac{1-\varepsilon}{e}. \tag{4.11}$$

Consider the differential equation

$$x'(t) + \sum_{i=1}^{m} c_i x(t - \Delta_i) = 0, \tag{4.12}$$

where $c_i, \Delta_i \in (0, +\infty)$, $i = 1, \dots, m$. Choose $\delta > 0$ such that, if

$$\sum_{j,i=1}^{m} \big(|c_i - c_j| + |\Delta_i - \Delta_j|\big) < \delta,$$

then

$$\sum_{i=1}^{m} c_i e^{\lambda\Delta_i} \leq \Big(1 + \frac{\varepsilon}{2}\Big) m \Big(\prod_{i=1}^{m} c_i\Big)^{\frac{1}{m}} e^{\frac{\lambda}{m}\sum\limits_{i=1}^{m}\Delta_i} \tag{4.13}$$

and

$$\frac{1-\varepsilon}{e} \leq \Big(\prod_{i=1}^{m} c_i\Big)^{\frac{1}{m}} \sum_{i=1}^{m} \Delta_i \leq \frac{1-\frac{\varepsilon}{2}}{e}. \tag{4.14}$$

By (4.13) and (4.14) we have

$$\max\left\{ \frac{\lambda}{\sum\limits_{i=1}^{m} c_i\, e^{\lambda\Delta_i}} : \lambda \in [0, +\infty) \right\} \geq$$

$$\geq \frac{1}{m(1+\frac{\varepsilon}{2})\big(\prod\limits_{i=1}^{m} c_i\big)^{\frac{1}{m}}} \max\left\{ \frac{\lambda}{e^{\frac{\lambda}{m}\sum\limits_{i=1}^{m}\Delta_i}} : \lambda \in [0, +\infty) \right\} =$$

$$= \frac{1}{m(1+\frac{\varepsilon}{2})\left(\prod\limits_{i=1}^{m} c_i\right)^{\frac{1}{m}}} \cdot \frac{m}{e\sum\limits_{i=1}^{m} \Delta_i} \geq \frac{1}{(1+\frac{\varepsilon}{2})(1-\frac{\varepsilon}{2})} = \frac{1}{1-\frac{\varepsilon^2}{4}} > 1.$$

According to the last inequality, it is obvious that $e^{-\lambda_0 \cdot t}$ is a solution of equation (4.12), where λ_0 is the root of equation $\lambda = \sum\limits_{i=1}^{m} c_i e^{\lambda \Delta_i}$. On the other hand, by (4.14), condition (4.11) holds, where $c_i = p_i(t)$ and $t - \Delta_i = \tau_i(t)$.

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ULAM STABILITY OF FRACTIONAL IMPULSIVE DIFFERENTIAL EQUATIONS WITH RIEMANN-LIOUVILLE INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper we study the existence and uniqueness of a solution for fractional impulsive differential equations with Riemann-Liouville integral boundary condition, using a fixed point theorem. Also, we discuss Ulam stability for these equations via some generalized singular Gronwall inequalities.

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Keywords: Ulam type stability; impulsive fractional differential equations; integral boundary conditions; fixed point theorem; generalized singular Gronwall inequality.

1. Introduction

Systems with short-term perturbations are often naturally described by impulsive differential equations (see the monographs by Bainov and Simeonov [2], Benchohra et al. [3] and Lakshmikantham et al. [15], and references therein). Besides, impulsive effects and delay effects appear widely in many real-world models (see [4] and [8]). On the other hand, fractional-order models are found to be more adequate than integer-order models in some real world problems. In fact, fractional differential equations appear naturally in a number of fields such as physics, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, etc. For details and examples we refer to [1], [5], and references therein.

The stability problem of functional equations was originally raised in 1940, by Ulam at Wisconsin University. The problem posed by Ulam was the following: "under what conditions there exist an additive mapping near an approximately additive mapping"? (for more details see [18]). The first answer to Ulam's question was given in 1941 by Hyers in the case of Banach spaces (see [9]). Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [17] obtained a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. As a matter of fact, the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability have been taken up by a number of mathematicians and the study of this area has grown to be one of the central subject in the mathematical analysis. For more details on the recent advances on the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of differential equations we refer to the monographs [10], [12] and the research papers [13], [14] and [19].

Recently, many existence results for nonlinear initial and nonlocal boundary value problems were established by using techniques of nonlinear analysis, such as Banach fixed point theorem and Leray-Schauder nonlinear alternative. In this paper, we study fractional impulsive differential equations with Riemann-Liouville integral boundary condition of the form

$$\begin{cases} {}^{C}D_t^q y(t) = f(t, y(t)), \quad t \in J' = J \setminus \{t_1, ..., t_m\}, \ J = [0, T], \\ \Delta y(t_k) = I_k(y(t_k^-)), \quad k = 1, 2, ..., m, \\ y(0) = \beta I^p y(\eta), \quad 0 < \eta < 1, \end{cases} \tag{1.1}$$

where $0 < q \leq 1$, ${}^{C}D^q$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given jointly continuous function, $I_k : \mathbb{R} \to \mathbb{R}$, $\beta \in \mathbb{R}$ is such that $\beta \neq \Gamma(p+1)/\eta^p$, I^p is the Riemann-Liouville fractional integral of order p, $0 < p < 1$, $0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = T$ and $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$, where $y(t_k^-) = \lim_{\epsilon \to 0^-} y(t_k + \epsilon)$ and $y(t_k^+) = \lim_{\epsilon \to 0^+} y(t_k + \epsilon)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively.

Integral boundary conditions are encountered in population dynamics, blood flow models, chemical engineering, cellular systems, heat transmission, plasma physics, thermoelasticity, etc. They come up when values of a function on the boundary are connected to its values inside the domain, they have physical significations such as total mass, moments, etc. Sometimes it is better to impose integral conditions because they lead to more precise measures than those proposed by a local condition (see [7]).

2. Preliminaries

In this section, we introduce some notation, definitions, and preliminary facts to be used throughout the paper. Let $C(J, \mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with norm $\|u\|_C = \sup_{t \in J} |u(t)|$, $u \in C(J, \mathbb{R})$. By $PC(J, \mathbb{X})$

we denote the set of functions defined by

$$PC(J,\mathbb{X}) = \{x : J \to J \mid x \in C((t_k, t_{k+1}], \mathbb{X}),\ k = 0, 1, ..., m \ \textit{and there exist}$$
$$x(t_k^-) \textit{ and } x(t_k^+), k = 1, 2, ..., m \textit{ with } x(t_k^-) = x(t_k)\},$$

and notice that $PC(J, \mathbb{X})$ becomes a Banach space with respect to the norm

$$\|x\|_{PC} = \sup_{t\in J} |x(t)|.$$

In what follows, $\mathbb{X}$ stands for Banach spaces and $\mathcal{B} \in \mathscr{P}(\mathbb{X})$ denotes the family of all its bounded sets. For description of these notion we refer the reader to Deimling [6].

Definition 2.1 ([11]). *A function $\alpha : \mathcal{B} \to \mathbb{R}^+$ defined by*

$$\alpha(B) = \inf\{d > 0 : B \text{ admits a finite covering by sets of diameter } \leq d\},\ B \in \mathcal{B}$$

is called the Kuratowski measure of noncompactness.

Definition 2.2 ([11]). *Let $\Omega \subset \mathbb{X}$ and let $F : \Omega \to \mathbb{X}$ be a continuous bounded map. We say that F is $\alpha-$Lipschitz if there exists $k \geq 0$ such that*

$$\alpha(F(B)) \leq k\alpha(B), \quad \forall B \subset \Omega.$$

If, in addition, $k < 1$, then we say that F is a strict $\alpha-$contraction.
We say that F is $\alpha-$condensing if

$$\alpha(F(B)) < \alpha(B), \quad \forall B \subset \Omega \textit{ with } \alpha(B) > 0.$$

In other words, if $\alpha(F(B)) \geq k\alpha(B)$ implies that $\alpha(B) = 0$.

The class of all strict $\alpha-$contractions $F : \Omega \to \mathbb{X}$ is denoted by $\mathscr{S}C_\alpha(\Omega)$, and the class of all $\alpha-$condensing maps $F : \Omega \to \mathbb{X}$ is denoted by $C_\alpha(\Omega)$.

We remark that $\mathscr{S}C_\alpha(\Omega) \subset C_\alpha(\Omega)$ and every $F \in C_\alpha(\Omega)$ is $\alpha-$Lipschitz with constant $k = 1$. Recall also that $F : \Omega \to \mathbb{X}$ is Lipschitz if there exists $k > 0$ such that

$$\|Fx - Fy\| \leq k\|x - y\|, \ \forall x, y \in \Omega,$$

and that F is a strict contraction if $k < 1$.

Next, we state some properties of the maps defined above, the proofs of which can be found in [11].

Proposition 2.1. *If $F, G : \Omega \to \mathbb{X}$ are $\alpha-$Lipschitz maps with constants k and k', respectively, then $F + G : \Omega \to \mathbb{X}$ is Lipschitz with constant $k + k'$.*

Proposition 2.2. *If $F : \Omega \to \mathbb{X}$ is compact, then F is $\alpha-$Lipschitz with constant $k = 0$.*

Proposition 2.3. *If $F : \Omega \to \mathbb{X}$ is Lipschitz with constant k, then F is $\alpha-$Lipschitz with the same constant k.*

Now we state a fixed point theorem which will be used in the proof of the main existence results. For details see [11].

Theorem 2.1 (Theorem 2, [11]). *Let $F : \mathbb{X} \to \mathbb{X}$ be $\alpha-$condensing and*

$$\mathscr{S} = \{x \in \mathbb{X} : \ \exists \lambda \in [0,1] \text{ such that } x = \lambda F x\}.$$

If $\mathscr{S}$ is a bounded set in $\mathbb{X}$, and so there exists $r > 0$ such that $\mathscr{S} \subset B_r(0)$, then F has at least one fixed point and the set of the fixed points of F lies in $B_r(0)$.

We also will need some basic definitions and properties of fractional calculus.

Definition 2.3. *If $h \in C([a,b])$ and $q > 0$, then the Riemann-Liouville fractional integral is defined by*

$$I^q_{a^+} h(t) = \frac{1}{\Gamma(q)} \int_a^t \frac{h(s)}{(t-s)^{1-q}} ds.$$

Definition 2.4. *Let $q \geq 0$ and $n = [q] + 1$. If $h \in AC^n([a,b])$, then the Caputo fractional derivative of order q of h defined by*

$${}^C D^q_{a^+} h(t) = \frac{1}{\Gamma(n-q)} \int_a^t \frac{h^{(n)}(s)}{(t-s)^{q-n+1}} ds,$$

exist almost everywhere on $[a,b]$, where $[q]$ is the integer part of q.

Lemma 2.1 ([21]). *Let $q > 0$, then the differential equation*

$${}^C D^q_t h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, $n = [q]+1$.

Lemma 2.2 ([21]). *Let $q > 0$, then*

$$I^q \, {}^C D^q_t h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, $n = [q] + 1$.

Finally, in order to discuss Ulam's type stability for the problem (1.1), we need a new generalized singular type Gronwall inequality with mixed type singular integral operator.

Lemma 2.3 (Theorem 1, [20], p. 1076). *Let $a(t)$ be a nonnegative function locally integrable on $0 \le t < T'$ for some $T' \le +\infty$, and let $g(t)$ be a nonnegative, nondecreasing continuous function defined on $0 \le t < T'$ such that $g(t) \le K$ for some constant K. Further, let $y(t)$ be a nonnegative and locally integrable on $0 \le t < T'$ function satisfying*

$$y(t) \le a(t) + g(t)\int_0^t (t-s)^{\gamma-1} y(s)ds, \quad t \in (0, T')$$

with some $\gamma > 0$. Then

$$y(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\gamma))^n}{\Gamma(n\gamma)}(t-s)^{n\gamma-1}a(s)\right] ds, \quad 0 \le t < T'.$$

Remark 2.1 (Corollary 2, [20], p. 1078). *Under the hypothesis of Lemma 2.3, let $a(t)$ be a nondecreasing function on $[0, T)$. Then*

$$y(t) \le a(t)E_\gamma(g(t)\Gamma(\gamma)t^\gamma),$$

where E_γ is the Mittage-Leffler function defined by (see [16])

$$E_\gamma(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\gamma+1)}, \quad z \in \mathbb{C}, \ Re(\gamma) > 0.$$

Lemma 2.4. *Let $u \in PC(J, \mathbb{R})$ satisfy the following inequality*

$$|u(t)| \le c_1(t) + c_2\int_0^t (t-s)^{\beta_1-1}|u(s)|ds + c_3\int_0^\eta (\eta-s)^{\beta_2-1}|u(s)|ds + \sum_{0<t_k<t} \theta_k |u(t_k^-)|,$$

where $\beta_1, \beta_2 \in (0,1)$, $c_1(t)$ is a nonnegative, continuous and nondecreasing function on J and $c_2, c_3, \theta_k \ge 0$ are constants. Then

(2.1)

$$|u(t)| \le c_1(t)\Big(1+\theta\left[E_{\beta_1}(c_2\Gamma(\beta_1)t^{\beta_1}) + E_{\beta_2}(c_3\Gamma(\beta_2)\eta^{\beta_2})\right]\Big)^k \Big(E_{\beta_1}(c_2\Gamma(\beta_1)t^{\beta_1}) + E_{\beta_2}(c_3\Gamma($$

for $t \in (t_k, t_{k+1}]$, where $\theta = \max\{\theta_k : k = 1, 2, ..., m\}$.

Proof. We apply induction on k. To this end, observe first that in view of Remark 2.1, we have

$$|u(t)| \le c_1(t)\Big(E_{\beta_1}(c_2\Gamma(\beta_1)t^{\beta_1}) + E_{\beta_2}(c_3\Gamma(\beta_2)\eta^{\beta_2})\Big) \text{ for } t \in [0, t_1], \tag{2.2}$$

(2.3)

$$|u(t)| \le \Big(c_1(t) + \sum_{j=1}^{k} \theta_j |u(t_j^-)|\Big)\Big(E_{\beta_1}(c_2\Gamma(\beta_1)t^{\beta_1}) + E_{\beta_2}(c_3\Gamma(\beta_2)\eta^{\beta_2})\Big) \text{ for } t \in (t_k, t_{k+1}].$$

It follows from (2.2) that the inequality (2.1) holds for $k = 0$. Next, assume that (2.1) holds for $k = j < m$. Then taking into account that $c_1(t)$ and E_{β_i}, $i = 1, 2$ are nondecreasing, by (2.3) for $t \in (t_{j+1}, t_{j+2}]$, we can write

$$|u(t)| \leq \left(c_1(t) + \sum_{i=1}^{j+1} \theta_i |u(t_i^-)|\right)\left(E_{\beta_1}(c_2\Gamma(\beta_1)t^{\beta_1}) + E_{\beta_2}(c_3\Gamma(\beta_2)\eta^{\beta_2})\right)$$

$$\leq \Bigg(c_1(t) + \sum_{i=1}^{j+1} \theta_i c_1(t_i^-)\left(1 + \theta\left[E_{\beta_1}(c_2\Gamma(\beta_1)(t_i^-)^{\beta_1}) + E_{\beta_2}(c_3\Gamma(\beta_2)\eta^{\beta_2})\right]\right)^{i-1}$$

$$\times \left[E_{\beta_1}(c_2\Gamma(\beta_1)(t_i^-)^{\beta_1}) + E_{\beta_2}(c_3\Gamma(\beta_2)\eta^{\beta_2}\right]\Bigg)\left(E_{\beta_1}(c_2\Gamma(\beta_1)t^{\beta_1}) + E_{\beta_2}(c_3\Gamma(\beta_2)\eta^{\beta_2})\right)$$

$$\leq \Bigg(c_1(t) + \theta\sum_{i=1}^{j+1} c_1(t)\left(1 + \theta\left[E_{\beta_1}(c_2\Gamma(\beta_1)(t)^{\beta_1}) + E_{\beta_2}(c_3\Gamma(\beta_2)\eta^{\beta_2})\right]\right)^{i-1}$$

$$\times \left[E_{\beta_1}(c_2\Gamma(\beta_1)(t)^{\beta_1}) + E_{\beta_2}(c_3\Gamma(\beta_2)\eta^{\beta_2}\right]\Bigg)\left(E_{\beta_1}(c_2\Gamma(\beta_1)t^{\beta_1}) + E_{\beta_2}(c_3\Gamma(\beta_2)\eta^{\beta_2})\right)$$

$$= c_1(t)\left(1 + \theta\left[E_{\beta_1}(c_2\Gamma(\beta_1)(t)^{\beta_1}) + E_{\beta_2}(c_3\Gamma(\beta_2)\eta^{\beta_2})\right]\right)^{j+1}\left(E_{\beta_1}(c_2\Gamma(\beta_1)t^{\beta_1}) + E_{\beta_2}(c_3\Gamma(\beta_2)\eta^{\beta_2})\right),$$

and the result follows. Lemma 2.4 is proved.

Lemma 2.5. *Let $u \in PC(J, \mathbb{R})$ satisfy the following inequality*

$$|u(t)| \leq a + b\int_0^t (t-s)^{\beta_1 - 1}|u(s)|ds + c\int_0^\eta (\eta - s)^{\beta_2 - 1}|u(s)|ds \quad \text{for } t \in J',$$

where $a, b, c \geq 0$ are constants. Then there exists a constant $M > 0$ such that

$$|u(t)| \leq M.$$

Proof. Let $m = \max_{t\in J}|u(t)|$, then we get

$$\begin{aligned} |u(t)| &\leq a + bm\int_0^t (t-s)^{\beta_1-1}ds + cm\int_0^\eta (\eta - s)^{\beta_2 - 1}ds \\ &\leq a + bm\frac{t^{\beta_1}}{\beta_1} + cm\frac{\eta^{\beta_2}}{\beta_2} \leq a + \left(\frac{bT^{\beta_1}}{\beta_1} + \frac{c}{\beta_2}\right)m = M, \end{aligned}$$

and the result follows. Lemma 2.5 is proved.

3. Existence and uniqueness results

Definition 3.1. *A function $u \in PC(J,\mathbb{R})$ whose $q-$derivative exists on J' is said to be a solution of the problem (1.1) if u satisfies ${}^CD_t^q u(t) = f(t, y(t))$ on J' and satisfies the conditions $\Delta y(t_k) = I_k(y(t_k^-))$, $k = 1,2,...,m$ and $y(0) = \beta I^p y(\eta)$, $0 < \eta < 1$.*

To prove the existence of solutions for problem (1.1), we first prove a number of auxiliary lemmas.

Lemma 3.1. *Let $0 < \eta \leq 1$ and $\beta \neq \frac{\Gamma(p+1)}{\eta^p}$, and let $h : J \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ be a continuous function. A function u is a solution of the fractional integral equation*

(3.1)
$$u(t) = \begin{cases} \varkappa \int_0^\eta (\eta - s)^{p+q-1} h(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds, & t \in (0, t_1] \\ I_1(u(t_1^-)) + \varkappa \int_0^\eta (\eta - s)^{p+q-1} h(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds, & t \in (t_1, t_2] \\ I_1(u(t_1^-)) + I_2(u(t_2^-)) + \varkappa \int_0^\eta (\eta - s)^{p+q-1} h(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds, & t \in (t_2, t \\ \vdots \\ \sum_{k=1}^m I_k(u(t_k^-)) + \varkappa \int_0^\eta (\eta - s)^{p+q-1} h(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds, & t \in (t_m, T], \end{cases}$$

where

$$\varkappa = \frac{\beta \Gamma(p+1)}{\Gamma(p+q)\left(\Gamma(p+1) - \beta \eta^p\right)}$$

if and only if u is a solution of the impulsive problem

$$\begin{cases} {}^CD_t^q u(t) = h(t), \quad t \in J \\ \\ \Delta u(t_k) = I_k(u(t_k^-)), \quad k = 1,2,...,m \\ \\ u(0) = \beta I^p u(\eta), \quad 0 < \eta < 1. \end{cases} \tag{3.2}$$

Proof. Assume that u satisfies (3.2). If $t \in [0, t_1]$, then

$$ {}^CD_t^q u(t) = h(t), \quad t \in (0, t_1] \quad \text{with} \quad u(0) = \beta I^p u(\eta). \tag{3.3}$$

By Lemma 2.2, with some constant $c_0 \in \mathbb{R}$, we get

$$u(t) \quad = \quad I^q h(t) - c_0 = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds - c_0.$$

Then we have

$$\begin{aligned} \beta I^p u(\eta) &= \frac{\beta}{\Gamma(p)} \int_0^\eta (\eta - \tau)^{p-1} u(\tau)d\tau \\ &= \frac{\beta}{\Gamma(p)\Gamma(q)} \int_0^\eta \left(\int_s^\eta (\eta - \tau)^{p-1} (\tau - s)^{q-1} d\tau \right) h(s)ds - \frac{\beta \eta^p}{\Gamma(p+1)} c_0 \\ &= \frac{\beta}{\Gamma(p)\Gamma(q)} \int_0^\eta (\eta - s)^{p+q-1} \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} h(s)ds - \frac{\beta \eta^p}{\Gamma(p+1)} c_0, \end{aligned}$$

where the integral

$$\begin{aligned}\int_s^\eta (\eta-\tau)^{p-1}(\tau-s)^{q-1}d\tau &= (\eta-s)^{p+q-1}\int_0^1 (1-z)^{p-1}z^{q-1}dz\\ &= (\eta-s)^{p+q-1}\mathbf{B}(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}(\eta-s)^{p+q-1}\end{aligned}$$

is evaluated with the help of substitution $\tau = s + z(\eta - s)$ and the definition of beta function $\mathbf{B}(p,q)$.

Using the boundary condition $u(0) = \beta I^p u(\eta)$, we get $u(0) = -c_0 = \beta I^p u(\eta)$, implying that

$$c_0 = \frac{\beta\Gamma(p+1)}{\beta\eta^p - \Gamma(p+1)}\int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}h(s)ds.$$

Hence

$$u(t) = \frac{\beta\Gamma(p+1)}{\Gamma(p+1)-\beta\eta^p}\int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds.$$

If $t \in (t_1, t_2]$, then we have

$$^C D_t^q u(t) = h(t) \text{ for } t \in (t_1,t_2] \text{ with } u(t_1^+) = u(t_1^-) + I_1(u(t_1^-)).$$

By Lemma 2.2, we get

$$\begin{aligned}u(t) &= u(t_1^+) - \frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds\\ &= u(t_1^-) + I_1(u(t_1^-)) - \frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds\\ &= I_1(u(t_1^-)) + \frac{\beta\Gamma(p+1)}{\Gamma(p+1)-\beta\eta^p}\int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds\end{aligned}$$

If $t \in (t_2, t_3]$, then again by Lemma 2.2, we obtain

$$u(t) = u(t_2^+) - \frac{1}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds$$

$$= u(t_2^-) + I_2(u(t_2^-)) - \frac{1}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds$$

$$= I_1(u(t_1^-)) + I_2(u(t_2^-)) + \frac{\beta\Gamma(p+1)}{\Gamma(p+1)-\beta\eta^p}\int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds$$

Finally, if $t \in (t_k, t_{k+1}]$, then by Lemma 2.2 we get

$$u(t) = \sum_{i=1}^k I_i(u(t_i^-)) + \varkappa\cdot\int_0^\eta (\eta-s)^{p+q-1}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds,$$

where $k = 1, 2, ..., m$.

Conversely, assume that u satisfies (3.1). If $t \in (0, t_1]$, then $u(0) = \beta I^p u(\eta)$, $0 < \eta < 1$, and using the fact that $^C D_t^q$ is the left inverse of I^q we get (3.3). If $t \in$

$(t_k, t_{k+1}]$, $k = 1, 2, ..., m$, then using the fact that the Caputo derivative of a constant is equal to zero, we obtain

$${}^C D_t^q u(t) = h(t), \quad t \in (t_k, t_{k+1}] \text{ with } u(t_k^+) = u(t_k^-) + I_k(u(t_1^-)), \quad k = 1, 2, ..., m.$$

This completes the proof of Lemma 3.1.

Now we are in a position to state and prove our existence result for problem (1.1), based on the following hypotheses.

(H1) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is jointly continuous.

(H2) For every $(t, u) \in J \times \mathbb{R}$, there exist $C_f, M_f > 0$ and $q_1 \in [0, 1)$ such that

$$|f(t, u)| \leq C_f |u|^{q_1} + M_f.$$

(H3) There exist constants $L_f > 0$ and $\lambda \in [0, 1)$ such that

$$|f(t, u) - f(t, v)| \leq L_f |u - v|^{\lambda}.$$

(H4) $I_k : \mathbb{R} \to \mathbb{R}$ and there exists a constant $\rho_I^k > 0$ such that

$$|I_k(u) - I_k(v)| \leq \rho_I^k |u - v|, \ \text{for every } u, v \in \mathbb{R}, \text{ and } k = 1, 2, ..., m.$$

Moreover, it holds $\rho_I = \sum_{i=1}^{m} \rho_I^i \in [0, 1)$.

(H5) For every $u \in \mathbb{R}$, there exist $C_I, M_I > 0$ and $q_2 \in [0, 1)$ such that

$$|I_k(u)| \leq C_I |u|^{q_2} + M_I.$$

Now we define the operators F, G and T as follows:

$F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ given by

$$(Fu)(t) = \sum_{i=1}^{k} (F^i u)(t), \quad t \in (t_k, t_{k+1}], \quad k = 1, 2, ..., m,$$

where $(F^i u)(t) = \sum_{i=1}^{k} I_i(u(t_i^-))$, $i = 1, 2, ..., m$.

$G : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ given by

$$(Gu)(t) = \frac{\beta \Gamma(p+1)}{\Gamma(p+q)\,(\Gamma(p+1) - \beta \eta^p)} \int_0^{\eta} (\eta - s)^{p+q-1} f(s, u(s)) ds$$
$$+ \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, u(s)) ds,$$

where $t \in (t_k, t_{k+1}]$, $k = 1, 2, ..., m$.

$\mathbb{T} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ given by $\mathbb{T}u = Fu + Gu$.

Thus, the existence of a solution for the impulsive problem (1.1) is equivalent to the existence of a fixed point for operator $\mathbb{T}$.

Lemma 3.2. *The operator $F : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is Lipschitz with constant ρ_I. Consequently F is $\alpha-$Lipschitz with the same constant ρ_I. Moreover, F satisfies the following growth condition: for every $u \in PC(J,\mathbb{R})$*

$$\|Fu\|_{PC} \leq m(C_I\|u\|_{PC}^{q_2} + M_I). \tag{3.4}$$

Proof. For $t \in (t_1, t_2]$, using (H4), we get

$$\begin{aligned}|(Fu)(t) - (Fv)(t)| &= |(F^1u)(t) - (F^1v)(t)| \\ &\leq \rho_I^1\|u - v\|_{PC},\end{aligned}$$

for every $u, v \in PC(J,\mathbb{R})$. In general, for $t \in (t_k, t_{k+1}]$, $k = 2, 3, ..., m.$, using (H4) step by step, we conclude that F^i is $\alpha-$Lipschitz with constant ρ_I^i. Next, using Proposition 2.2, it is easy to see that F is $\alpha-$Lipschitz with constant $\sum_{i=1}^{m}\rho_I^i$. As for the relation (3.4), it is a simple consequence of (H5). Lemma 3.2 is proved.

Lemma 3.3. *The operator $G : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is continuous and satisfies the following growth condition:*

$$\|Gu\|_{PC} \leq \left(\frac{|\beta|\Gamma(p+1)\eta^{p+q}}{\Gamma(p+q+1)|\Gamma(p+1) - \beta\eta^p|} + \frac{T^q}{\Gamma(q+1)}\right)(C_f\|u\|_{PC}^{q_1} + M_f), \tag{3.5}$$

for every $u \in PC(J,\mathbb{R})$.

Proof. We check the continuity of the operator G on $C((t_k, t_{k+1}], \mathbb{R})$, $k = 2, 3, ..., m$, step by step. To this end, let $\{u_n\}$ be a sequence on a bounded set $\mathfrak{B}_{K_1} := \{\|u\|_{C((0,t_1],\mathbb{R})} \leq K_1 : u \in C((0,t_1],\mathbb{R})\} \subseteq C((0,t_1],\mathbb{R})$ such that $u_n \to u$ in $\mathfrak{B}_{K_1}$.

We have to show that $\|Gu_n - Gu\|_{C((0,t_1],\mathbb{R})} \to 0$. First, it is easy to show that $\Lambda_f = \max_{s\in J}|f(s, u_n(s)) - f(s, u(s))| \to 0$ as $n \to \infty$ due to the continuity of f. Then for all $t \in [0, t_1]$, we have

$$\begin{aligned}&|(Gu_n)(t) - (Gu)(t)| \leq \\ &\leq \frac{|\beta|\Gamma(p+1)}{\Gamma(p+q)|\Gamma(p+1) - \beta\eta^p|}\int_0^{\eta}(\eta - s)^{p+q-1}|f(s, u_n(s)) - f(s, u(s))|ds \\ &\quad + \frac{1}{\Gamma(q)}\int_0^t (t - s)^{q-1}|f(s, u_n(s)) - f(s, u(s))|ds \\ &\leq \left(\frac{|\beta|\Gamma(p+1)\eta^{p+q}}{\Gamma(p+q+1)|\Gamma(p+1) - \beta\eta^p|} + \frac{T^q}{\Gamma(q+1)}\right)\Lambda_f.\end{aligned}$$

Therefore, $\|Gu_n - Gu\|_{C((0,t_1],\mathbb{R})} \to 0$ as $n \to \infty$, implying that G is continuous on $[0, t_1]$. Repeating the above process step by step, one can show that the operator G is continuous on $(t_k, t_{k+1}]$ for $k = 2, 3, ..., m$. Finally, the relation (3.5) is a simple consequence of (H2). Lemma 3.3 is proved.

Lemma 3.4. *The operator $G : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is compact. Consequently, G is $\alpha-$Lipschitz with zero constant.*

Proof. In order to prove the compactness of G, we consider a bounded set $\mathfrak{D} \subset \mathfrak{B}_{K_1} \subseteq C((t_k, t_{k+1}], \mathbb{R})$, $k = 1, 2, ..., m$, and use Arzela-Ascoli theorem to show that $G(\mathfrak{D})$ is relatively compact in $C((t_k, t_{k+1}], \mathbb{R})$ for $k = 2, 3, ..., m$.

For $t \in [0, t_1]$, let $\{u_n\}$ be a sequence on $\mathfrak{D} \subset \mathfrak{B}_{K_1} \subseteq C((0, t_1], \mathbb{R})$. By (3.5), for every $u_n \in \mathfrak{D}$, we have

$$\|Gu\|_{C((0,t_1],\mathbb{R})} \leq \left(\frac{|\beta|\Gamma(p+1)\eta^{p+q}}{\Gamma(p+q+1)|\Gamma(p+1) - \beta\eta^p|} + \frac{T^q}{\Gamma(q+1)}\right)(C_f\|u\|^{q_1}_{C((0,t_1],\mathbb{R})} + M_f),$$

implying that $G(\mathfrak{D})$ is bounded in $C((0, t_1], \mathbb{R})$.

Now we prove that $\{Gu_n\}$ is equicontinuous. For $0 \leq \tau_1 < \tau_2 \leq t_1$ we can write

$$\begin{aligned}
|(Gu_n)(\tau_1) - (Gu_n)(\tau_2)| &\leq \tfrac{1}{\Gamma(q)} \int_0^{\tau_1} \left((\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1}\right) |f(s, u_n(s))| ds \\
&\quad + \tfrac{1}{\Gamma(q)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{q-1} |f(s, u_n(s))| ds \\
&\leq \tfrac{1}{\Gamma(q)} \int_0^{\tau_1} \left((\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1}\right) (C_f|u_n(s)|^{q_1} + M_f) ds \\
&\quad + \tfrac{1}{\Gamma(q)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{q-1} (C_f|u_n(s)|^{q_1} + M_f) ds \\
&\leq \tfrac{(C_f K_1^{q_1} + M_f)}{\Gamma(q+1)} \left(\tau_1^q - \tau_2^q + 2(\tau_2 - \tau_1)^q\right) \leq \tfrac{2(C_f K_1^{q_1} + M_f)(\tau_2 - \tau_1)^q}{\Gamma(q+1)}.
\end{aligned}$$

As $\tau_2 \to \tau_1$, the right-hand side of the above inequality tends to zero, implying that $\{Gu_n\}$ is equicontinuous, and $G(\mathfrak{D}) \subset C((0, t_1], \mathbb{R})$ satisfies the hypothesis of Arzela-Ascoli theorem. Therefore, $G(\mathfrak{D})$ is relatively compact in $C((0, t_1], \mathbb{R})$.

For each $(t_k, t_{k+1}]$, $k = 2, 3, ..., m$, repeating the above process, we can obtain the compactness of the operator G on $C((t_k, t_{k+1}], \mathbb{R})$ for $k = 2, 3, ..., m$. Finally, by Proposition 2.3, G is $\alpha-$Lipschitz with zero constant. Lemma 3.4 is proved.

Now we are in position to prove the main result of this section.

Theorem 3.1. *Assume that hypotheses (H1)-(H5) hold, then the impulsive problem (1.1) has at least one solution $u \in PC(J,\mathbb{R})$ and the set of solutions of that problem is bounded in $PC(J,\mathbb{R})$.*

Proof. Let $F, G, \mathbb{T} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ be the operators defined at the beginning of this section. Recall that they are continuous and bounded. Moreover, F is α−Lipschitz with constant $\rho_I \in [0,1)$ and G is α−Lipschitz with zero constant (see Lemmas 3.2 - 3.4). Also, by Proposition 2.2, the operator $\mathbb{T}$ is a strict α−contraction with constant ρ_I. We set

$$S = \{u \in PC(J,\mathbb{R}) : \exists\ \lambda \in [0,1] \textit{ such that } u = \lambda \mathbb{T} u\},$$

and show that S is bounded in $PC(J,\mathbb{R})$. Consider $u \in S$ and $\lambda \in [0,1]$ such that $u = \lambda \mathbb{T} u$. It follows from (3.4) and (3.5) that

$$\begin{aligned}\|u\|_{PC} &\leq \lambda(\|Fu\|_{PC} + \|Gu\|_{PC}) \leq \lambda(m(C_I\|u\|_{PC}^{q_2} + M_I) + \\ &+ \left(\frac{|\beta|\Gamma(p+1)\eta^{p+q}}{\Gamma(p+q+1)|\Gamma(p+1)-\beta\eta^p|} + \frac{T^q}{\Gamma(q+1)}\right)(C_f\|u\|_{PC}^{q_1} + M_f)).\end{aligned} \tag{3.6}$$

This inequality, together with $q_1 < 1$ and $q_2 < 1$, shows us that S is bounded in $PC(J,\mathbb{R})$. If not, we suppose by contradiction that $\mu := \|u\|_{PC} \to \infty$. Then dividing both sides of (3.6) by μ, and passing to the limit as $\mu \to \infty$, we get

$$1 \leq \lim_{\mu\to\infty} \frac{m(C_I\mu^{q_2} + M_I) + \left(\frac{|\beta|\Gamma(p+1)\eta^{p+q}}{\Gamma(p+q+1)|\Gamma(p+1)-\beta\eta^p|} + \frac{T^q}{\Gamma(q+1)}\right)(C_f\mu^{q_1} + M_f)}{\mu} = 0. \tag{3.7}$$

This is a contradiction. Consequently, by Theorem 2.1 we conclude that $\mathbb{T}$ has at least one fixed point and the set of the fixed points of $\mathbb{T}$ is bounded in $PC(J,\mathbb{R})$. Theorem 3.1 is proved.

Finally, we prove a uniqueness result that follows. In order to obtain the uniqueness of solutions, instead of $(H3)$ we impose the following assumption:

$(H3')$ There exists a constant $L_f > 0$ such that

$$|f(t,u) - f(t,v)| \leq L_f|u - v|, \quad \textit{for each } t \in J \textit{ and } u, v \in \mathbb{R}.$$

Theorem 3.2. *Assume that (H1), (H2), $(H3')$, (H4) and (H5) hold, then the problem (1.1) has a unique solution $u \in PC(J,\mathbb{R})$.*

Proof. By Theorem 3.1 the problem (1.1) has a solution $u(\cdot)$ in $PC(J,\mathbb{R})$. Let $v(\cdot)$ be another solution of problem (1.1). In view of $(H3')$ and $(H4)$, we can apply

Lemma 3.1 to obtain

$$\begin{aligned}|u(t)-v(t)| &\leq \rho_I\|u-v\|_{PC}+\frac{\beta\Gamma(p+1)L_f}{\Gamma(p+q)|\Gamma(p+1)-\beta\eta^p|}\int_0^\eta(\eta-s)^{p+q-1}|u(s)-v(s)|ds\\ &+\frac{L_f}{\Gamma(q)}\int_0^t(t-s)^{q-1}|u(s)-v(s)|ds,\end{aligned}$$

which implies that

$$\begin{aligned}(1-\rho_I)|u(t)-v(t)| &\leq (1-\rho_I)\|u-v\|_{PC}\\ &\leq \frac{\beta\Gamma(p+1)L_f}{\Gamma(p+q)|\Gamma(p+1)-\beta\eta^p|}\int_0^\eta(\eta-s)^{p+q-1}|u(s)-v(s)|ds\\ &+\frac{L_f}{\Gamma(q)}\int_0^t(t-s)^{q-1}|u(s)-v(s)|ds.\end{aligned}$$

Thus

$$\begin{aligned}|u(t)-v(t)| &\leq \frac{\beta\Gamma(p+1)L_f}{(1-\rho_I)\Gamma(p+q)|\Gamma(p+1)-\beta\eta^p|}\int_0^\eta(\eta-s)^{p+q-1}|u(s)-v(s)|ds\\ &+\frac{L_f}{(1-\rho_I)\Gamma(q)}\int_0^t(t-s)^{q-1}|u(s)-v(s)|ds,\end{aligned}$$

which, in view of Lemma 2.5, implies the uniqueness of u. Theorem 3.2 is proved.

4. Ulam stability results

In this section, we consider the Ulam stability of impulsive fractional differential equation (1.1). Let ϵ be a positive real number and let $\varphi : J \to \mathbb{R}^+$ be a continuous function. We consider the following inequalities

$$\text{(4.1)}\qquad \begin{cases}\left|{}^CD_t^q y(t)-f(t,y(t))\right|\leq\epsilon, & t\in J'\\ \left|\Delta y(t_k)-I_k(y(t_k^-))\right|\leq\epsilon, & k=1,2,\dots,m,\end{cases}$$

$$\text{(4.2)}\qquad \begin{cases}\left|{}^CD_t^q y(t)-f(t,y(t))\right|\leq\varphi(t), & t\in J'\\ \left|\Delta y(t_k)-I_k(y(t_k^-))\right|\leq\varphi(t), & k=1,2,\dots,m,\end{cases}$$

$$\text{(4.3)}\qquad \begin{cases}\left|{}^CD_t^q y(t)-f(t,y(t))\right|\leq\epsilon\varphi(t), & t\in J'\\ \left|\Delta y(t_k)-I_k(y(t_k^-))\right|\leq\epsilon\varphi(t), & k=1,2,\dots,m,\end{cases}$$

Definition 4.1. *Equation (1.1) is said to be Ulam-Hyers stable if there exists a real number $C_{f,m}>0$ such that for each $\epsilon>0$ and for each solution $y\in PC(J,\mathbb{R})$ of the inequality (4.1), there exists a solution $x\in PC(J,\mathbb{R})$ of equation (1.1) satisfying*

$$|y(t)-x(t)|\leq C_{f,m}\epsilon,\quad t\in J.$$

Definition 4.2. *Equation (1.1) is said to be generalized Ulam-Hyers stable if there exists $\theta_{f,m} \in PC(J, \mathbb{R}^+)$ with $\theta_{f,m}(0) = 0$ such that for each solution $y \in PC(J, \mathbb{R})$ of the inequality (4.1), there exists a solution $x \in PC(J, \mathbb{R})$ of equation (1.1) satisfying*

$$|y(t) - x(t)| \leq \theta_{f,m}(\epsilon), \quad t \in J.$$

Definition 4.3. *Equation (1.1) is said to be Ulam-Hyers-Rassias stable with respect to φ if there exists $C_{f,m,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in PC(J, \mathbb{R})$ of the inequality (4.3), there exists a solution $x \in PC(J, \mathbb{R})$ of equation (1.1) satisfying*

$$|y(t) - x(t)| \leq C_{f,m,\varphi}\epsilon\varphi(t), \quad t \in J.$$

Definition 4.4. *Equation (1.1) is said to be generalized Ulam-Hyers-Rassias stable with respect to φ if there exists $C_{f,m,\varphi} > 0$ such that for each solution $y \in PC(J, \mathbb{R})$ of the inequality (4.2), there exists a solution $x \in PC(J, \mathbb{R})$ of equation (1.1) satisfying*

$$|y(t) - x(t)| \leq C_{f,m,\varphi}\varphi(t), \quad t \in J.$$

Remark 4.1. *It is clear that: (i) Definition 4.1 $\Longrightarrow$ Definition 4.2; (ii) Definition 4.3 $\Longrightarrow$ Definition 4.4; (iii) Definition 4.3 for $\varphi(t) = 1 \Longrightarrow$ Definition 4.1.*

Remark 4.2. *A function $y \in PC(J, \mathbb{R})$ is a solution of the inequality (4.1) if and only if there exist a function $g \in PC(J, \mathbb{R})$ and a sequence g_k, $k = 1, 2, ..., m$ (which depends on y) such that*

(i) $|g(t)| \leq \epsilon$, $t \in J$ *and* $|g_k| \leq \epsilon$, $k = 1, 2, ..., m$,

(ii) ${}^C D_t^q y(t) = f(t, y(t)) + g(t)$, $t \in J'$,

(iii) $\Delta y(t_k) = I_k(y(t_k^-)) + g_k$, $k = 1, 2, ..., m$.

One can make similar remarks for inequalities (4.2) and (4.3).

So, the Ulam stabilities of the impulsive fractional differential equations are some special types of data dependence of the solutions of impulsive fractional differential equations.

Remark 4.3. *Let $0 < q < 1$. If $y \in PC(J, \mathbb{R})$ is a solution of the inequality (4.1), then y is a solution of the following integral inequality*

$$\left| y(t) - \sum_{i=1}^{k} I_i(y(t_i^-)) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s))\, ds \right.$$
$$\left. - \frac{\beta\Gamma(p+1)}{\Gamma(p+q)(\Gamma(p+1) - \beta\eta^p)} \int_0^\eta (\eta - s)^{p+q-1} f(s, y(s))\, ds \right|$$

$$\leq \left(m + \frac{t^q}{\Gamma(q+1)} + \frac{\beta\Gamma(p+1)\eta^{p+q}}{\Gamma(p+q+1)(\Gamma(p+1)-\beta\eta^p)}\right)\epsilon.$$

Indeed, by Remark 4.2 we have

$$\begin{cases} {}^C D_t^q y(t) = f(t, y(t)) + g(t), & t \in J' \\ \Delta y(t_k) = I_k(y(t_k^-)) + g_k, & k = 1, 2, ..., m \\ y(0) = \beta I^p y(\eta). \end{cases}$$

Then, we have

$$\begin{aligned} y(t) &= \sum_{i=1}^{k} I_i(y(t_i^-)) + \sum_{i=1}^{k} g_i \\ &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s))\, ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s) ds \\ &+ \frac{\beta\Gamma(p+1)}{\Gamma(p+q)(\Gamma(p+1)-\beta\eta^p)} \int_0^\eta (\eta-s)^{p+q-1} f(s, y(s))\, ds \\ &+ \frac{\beta\Gamma(p+1)}{\Gamma(p+q)(\Gamma(p+1)-\beta\eta^p)} \int_0^\eta (\eta-s)^{p+q-1} g(s) ds. \end{aligned}$$

Hence we can write

$$\begin{aligned} &\left| y(t) - \sum_{i=1}^{k} I_i(y(t_i^-)) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s))\, ds \right. \\ &\qquad \left. - \frac{\beta\Gamma(p+1)}{\Gamma(p+q)(\Gamma(p+1)-\beta\eta^p)} \int_0^\eta (\eta-s)^{p+q-1} f(s, y(s))\, ds \right| \\ &\leq \sum_{i=1}^{k} |g_i| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |g(s)| ds + \varkappa \int_0^\eta (\eta-s)^{p+q-1} |g(s)| ds \\ &\leq \left(m + \frac{t^q}{\Gamma(q+1)} + \frac{\beta\Gamma(p+1)\eta^{p+q}}{\Gamma(p+q+1)(\Gamma(p+1)-\beta\eta^p)}\right)\epsilon \end{aligned}$$

Similar remarks can be made for the solutions of inequalities (4.2) and (4.3). Now we are going to prove the main result of this section, stating that equation (1.1) is generalized Ulam-Hyers-Rassias stable.

Theorem 4.1. *Let the assumptions $(H1), (H2), (H3'), (H4)$ and $(H5)$ be fulfilled. Suppose there exist $\lambda_\varphi, \gamma_\varphi > 0$ such that for all $t \in J$*

$$\begin{aligned} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \varphi(s) ds &\leq \lambda_\varphi \varphi(t), \\ \frac{\beta\Gamma(p+1)}{\Gamma(p+q)(\Gamma(p+1)-\beta\eta^p)} \int_0^\eta (\eta-s)^{p+q-1} \varphi(s) ds &\leq \gamma_\varphi \varphi(t), \end{aligned}$$

where $\varphi \in C(J, \mathbb{R}^+)$ is nondecreasing. Then equation (1.1) is generalized Ulam-Hyers-Rassias stable.

Proof. Let $y \in PC(J, \mathbb{R})$ be a solution of the inequality (4.2). By Theorem 3.2 there is a unique solution of the impulsive Cauchy problem

$$\begin{cases} {}^{C}D_t^q x(t) = f(t, x(t)), \quad t \in J' \\ \Delta x(t_k) = I_k(x(t_k^-)) + g_k, \; k = 1, 2, ..., m \\ x(0) = \beta I^p y(\eta). \end{cases} \tag{4.4}$$

Then we have

$$x(t) = \begin{cases} \gamma \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} f(s, x(s))\, ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds, \text{for } t \in (0, t_1] \\ I_1(x(t_1^-)) + \gamma \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} f(s, x(s))\, ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds, \\ \text{for } t \in (t_1, t_2] \\ I_1(x(t_1^-)) + I_2(x(t_2^-)) + \gamma \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} f(s, x(s))\, ds + \\ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds, \text{ for } t \in (t_2, t_3] \\ \vdots \\ \sum_{k=1}^m I_k(x(t_k^-)) + \gamma \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} f(s, x(s))\, ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds, \\ \text{for } t \in (t_m, T] \end{cases}$$

where

$$\gamma = \frac{\beta\Gamma(p+1)}{\Gamma(p+1) - \beta\eta^p}.$$

By differentiating inequality (4.2) for each $t \in (t_k, t_{k+1}]$, we obtain

$$\left| y(t) - \sum_{i=1}^k I_i(y(t_i^-)) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s))\, ds \right.$$
$$\left. + \frac{\beta\Gamma(p+1)\eta^{p+q}}{\Gamma(p+q+1)(\Gamma(p+1) - \beta\eta^p)} \int_0^\eta (\eta-s)^{p+q-1} f(s, y(s))\, ds \right|$$
$$\leq \sum_{k=1}^m |g(t_k^-)| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\varphi(s) ds + \frac{\beta\Gamma(p+1)\eta^{p+q}}{\Gamma(p+q+1)(\Gamma(p+1) - \beta\eta^p)} \times$$
$$\times \int_0^\eta (\eta-s)^{p+q-1}\varphi(s) ds \leq (m + \lambda_\varphi + \gamma_\varphi)\, \varphi(t), \quad t \in J.$$

Hence for each $t \in (t_k, t_{k+1}]$, we can write

$$\begin{aligned}
|x(t)-y(t)| \leq & \Bigg| y(t) - \sum_{i=1}^{k} I_i(x(t_i^-)) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \\
+ & \frac{\beta\Gamma(p+1)}{\Gamma(p+q)\,(\Gamma(p+1)-\beta\eta^p)} \int_0^\eta (\eta-s)^{p+q-1} f(s, x(s))\, ds \Bigg| \\
\leq & \Bigg| y(t) - \sum_{i=1}^{k} I_i(y(t_i^-)) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s))\, ds \\
+ & \frac{\beta\Gamma(p+1)}{\Gamma(p+q)\,(\Gamma(p+1)-\beta\eta^p)} \int_0^\eta (\eta-s)^{p+q-1} f(s, y(s))\, ds \Bigg| \\
+ & \sum_{i=1}^{k} |I_i(y(t_i^-)) - I_i(x(t_i^-))| \\
+ & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,y(s)) - f(s,x(s))|\, ds \\
+ & \frac{|\beta|\Gamma(p+1)}{\Gamma(p+q)\,|\Gamma(p+1)-\beta\eta^p|} \int_0^\eta (\eta-s)^{p+q-1} |f(s,y(s)) - f(s,x(s))|\, ds \\
\leq & (m+\lambda_\varphi+\gamma_\varphi)\,\varphi(t) + \sum_{k=1}^{m} \rho_I^k |(y(t_k^-)) - (x(t_k^-))| \\
+ & \frac{L_f}{\Gamma(q)} \int_0^t (t-s)^{q-1} |y(s)-x(s)|\, ds \\
+ & \frac{|\beta|\Gamma(p+1)L_f}{\Gamma(p+q)\,|\Gamma(p+1)-\beta\eta^p|} \int_0^\eta (\eta-s)^{p+q-1} |y(s)-x(s)|\, ds.
\end{aligned}$$

By Lemma 2.4, there exists a constant $M_{f,m}^* > 0$ independent of $(\lambda_\varphi + \gamma_\varphi)\,\varphi(t)$ such that

$$|y(s) - x(s)| \leq M_{f,m}^*(m+\lambda_\varphi+\gamma_\varphi)\,\varphi(t), \quad t \in J.$$

Thus, equation (1.1) is generalized Ulam-Hyers-Rassias stable. Theorem 4.1 is proved.

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HARDY-TYPE INEQUALITIES FOR FUNCTIONS WHOSE FOURIER TRANSFORMS HAVE GAPS

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Abstract. The original proof of the Littlewood conjecture was a special case of a more general inequality of functions whose Fourier coefficients have gaps. In this article, we prove similar inequalities, but treating the Fourier transform of a function integrable on the real line, rather than on the unit circle.

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1. Introduction

The problem of finding a lower bound of the L^1-norm of exponential sums, known as a Littlewood conjecture, was posed by Littlewood [2], stating that a constant K exists such that for any set $\{n_1 < n_2 < ... < n_N\} \subset \mathbb{Z}$,

$$\left\| \sum_{k=1}^{N} e^{in_k t} \right\|_1 \geq K \log N, \tag{1.1}$$

where the L^1-norm is taken on $\mathbb{T}$, that is, over unit circle. In [5], the Littlewood conjecture was proved as a special case of the following general result.

Theorem 1.1. *(McGehee, Pigno and Smith) There is an absolute constant $c > 0$ such that for any function $f \in L^1(\mathbb{T})$ whose spectrum is contained in the set $\{n_1 < n_2 < ...\} \subset \mathbb{Z}$ we have*

$$\sum_{k=1}^{\infty} \frac{|\hat{f}(n_k)|}{k} \leq c\|f\|_1, \tag{1.2}$$

where $\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-int}dt$ is the $n-th$ Fourier coefficient of f.

In this context, the spectrum of f is the support of $\hat{f}$. Since then, many attempts have been done to generalize the original Hardy's inequality, which is given by (1.2) with $n_k = k$.

In this paper we prove the corresponding inequality of (1.2) for functions $f \in L^1(\mathbb{R})$ and their Fourier transforms $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$. To this end, we will prove first a continuous version of the inequality

$$(1.3) \qquad \sum_{j=1}^{\infty}\left(4^{-j}\sum_{4^{j-1}\leq k<4^j}|\hat{f}(n_k)|^2\right)^{1/2} \leq C\|f\|_1,$$

for all $f \in L^1(\mathbb{T})$ whose spectrum lies in $\{n_1, n_2, \cdots\} \subset \mathbb{Z}$. This inequality was proved in [9] as a gapped version of the following inequality of [3]:

$$(1.4) \quad \sum_{j=1}^{\infty}\left(4^{-j}\sum_{n=4^{j-1}}^{4^j-1}|\hat{f}(n)|^2\right)^{1/2} \leq C\|f\|_1 + C\sum_{j=1}^{\infty}\left(4^{-j}\sum_{n=4^{j-1}}^{4^j-1}|\hat{f}(-n)|^2\right)^{1/2}$$

for all functions $f \in L^1(\mathbb{T})$.

We emphasize here that although inequalities (1.3) and (1.4) look very similar, the authors in [3] and [9] used completely different constructions to prove these inequalities. In [9] the author used the construction applied in [5] to prove the Littlewood conjecture, while [3] used what we call the algebraic construction. In this paper we use the algebraic construction to prove some gapped versions of such inequalities. Hence, the importance of this paper is two folded: the results themselves and the treatment of the algebraic construction with gaps.

We refer the reader to [1] where these two, and two other constructions were reported as alternatives to prove the Littlewood conjecture.

It is worth to note that the recent proofs of inequalities of type (1.3) and (1.4) used a duality idea, where a bounded function with certain decay properties must be constructed, a powerful idea that has been used extensively on the circle.

In our recent works we have focused on how to deal with such inequalities on the real-line, that is, when having a function $f \in L^1(\mathbb{R})$ rather than $L^1(\mathbb{T})$. We refer the reader to [6] – [8] for the study of the real-line versions of Hardy's inequality.

In particular, in [8] we have proved the continuous version of (1.4), stating that a constant $C > 0$ exists such that for all $f \in L^1(\mathbb{R})$,

$$(1.5) \quad \sum_{j=1}^{\infty}\left(4^{-j}\int_{4^{j-1}}^{4^j}|\hat{f}(\xi)|^2d\xi\right)^{1/2} \leq C\|f\|_1 + C\sum_{j=1}^{\infty}\left(4^{-j}\int_{4^{j-1}}^{4^j}|\hat{f}(-\xi)|^2d\xi\right)^{1/2}.$$

Now we propose the following question: What are the real-line (continuous) versions of the inequalities (1.1) – (1.3)?

Observe that an analogue of the Littlewood original conjecture (1.1) would be

$$\left\|\int_E e^{i\xi t} d\xi\right\|_1 \geq K \log(A),$$

where E is a subset of $\mathbb{R}$ with a finite positive Lebesgue measure A, and K is an absolute constant. However, this inequality is trivially true by virtue of the following proposition.

Proposition 1.1. *Let $E \subset \mathbb{R}$ be a set of finite positive Lebesgue measure $m(E)$. Then,*

$$\left\|\int_E e^{i\xi t} d\xi\right\|_1 = \infty.$$

Proof. Let f be the characteristic function of the set E. Then we have $\int_E e^{i\xi t} d\xi = \hat{f}(-t)$. We claim that $\|\hat{f}\|_1 = \infty$. Otherwise we would have $\hat{f} \in L^1(\mathbb{R})$ and f is continuous. But f, being a characteristic function, is continuous if and only if $E = \mathbb{R}$, contradicting the fact that $m(E) < \infty$. □

This concludes the study of the continuous version of (1.1). The purpose of this paper is to present the continuous versions of inequalities (1.2) and (1.3). The corresponding results are stated in Section 3 (see (3.1) and (3.2) below).

To state the aimed results, we first introduce some notation.

For $f \in L^1(\mathbb{R})$, we set $\operatorname{supp}(\hat{f}) = \{\xi \in \mathbb{R} : \hat{f}(\xi) \neq 0\}$ and suppose that $\hat{f}(\xi) = 0$ for all $\xi \leq \xi_0$ with some ξ_0.

We then define a new sequence $\{b_j, j \geq 1\}$ by setting

$$b_j = \inf\left\{b : m\left((b_{j-1}, b) \cap \operatorname{supp}(\hat{f})\right) \geq 3 \times 4^{j-1}\right\}, \quad j \geq 1,$$

where $b_0 = \xi_0$ and $m(\cdot)$ is the Lebesgue measure.

Next, we define a new sequence $\{I_j\}$ of disjoint sets by

$$I_j = (b_{j-1}, b_j) \cap \operatorname{supp}(\hat{f}), \quad j = 1, 2, 3, \cdots.$$

We remark that at each step j of the construction, if

$$m\left((b_{j-1}, b) \cap \operatorname{supp}(\hat{f})\right) < 3 \times 4^{j-1}$$

for all $b > b_{j-1}$, then we set $b_j = \sup \operatorname{supp}(\hat{f})$ and we stop the process to get finitely many I_j's.

Thus, for $f \in L^1(\mathbb{R})$ satisfying $\hat{f}(\xi) = 0$ for all $\xi \leq \xi_0$ with some ξ_0, we have constructed the sets $\{I_j\}$, possibly finitely many, with the following properties:

i) $\operatorname{supp}(\hat{f}) = \bigcup_j I_j$.

ii) If $x \in I_j$ and $y \in I_{j+1}$, then $x < y$.

iii) $m(I_j) = 3 \times 4^{j-1}$, except possibly for the last one I_n, if there are finitely many of them, where we would have $m(I_n) \leq 3 \times 4^{n-1}$.

iv) If there are finitely many of the I_j, say $\{I_1, \cdots, I_n\}$, then by convention let $I_{n+1} = I_{n+2} = \cdots = \phi$.

If $f \in L^1(\mathbb{R})$ satisfies the properties i)-iv), then we refer to it as a gapped function with partition $\{I_j\}$.

Keeping these notation in mind, we prove the continuous version of (1.3). That is, for functions f possessing the above properties, we prove existence of a constant $C > 0$ such that

$$\sum_{j=1}^{\infty}\left(4^{-j}\int_{I_j}|\hat{f}(\xi)|^2 d\xi\right)^{1/2} \leq C\|f\|_1 \tag{1.6}$$

In this context, C is an absolute constant that does not depend on f nor on the partition $\{I_j\}$. It is worth to mention the advantage of (1.6) over (1.5). If, for example, $\hat{f}(\xi) = 0$ for all $\xi < 4^{100}$, then in (1.5) the first nonzero integral will be multiplied by 4^{-99}, while in the new form, this integral will be multiplied by 4^{-1}. Although we can shift the first block to be multiplied by 4^{-1} in (1.5), there is no way then we shift all other blocks. However, inequality (1.6) shifts and merges the support of $\hat{f}$ to behave like a function whose support is continuously extended over the real line. We remark that the ideas of the forthcoming proofs are similar to those of [8].

2. Preliminaries

Let $f \in L^1$ be a gapped function with partition $\{I_j\}$, and assume that $\hat{f}$ is of compact support. For $j \geq 1$ we define the following sequence of functions:

$$f_j(x) = \frac{1}{\sqrt{2\pi}}4^{-j/2}\left(\int_{I_j}|\hat{f}(\xi)|^2 d\xi\right)^{-1/2}\int_{I_j}\hat{f}(\xi)e^{i\xi x}d\xi. \tag{2.1}$$

Since $\mathrm{supp}(\hat{f})$ is assumed to be compact, for large j we have $f_j = 0$ and $I_j = \phi$. The following lemma gives the basic properties of the sequence $\{f_j\}$. The proof of this lemma is similar to the proof given in [8], and so is omitted.

Lemma 2.1. *Let f_j be as above. Then* $\|f_j\|_2 = 4^{-j/2}$, *unless* $f_j = 0$, $\|f_j\|_\infty \leq 1$, *and*

$$\hat{f}_j(\xi) = \sqrt{2\pi}4^{-j/2}\left(\int_{I_j}|\hat{f}(\xi)|^2 d\xi\right)^{-1/2} g_j(\xi), \tag{2.2}$$

where

$$g_j(\xi) = \begin{cases} \hat{f}(\xi), & \xi \in I_j \\ 0, & \textit{otherwise}. \end{cases} \tag{2.3}$$

Now we construct a new sequence of functions $\{F_j,\ j \geq 0\}$ as follows. Put $F_0 = 0$ and for $j \geq 0$ define

$$F_{j+1} = \frac{\epsilon}{2} f_{j+1} + (1 - \epsilon^2 |f_{j+1}|^2) F_j - \frac{\epsilon}{2} \overline{f}_{j+1} F_j^2, \tag{2.4}$$

where $0 < \epsilon < 1$ is a number to be specified later.

Since $\hat{f}$ is of compact support, we have $f_j \equiv 0$ for large j. Therefore, there exists an index k such that $F_k = F_{k+1} = F_{k+2} = \cdots$. Let $F = \frac{2}{\epsilon} F_k$.

Some properties of the sequence $\{F_j,\ j \geq 0\}$ are given in the following sequence of lemmas. Again, the proofs of these results are similar, and some times identical to those in [8]. Hence, we omit the proofs, unless it is necessary.

Lemma 2.2. *For each $j \geq 0$, we have $\|F_j\|_\infty \leq 1$.*

Lemma 2.3. *For each $j \geq 0$, we have* $\mathrm{spec}(F_j) \subset (A_j, b_j)$ *for some $A_j \in \mathbb{R}$. Here* $\mathrm{spec}(F) = \mathrm{supp}(\hat{F})$.

Proof. We proceed by induction on j. The result is true for F_0 because $\mathrm{spec}(F_0) = \phi$. Suppose that $\mathrm{spec}(F_j) \subset (A_j, b_j)$ for some $A_j \in \mathbb{R}$. Observe that f_j is a scalar multiple of the Fourier transform of g_j, where g_j is given in (2.3). Therefore, $\mathrm{spec}(f_j) = I_j \subset (b_{j-1}, b_j)$, and using the fact $b_j < b_{j+1}$, we can write

$$\mathrm{spec}(f_{j+1}) \subset (b_j, b_{j+1}); \qquad \mathrm{spec}(F_j) \subset (A_j, b_j)$$

$$\mathrm{spec}(|f_{j+1}|^2 F_j) \subset \mathrm{spec}(f_{j+1}) + \mathrm{spec}(\overline{f}_{j+1}) + \mathrm{spec}(F_j)$$

$$\subset (b_j, b_{j+1}) + (-b_{j+1}, -b_j) + (A_j, b_j) \subset (b_j - b_{j+1} + A_j, b_{j+1})$$

$$\mathrm{spec}(\overline{f}_{j+1} F_j^2) \subset -\mathrm{spec}(f_{j+1}) + 2\mathrm{spec}(F_j) \subset (-b_{j+1}, -b_j) + (2A_j, 2b_j)$$

$$\subset (-b_{j+1} + 2A_j, b_j) \subset (-b_{j+1} + 2A_j, b_{j+1}).$$

Therefore $\mathrm{spec}(F_{j+1}) \subset (A_{j+1}, b_{j+1})$,
where $A_{j+1} = \min\{b_j, A_j, b_j - b_{j+1} + A_j, -b_{j+1} + 2A_j\}$. This completes the proof. □

Lemma 2.4. *For any $k > j \geq 1$ we have*

$$\left(\int_{\xi \geq b_{j-1}} |\hat{F}_k(\xi)|^2 d\xi \right)^{1/2} \leq 16\epsilon\sqrt{2\pi} 4^{-j/2}. \tag{2.5}$$

Lemma 2.5. *Let $j \geq 1$ and $k > j$. Then*

$$\left(\int_{I_j} |\hat{F}_k(\xi) - \frac{\epsilon}{2}\hat{f}_j(\xi)|^2 d\xi\right)^{1/2} \leq 18\epsilon^2 4^{-j/2}. \tag{2.6}$$

Now, recall that $F = \frac{2}{\epsilon}F_k$ for some $k \geq 1$.

Lemma 2.6. *Let $F = \frac{2}{\epsilon}F_k$, $k \geq 1$, and let $\epsilon := \frac{1}{72\sqrt{2\pi}}$. The following inequalities hold:*

$$\|F\|_\infty \leq c, \text{ where } c = 144\sqrt{2\pi}, \tag{2.7}$$

$$|\hat{F}(\xi) - \hat{f}_j(\xi)| \leq \frac{1}{2}4^{-j} \text{ for } \xi \in I_j. \tag{2.8}$$

The following basic result will be needed in our proofs in Section 3.

Lemma 2.7. *If $f, g \in L^2$, then*

$$\int_{\mathbb{R}} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi.$$

Observe that if $f \in L^1$ is such that $\hat{f}$ is compactly supported, then $\hat{f} \in L^2$. Then Plancherel theorem guarantees that $f \in L^2$.

Now, if F is as above, then $F \in L^2$, and hence by lemma 2.7

$$\int_{\mathbb{R}} f(x)\overline{F(x)}dx = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi. \tag{2.9}$$

3. The main results

Now we are ready to prove our first main result, which is the continuous version of (1.3). Notice that although the proof is identical to that of (1.5), we present it here for completeness.

Theorem 3.1. *There exists an absolute constant $C > 0$, such that for all gapped $f \in L^1(\mathbb{R})$ with partition $\{I_j\}$,*

$$\sum_{j=1}^{\infty}\left(4^{-j}\int_{I_j} |\hat{f}(\xi)|^2 d\xi\right)^{1/2} \leq C\|f\|_1. \tag{3.1}$$

Proof. We first prove the result for $f \in L^1$ whose Fourier transform $\hat{f}$ is of compact support. Let f_j and F be as above. Recall (2.7) and observe that (2.9) holds because

$\hat{f}$ is of compact support. Therefore,

$$\begin{aligned} c\|f\|_1 &\geq \left|\int_{\mathbb{R}} f(x)\overline{F(x)}dx\right| = \frac{1}{2\pi}\left|\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi\right| \\ &= \frac{1}{2\pi}\left|\int_{\cup I_j} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi\right| \geq \frac{1}{2\pi}\Re\left(\int_{\cup I_j} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi\right) = \frac{1}{2\pi}\sum_{j=1}^{\infty}\Re\left(\int_{I_j} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi\right). \end{aligned}$$

But in view of (2.8), for $\xi \in I_j$ we have

$$\left|\overline{\hat{F}(\xi)} - \overline{\hat{f}_j(\xi)}\right| \leq \frac{1}{2}4^{-j},$$

and hence

$$\left|\overline{\hat{F}(\xi)}\hat{f}(\xi) - \overline{\hat{f}_j(\xi)}\hat{f}(\xi)\right| \leq \frac{1}{2}4^{-j}|\hat{f}(\xi)|,$$

implying that

$$\Re\left(\overline{\hat{f}_j(\xi)}\hat{f}(\xi) - \overline{\hat{F}(\xi)}\hat{f}(\xi)\right) \leq \frac{1}{2}4^{-j}|\hat{f}(\xi)|.$$

Consequently, for $\xi \in I_j$, we have

$$\begin{aligned} \Re\left(\overline{\hat{F}(\xi)}\hat{f}(\xi)\right) &\geq \Re\left(\hat{f}(\xi)\overline{\hat{f}_j(\xi)}\right) - \frac{1}{2}4^{-j}|\hat{f}(\xi)| = \\ &= \sqrt{2\pi}4^{-j/2}\left(\int_{I_j} |\hat{f}(\tau)|^2 d\tau\right)^{-1/2} |\hat{f}(\xi)|^2 - \frac{1}{2}4^{-j}|\hat{f}(\xi)|, \end{aligned}$$

where we have used (2.2) to obtain the last line.

Integrate both sides and then use Cauchy-Schwartz inequality to get

$$\begin{aligned} \int_{I_j} \Re\left(\overline{\hat{F}(\xi)}\hat{f}(\xi)d\xi\right) &\geq \sqrt{2\pi}4^{-j/2}\left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi\right)^{1/2} - \frac{4^{-j}}{2}\int_{I_j} |\hat{f}(\xi)|d\xi \\ &\geq \sqrt{2\pi}4^{-j/2}\left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi\right)^{1/2} - \frac{4^{-j}}{2}\left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi\right)^{1/2}\left(\int_{I_j} d\xi\right)^{1/2} \\ &= \left(\sqrt{2\pi} - \frac{\sqrt{3}}{4}\right)\left(4^{-j}\int_{I_j} |\hat{f}(\xi)|^2 d\xi\right)^{1/2} \geq \left(4^{-j}\int_{I_j} |\hat{f}(\xi)|^2 d\xi\right)^{1/2}. \end{aligned}$$

This proves (3.1) with $C = 2\pi c$. This completes the proof of the theorem in the case where $\hat{f}$ is compactly supported.

For the general case, let $f \in L^1$, then apply the above arguments to $f * K_\lambda$, where K_λ is the Fejer kernel of order λ. □

To proceed to the next result, we recall Hardy's inequality and its gapped version (1.2), and (1.5) and its gapped version (3.1).

It is natural to propose a gapped version of the real-line Hardy inequality that states

$$\int_0^\infty \frac{|\hat{f}(\xi)|}{\xi} d\xi \leq c\|f\|_1, \tag{3.2}$$

for $f \in L^1(\mathbb{R})$ satisfying $\hat{f}(\xi) = 0$ for all $\xi < 0$. See [8] for a proof, deduced from (1.5). To state and prove a gapped version of this inequality, we look at the gapped generalization of the discrete inequality in the following way: the inequality (1.2) can be thought of as

$$\sum_{k=1}^{\infty} \frac{|\hat{f}(n_k)|}{g(k)} \leq C\|f\|_1,$$

where g is the mapping $g(n_k) = k$, which maps the supporting integers into the original set of integers.

To realize this idea in the real-line case, we need the following setup.

Observe first that I_j is open, being the intersection of two open sets. Hence, we can write

$$I_j = \bigcup_{k=1}^{n_j} (\alpha_{j,k}, \beta_{j,k}),$$

for some $\alpha_{j,k}, \beta_{j,k} \in \mathbb{R}$ such that $\sum_{k=1}^{n_j} (\beta_{j,k} - \alpha_{i,k}) \leq 3 \times 4^{j-1}$. Consequently, we can find $\{\gamma_{j,k}\}$ and $\{\eta_{j,k}\}$ to satisfy

$$4^{j-1} \leq \gamma_{j,k} < \eta_{j,k} \leq 4^j \text{ and } \eta_{j,k} - \gamma_{j,k} = \beta_{j,k} - \alpha_{j,k}.$$

For each j and $k = 1, \dots, n_j$, let $g_{j,k} : (\alpha_{j,k}, \beta_{j,k}) \longrightarrow (\gamma_{j,k}, \eta_{j,k})$ be defined by

$$g_{j,k}(\xi) = \xi - \alpha_{j,k} + \gamma_{j,k}.$$

Then we set

$$g_j = \sum_{k=1}^{n_j} g_{j,k}\chi_{j,k} \text{ and } g = \sum_j g_j,$$

where $\chi_{j,k}$ is the characteristic function of $(\alpha_{j,k}, \beta_{j,k})$.

Now we are in position to state our result that gives a gapped version of the continuous Hardy inequality (3.2).

Theorem 3.2. *Let f be a gapped function with partition $\{I_j\}$, and let g be as above. Then for some absolute constant C', we have*

$$\int_{\mathbb{R}} \frac{|\hat{f}(\xi)|}{g(\xi)} d\xi \leq C'\|f\|_1 \tag{3.3}$$

Proof. Since $\mathrm{supp}(\hat{f}) \subset \bigcup_j I_j$, we have

$$\int_{\mathbb{R}} \frac{|\hat{f}(\xi)|}{g(\xi)}d\xi = \sum_j \int_{I_j} \frac{|\hat{f}(\xi)|}{g(\xi)}d\xi \le \sum_j \left(\int_{I_j} \frac{1}{g_j^2(\xi)}d\xi\right)^{1/2} \left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi\right)^{1/2}$$

$$= \sum_j \left(\sum_{k=1}^{n_j} \int_{\alpha_{j,k}}^{\beta_{j,k}} \frac{1}{g_{j,k}^2(\xi)}d\xi\right)^{1/2} \left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi\right)^{1/2}.$$

Making the substitution $g_{j,k}(\xi) = \tau$ in the first integral, we can write

$$\int_{\mathbb{R}} \frac{|\hat{f}(\xi)|}{g(\xi)}d\xi \le \sum_j \left(\sum_{k=1}^{n_j} \int_{\gamma_{j,k}}^{\eta_{j,k}} \frac{1}{\tau^2}d\tau\right)^{1/2} \cdot \left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi\right)^{1/2}$$

$$= \sum_j \left(\sum_{k=1}^{n_j} \frac{\eta_{j,k} - \gamma_{j,k}}{\eta_{j,k}\gamma_{j,k}}\right)^{1/2} \left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi\right)^{1/2}$$

$$\le \sum_j \left(\frac{1}{4^{2j-2}} \sum_{k=1}^{n_j} (\eta_{j,k} - \gamma_{j,k})\right)^{1/2} \left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi\right)^{1/2}$$

$$\le \sum_j \left(\frac{3 \times 4^{j-1}}{4^{2j-2}}\right)^{1/2} \left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi\right)^{1/2} \le \sqrt{12}\sum_j \left(4^{-j} \int_{I_j} |\hat{f}(\xi)|^2 d\xi\right)^{1/2} \le C'\|f\|_1,$$

where $C' = \sqrt{12}C$, and (3.3) follows. □

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1

О РАЗРЕШИМОСТИ ОДНОГО КЛАССА НЕЛИНЕЙНЫХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВТОРОГО ПОРЯДКА В ПРОСТРАНСТВЕ $W_1^2(\mathbb{R}^+)$

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Аннотация. В настоящей заметке исследуется начально-краевая задача на положительной полупрямой для нелинейного интегро-дифференциального уравнения второго порядка с некомпактным оператором Гаммерштейна. Указанная задача имеет непосредственное применение в нелокальных задачах физической кинетики. Доказывается существование неотрицательного (тождественно ненулевого) решения этого уравнения в пространстве $W_1^2(\mathbb{R}^+)$. В конце работы приведены соответствующие частные примеры указанных уравнений, имеющих самостоятельный теоретический и прикладной интерес.

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Ключевые слова: нелинейное уравнение; нелинейный оператор; пространство Соболева; сходимость; монотонность.

1. Введение

Нелинейными интегро-дифференциальными уравнениями вида

$$-y'' + \mu y = \int_0^\infty K(x-t)H(t, y(t))dt, \quad x \in \mathbb{R}^+, \tag{1.1}$$

относительно искомой функции $y(x)$, описывается ряд задач современного естествознания. В частности, такие уравнения возникают в задачах теории нелокального взаимодействия, в кинетической теории газов (задача о скин-эффекте), в квантовой механике и т.д. (см. [1-5]).

В частном линейном случае, когда $H(t, u) \equiv u$, $t \in \mathbb{R}^+$ уравнение (1.1) при различных ограничениях на μ и K исследовалось в работах [3-5].

[1]Работа выполнена при финансовой поддержке ГКН МОН РА в рамках научного проекта № SCS 13YR-1A0003.

В случае, когда $H(t,u) \equiv G(u)$, где функция $G(u)$ на некотором отрезке $[0,\eta]$ удовлетворяет следующим условиям:

$$\begin{aligned} &G(0)=0, \quad G(\eta)=\eta, \\ &G(u)\geq u, \quad u\in[0,\eta], \quad G\in C[0,\eta], \quad G \uparrow \quad \text{на} \quad [0,\eta], \end{aligned} \tag{1.2}$$

$\mu-$ числовой параметр, ядро $K-$ непрерывная функция на $\mathbb{R}$, удовлетворяющая условиям:

$$0\leq K\in L_1(\mathbb{R})\cap L_\infty(\mathbb{R}), \qquad \int\limits_{-\infty}^{+\infty} K(\tau)d\tau=\mu>0, \tag{1.3}$$

$$K(-\tau)>K(\tau), \quad \tau\in\mathbb{R}^+, \tag{1.4}$$

$$\int\limits_{-\infty}^{+\infty} |\tau|^j K(\tau)d\tau<+\infty, \quad j=1,2 \tag{1.5}$$

уравнение (1.1) было изучено в недавней работе одного из авторов в пространстве Соболева $W^2_\infty(\mathbb{R}^+)$ (см. [6]). В указанной работе доказано, что уравнение (1.1) с краевыми условиями

$$y(0)=0, \quad y\in W^2_\infty(\mathbb{R}^+)$$

имеет неотрицательное (тождественно ненулевое) монотонно возрастающее решение $y(x)$ с пределом η на бесконечности.

В настоящей работе исследуется следующая краевая задача для уравнения (1.1):

$$\begin{cases} -y''+\mu y=\int\limits_0^\infty K(x-t)H(t,y(t))dt, \quad x\in\mathbb{R}^+, \\ y(0)=0, \quad y\in W^2_1(\mathbb{R}^+), \end{cases} \tag{1.6}$$

и доказывается, что при некоторых ограничениях на $H(t,u)$ уравнение (1.1) с начальным условием $y(0)=0$ имеет неотрицательное (тождественно ненулевое) решение из пространства $W^2_1(\mathbb{R}^+)$.

В конце работы для иллюстрации полученного результата будут приведены примеры функции $H(t,u)$, для которых выполняются условия сформулированной теоремы.

2. Обозначения и вспомогательные факты

2.1. О преобразованиях Лапласа ядерных функций. Пусть ядро K удовлетворяет условиям (1.3)-(1.5). Введем следующую функцию R, определенную на множестве $\mathbb{R}$:

$$R(x)=\int_0^\infty e^{-\lambda z}\int_0^\infty e^{-\lambda\tau}K(x-z+\tau)d\tau dz,\quad x\in\mathbb{R}, \tag{2.1}$$

где $\lambda\equiv\sqrt{\mu}$. Заметим, что

$$R\in L_1(\mathbb{R})\cap L_\infty(\mathbb{R}),\quad R(x)\ge 0,\quad x\in\mathbb{R},\quad \int_{-\infty}^{+\infty}R(\tau)d\tau=\lambda^2=\mu, \tag{2.2}$$

$$\int_{-\infty}^{+\infty}|\tau|^jR(\tau)d\tau<+\infty,\quad j=1,2,\quad \nu(R)\equiv\int_{-\infty}^{+\infty}\tau R(\tau)d\tau=\frac{\nu(K)}{\lambda^2}. \tag{2.3}$$

Действительно, последние факты легко следуют из (2.1),(1.3)-(1.5) с учетом теоремы Фубини (см.[7]).

Обозначим через $\Phi(p)$ преобразование Лапласа функции $R(t)$ на $(-\infty,0)$:

$$\Phi(p)=\int_{-\infty}^0 R(t)e^{pt}dt,\quad p\in\mathbb{R}^+. \tag{2.4}$$

Из (2.4) следует, что

$$\Phi(p)\quad\downarrow\ \text{на}\ [0,+\infty),\quad \Phi\in C[0,+\infty)\quad\text{и}\quad \Phi(+\infty)=0.$$

Ниже докажем, что

$$\Phi(0)\ge\frac{\alpha}{2\lambda^2}>0,\quad\text{где}\quad \alpha=\|K\|_-=\int_{-\infty}^0 K(t)dt. \tag{2.5}$$

Действительно,

$$\int_{-\infty}^0 R(x)dx=\int_{-\infty}^0\int_0^\infty e^{-\lambda z}\int_0^\infty e^{-\lambda\tau}K(x-z+\tau)dzd\tau dx=$$

$$=\int_0^\infty e^{-\lambda z}\int_0^\infty e^{-\lambda\tau}\int_{-\infty}^0 K(x-z+\tau)dxd\tau dz=$$

$$=\int_0^\infty e^{-\lambda z}\int_0^\infty e^{-\lambda\tau}\int_{-\infty}^{\tau-z}K(t)dtd\tau dz\ge\int_0^\infty e^{-\lambda z}\int_z^\infty e^{-\lambda\tau}\int_{-\infty}^{\tau-z}K(t)dtd\tau dz\ge$$

$$\ge\int_{-\infty}^0 K(t)dt\cdot\int_0^\infty e^{-\lambda z}\int_z^\infty e^{-\lambda\tau}d\tau dz=\frac{\alpha}{2\lambda^2}>0.$$

Следовательно, по теореме Коши можем утверждать, что для всякого $\varepsilon \in (0,1)$ существует число p_ε (причем единственное) такое, что

$$\Phi(p_\varepsilon) = \frac{\alpha\varepsilon}{2\lambda^2}. \tag{2.6}$$

Зафиксируем число ε (следовательно, и p_ε) для дальнейшего изложения.

2.2. **О некоторых неоднородных интегральных уравнениях Винера-Хопфа.** Пусть $\beta(t)$– определенная на $\mathbb{R}^+$ функция, удовлетворяющая следующим условиям:

$$\beta \in L_1(\mathbb{R}^+) \cap L_\infty(\mathbb{R}^+), \quad m_1(\beta) \equiv \int_0^\infty x\beta(x)dx < +\infty, \tag{2.7}$$

$$\beta(x) \geq \frac{2\lambda^2}{\alpha\varepsilon}\rho_\varepsilon(x), \quad x \in \mathbb{R}^+, \tag{2.8}$$

где

$$\rho_\varepsilon(x) \equiv \int_0^x e^{-\lambda(x-\tau)}e^{-p_\varepsilon\tau}d\tau \equiv \frac{e^{-p_\varepsilon x} - e^{-\lambda x}}{\lambda - p_\varepsilon}. \tag{2.9}$$

Рассмотрим следующее неоднородное уравнение Винера-Хопфа:

$$f(x) = g(x) + \int_0^\infty R(x-t)f(t)dt, \quad x \in \mathbb{R}^+ \tag{2.10}$$

относительно вещественной измеримой функции $f(x)$ со свободным членом

$$g(x) = \int_0^\infty T(x-t)\beta(t)dt, \quad x \in \mathbb{R}^+, \tag{2.11}$$

и ядром

$$T(\tau) = \int_0^\infty e^{-\lambda z}K(\tau+z)dz = \int_\tau^\infty K(u)e^{-\lambda(u-\tau)}du, \quad \tau \in \mathbb{R}. \tag{2.12}$$

В силу свойств (1.3)-(1.5) ядра K и (2.7)-(2.8) функции $\beta(x)$ можно убедиться, что

$$g \in L_1(\mathbb{R}^+), \quad m_1(g) < +\infty, \quad \nu(T) < 0. \tag{2.13}$$

Так как ядро R уравнения (2.10) обладает свойствами (2.2)-(2.3), то из результатов работы [8] (стр.193, лемма 3.6) следует, что уравнение (2.10) имеет неотрицательное суммируемое и существенно ограниченное решение $f(x)$, причем $\lim\limits_{x\to\infty} f(x) = 0$. Ниже докажем, что это решение снизу оценивается функцией $e^{-p_\varepsilon x}$, $x \in \mathbb{R}^+$.

Действительно, в силу (2.6),(2.8),(2.12) из (2.10) будем иметь:

$$f(x) \geq g(x) = \int_0^\infty T(x-t)\beta(t)dt \geq \frac{2\lambda^2}{\alpha\varepsilon}\int_0^\infty T(x-t)\int_0^t e^{-\lambda(t-\tau)}e^{-p_\varepsilon\tau}d\tau dt =$$

$$= \frac{2\lambda^2}{\alpha\varepsilon}\int_0^\infty e^{-p_\varepsilon\tau}\int_\tau^\infty T(x-t)e^{-\lambda(t-\tau)}dtd\tau = \frac{2\lambda^2}{\alpha\varepsilon}\int_0^\infty e^{-p_\varepsilon\tau}\int_0^\infty T(x-\tau-z)e^{-\lambda z}dzd\tau =$$

$$= \frac{2\lambda^2}{\alpha\varepsilon}\int_0^\infty e^{-p_\varepsilon\tau}\int_0^\infty e^{-\lambda z}\int_0^\infty K(x-\tau-z+v)e^{-\lambda v}dvdzd\tau =$$

$$= \frac{2\lambda^2}{\alpha\varepsilon}\int_0^\infty e^{-p_\varepsilon\tau}R(x-\tau)d\tau = \frac{2\lambda^2}{\alpha\varepsilon}\int_{-\infty}^x R(z)e^{-p_\varepsilon(x-z)}dz \geq$$

$$\geq \frac{2\lambda^2}{\alpha\varepsilon}e^{-p_\varepsilon x}\int_{-\infty}^0 R(z)e^{p_\varepsilon z}dz = \frac{2\lambda^2}{\alpha\varepsilon}e^{-p_\varepsilon x}\Phi(p_\varepsilon) = e^{-p_\varepsilon x}.$$

Итак,

$$f(x) \geq g(x) \geq e^{-p_\varepsilon x}. \tag{2.14}$$

Двусторонняя оценка (2.14) существенным образом будет использована в дальнейших рассуждениях.

2.3. О факторизации некоторых интегральных операторов Винера-Хопфа.

Пусть $E-$ одно из следующих банаховых пространств: $L_p(\mathbb{R}^+)$, $1 \leq p \leq +\infty$, $C_l(\mathbb{R}^+)$, $C_0(\mathbb{R}^+)$. Рассмотрим следующий интегральный оператор Винера-Хопфа:

$$(\mathcal{R}f)(x) = \int_0^\infty R(x-t)f(t)dt, \quad f \in E, \quad x \in \mathbb{R}^+ \tag{2.15}$$

с ядром $R(x)-$ обладающим свойствами (2.2)-(2.3). Тогда, как известно (см. [8]), $\mathcal{R}$ действует в каждом из пространств E, причем оператор $\mathcal{I}-\mathcal{R}$ (где $\mathcal{I}-$ единичный оператор) допускает следующую факторизацию:

$$\mathcal{I}-\mathcal{R} = (\mathcal{I}-\mathcal{V}_-)(\mathcal{I}-\mathcal{V}_+), \tag{2.16}$$

где $\mathcal{V}_\pm-$ соответственно верхние и нижние операторы Вольтерра следующей структуры:

$$(\mathcal{V}_+f)(x) = \int_0^x v_+(x-t)f(t)dt, \quad (\mathcal{V}_-f)(x) = \int_x^\infty v_-(t-x)f(t)dt.$$

Здесь ядра $v_\pm$ – неотрицательные суммируемые существенно ограниченные функции на $\mathbb{R}^+$ и определяются из следующей системы нелинейных уравнений факторизации:

$$v_\pm(x) = R(\pm x) + \int_0^\infty v_\pm(t)v_\mp(x+t)dt, \quad x \in \mathbb{R}^+, \tag{2.17}$$

причем

$$\gamma_- = \int_0^\infty v_-(\tau)d\tau = 1, \quad \gamma_+ = \int_0^\infty v_+(\tau)d\tau \in (0,1).$$

2.4. Об однородных уравнениях Винера-Хопфа. Рассмотрим следующую начальную задачу для однородного уравнения Винера-Хопфа:

$$S(x) = \int_0^\infty R(x-t)S(t)dt, \quad x \in \mathbb{R}^+, \tag{2.18}$$

$$S(0) = 1, \tag{2.19}$$

где R– задается согласно формуле (2.1).

Так как ядерная функция R обладает свойствами (2.2)-(2.3), то согласно теореме Линдли (см. [9]), начальная задача (2.18)-(2.19) имеет положительное монотонно возрастающее и существенно ограниченное на $\mathbb{R}^+$ решение, причем

$$S(x) \;\to\; \frac{1}{1-\gamma_+}, \quad \text{когда } x \to +\infty. \tag{2.20}$$

Из леммы 2 работы [10] следует также, что

$$\frac{1}{1-\gamma_+} - S \in L_1(\mathbb{R}^+). \tag{2.21}$$

3. Основной результат

3.1. Условия на функцию $H(t,u)$. Обозначим через η_0 следующее число:

$$\eta_0 \equiv \frac{\sup\limits_{x\geq 0} f(x)}{\lambda(1-\gamma_+)}. \tag{3.1}$$

Пусть h– определенная на $\mathbb{R}$ вещественная и измеримая функция, удовлетворяющая следующим условиям: существует число $\eta \geq \eta_0$ такое, что

$$\begin{array}{ll} 1) & h \in C[0,\eta], \quad h \uparrow \text{ на } [0,\eta], \\ 2) & h(0)=0, \quad h(\eta)=\eta, \quad h(u) \leq u, \quad u \in [0,\eta]. \end{array} \tag{3.2}$$

Ниже убедимся, что $\eta > \rho_\varepsilon(t)$, $t \in \mathbb{R}^+$, где ρ_ε задается посредством (2.9).

Действительно, с учетом (2.10), (2.14) из (3.1) имеем:

$$\rho_\varepsilon(t) \equiv \int_0^t e^{-\lambda(t-\tau)} e^{-p_\varepsilon \tau} d\tau < \eta\lambda \int_0^t e^{-\lambda(t-\tau)} d\tau = \eta - \eta e^{-\lambda t} \leq \eta,$$

так как

$$\eta\lambda \geq \frac{\sup\limits_{x\geq 0} f(x)}{1-\gamma_+} > f(x) \geq g(x) \geq e^{-p_\varepsilon x}.$$

Относительно функции $H(t,u)$ в дальнейшем будем предполагать выполнение следующих условий:

a) при каждом фиксированном $t \in \mathbb{R}^+$ функция $H(t,u) \uparrow$ по u на $[\rho_\varepsilon(t), \eta]$,

b) функция $H(t,u)$ удовлетворяет условию Каратеодори по аргументу u на множестве $\mathbb{R}^+ \times [0,\eta]$, т.е. при каждом фиксированном $u \in [0,\eta]$ функция $H(t,u)$ измерима по t на $\mathbb{R}^+$ и почти при всех $t \in \mathbb{R}^+$ эта функция непрерывна по u на $[0,\eta]$,

c) существует число $\varepsilon \in (0,1)$ такое, что выполняются следующие неравенства:

$$H(t,\rho_\varepsilon(t)) \geq c_\varepsilon \rho_\varepsilon(t), \qquad H(t,u) \leq h(u) + \beta(t), \qquad t \in \mathbb{R}^+, \ \ u \in [\rho_\varepsilon(t), \eta],$$

где

$$c_\varepsilon \equiv \frac{2\lambda^2}{\alpha\varepsilon}. \tag{3.3}$$

В следующем пункте будет доказано, что если функция $H(t,u)$ удовлетворяет условиям $a)-c)$, то задача (1.6) имеет неотрицательное (нетривиальное) решение.

3.2. Формулировка и доказательство основного результата. Справедлив следующий результат.

Теорема 3.1. *Пусть ядро K удовлетворяет условиям* (1.3)-(1.5), *а функция $H(t,u)-$ условиям $a)-c)$. Тогда задача* (1.6) *имеет неотрицательное ненулевое решение $y(x)$, $x \in \mathbb{R}^+$. Более того, если $x > 0$, то $y(x) > 0$.*

Доказательство разобьем на следующие шаги:

I шаг. Сведение задачи (1.6) к нелинейному интегральному уравнению. Умножим обе части уравнения (1.1) на $e^{-\lambda x}$, $x > 0$ и проинтегрируем по x от τ до $+\infty$. Умножая обе части полученного равенства на $e^{\lambda\tau}$, $\tau > 0$ и имея ввиду, что $y(+\infty) = y'(+\infty) = 0$ (что следует из $y \in W_1^2(\mathbb{R}^+)$) получим:

$$y' + \lambda y = \int_\tau^\infty e^{-\lambda(x-\tau)} \int_0^\infty K(x-t)H(t,y(t))dtdx, \quad x \in \mathbb{R}^+. \tag{3.4}$$

Изменяя в правой части (3.4) порядок интегрирования, в силу (1.6) приходим к следующей задаче:

$$y' + \lambda y = \int_0^\infty T(x-t)H(t, y(t))dt, \quad x \in \mathbb{R}^+, \tag{3.5}$$

$$y(0) = 0, \quad y \in W_1^2(\mathbb{R}^+), \tag{3.6}$$

где $T(x)$ задается согласно (2.12).

Обозначим через

$$\varphi(x) = y'(x) + \lambda y(x). \tag{3.7}$$

Тогда, учитывая (3.6), относительно искомой функции $\varphi \in W_1^1(\mathbb{R}^+)$ приходим к следующему нелинейному интегральному уравнению:

$$\varphi(x) = \int_0^\infty T(x-t)H\left(t, \int_0^t e^{-\lambda(t-\tau)}\varphi(\tau)d\tau\right)dt, \quad x \in \mathbb{R}^+. \tag{3.8}$$

II шаг. Построение положительного и ограниченного решения одного вспомогательного нелинейного интегрального уравнения. Рассмотрим следующее нелинейное вспомогательное уравнение:

$$F(x) = \int_0^\infty T(x-t)\,\bar{h}\left(\int_0^t e^{-\lambda(t-\tau)}(F(\tau) - f(\tau))d\tau\right)dt + Q(x), \quad x \in \mathbb{R}^+, \tag{3.9}$$

где функция $\bar{h}$ порождается с помощью h следующим образом $\bar{h} = \eta - h(\eta - u)$ и обладает свойствами:

$$\begin{aligned} &\bar{h}(0) = 0, \quad \bar{h}(\eta) = \eta, \\ &\bar{h}(u) \geq u, \quad \bar{h} \in C[0, \eta] \quad \bar{h} \uparrow \text{ на } [0, \eta], \end{aligned} \tag{3.10}$$

а

$$Q(x) = \int_0^\infty R(x-t)f(t)dt, \quad x \in \mathbb{R}^+, \tag{3.11}$$

где $f(x)-$ решение неоднородного уравнения (2.10).

Введем следующие последовательные приближения:

(3.12)

$$F_{n+1}(x) = \int_0^\infty T(x-t)\,\bar{h}\left(\int_0^t e^{-\lambda(t-\tau)}(F_n(\tau) - f(\tau))d\tau\right)dt + Q(x), \quad x \in \mathbb{R}^+$$

$$F_0(x) = f(x) + \lambda\eta, \quad n = 0, 1, 2, \ldots.$$

Индукцией по n докажем, что

$$F_n(x) \downarrow \text{ по } n, \; x \in \mathbb{R}^+, \; n = 0, 1, 2, \dots, \tag{3.13}$$

$$F_n(x) \geq \lambda\eta(1 - \gamma_+)S(x) \equiv \widetilde{S}(x), \quad x \in \mathbb{R}^+, \; n = 0, 1, 2, \dots. \tag{3.14}$$

В случае, когда $n = 0$ в силу (3.10),(2.10),(2.11) из (3.12) имеем:

$$F_1(x) = \int_0^\infty T(x-t)\,\overline{h}\left(\lambda\eta\int_0^t e^{-\lambda(t-\tau)}d\tau\right)dt + Q(x) \leq \int_0^\infty T(x-t)\,\overline{h}(\eta)dt + Q(x) \leq$$

$$\leq \lambda\eta + Q(x) \leq \lambda\eta + f(x) = F_0(x).$$

С другой стороны, имеем $F_0(x) = f(x) + \lambda\eta \geq \lambda\eta \geq \widetilde{S}(x)$, $x \in \mathbb{R}^+$ и

$$F_1(x) \geq \int_0^\infty T(x-t)\,\overline{h}\left(\int_0^t e^{-\lambda(t-\tau)}(\widetilde{S}(\tau) - f(\tau))d\tau\right)dt + Q(x) \geq$$

$$\geq \int_0^\infty T(x-t)\int_0^t e^{-\lambda(t-\tau)}(\widetilde{S}(\tau) - f(\tau))d\tau dt + Q(x) =$$

$$= \int_0^\infty R(x-\tau)(\widetilde{S}(\tau) - f(\tau))d\tau + Q(x) = \widetilde{S}(x).$$

Теперь предполагая, что $F_n(x) \leq F_{n-1}(x)$ и $F_n(x) \geq \widetilde{S}(x)$ при некотором $n \in \mathbb{N}$, учитывая монотонность функции $\overline{h}$ на отрезке $[0, \eta]$, из (3.12) получим:

$$F_{n+1}(x) \leq \int_0^\infty T(x-t)\,\overline{h}\left(\int_0^t e^{-\lambda(t-\tau)}(F_{n-1}(\tau) - f(\tau))d\tau\right)dt + Q(x) = F_n(x)$$

и

$$F_{n+1}(x) \geq \int_0^\infty T(x-t)\,\overline{h}\left(\int_0^t e^{-\lambda(t-\tau)}(\widetilde{S}(\tau)) - f(\tau))d\tau\right)dt + Q(x) \geq$$

$$\geq \int_0^\infty R(x-\tau)(\widetilde{S}(\tau) - f(\tau))d\tau + Q(x) = \widetilde{S}(x).$$

Итак, из (3.13), (3.14) следует, что последовательность функций $\{F_n(x)\}_{n=0}^\infty$ имеет поточечный предел, когда $n \to \infty$: $\lim\limits_{n\to\infty} F_n(x) = F(x)$, причем

$$\widetilde{S}(x) \leq F(x) \leq \lambda\eta + f(x), \quad x \in \mathbb{R}^+. \tag{3.15}$$

В силу того, что $\overline{h} \in C[0, \eta]$, из теоремы Б. Леви (см. [7]) следует, что функция $F(x)$ удовлетворяет уравнению (3.9).

Так как

$$\lambda\eta - \widetilde{S}(x) = \lambda\eta(1-\gamma_+)\left(\frac{1}{1-\gamma_+} - S(x)\right) \in L_1^0(\mathbb{R}^+)\cap L_\infty(\mathbb{R}^+)$$

(см. формулу (2.21)) и $f \in L_1^0(\mathbb{R}^+)\cap L_\infty(\mathbb{R}^+)$, то из следующего двустороннего неравенства $0 \le \lambda\eta + f(x) - F(x) \le \lambda\eta + f(x) - \widetilde{S}(x)$, $x \in \mathbb{R}^+$ следует, что

$$\lambda\eta + f - F \in L_1^0(\mathbb{R}^+)\cap L_\infty(\mathbb{R}^+), \tag{3.16}$$

где $L_1^0(\mathbb{R}^+) = \{f \in L_1(\mathbb{R}^+) : \lim\limits_{x\to\infty} f(x) = 0\}$.

III шаг. Итерационный процесс для нелинейного интегрального уравнения (3.8). Введем следующие последовательные приближения для уравнения (3.8):

$$\varphi_{n+1}(x) = \int_0^\infty T(x-t)H\left(t, \int_0^t e^{-\lambda(t-\tau)}\varphi_n(\tau)d\tau\right)dt, \quad x \in \mathbb{R}^+, \tag{3.17}$$
$$\varphi_0(x) = e^{-p_\varepsilon x}, \quad n = 0, 1, 2, \ldots.$$

Индукцией по n докажем:

$$I)\ \varphi_n(x) \quad \uparrow \ \text{по} \quad n, \quad II)\ \varphi_n(x) \le \lambda\eta + f(x) - F(x), \quad x \in \mathbb{R}^+. \tag{3.18}$$

Сначала убедимся в достоверности фактов $I)$ и $II)$ в случае $n = 0$. Из условия c) сформулированной теоремы следует:

$$\varphi_1(x) \ge \frac{2\lambda^2}{\alpha\varepsilon}\int_0^\infty T(x-t)\rho_\varepsilon(t)dt = \frac{2\lambda^2}{\alpha\varepsilon}\int_0^\infty T(x-t)\int_0^t e^{-\lambda(t-\tau)}e^{-p_\varepsilon\tau}d\tau dt =$$

$$= \frac{2\lambda^2}{\alpha\varepsilon}\int_0^\infty e^{-p_\varepsilon\tau}\int_0^\infty T(x-\tau-z)e^{-\lambda z}dzd\tau = \frac{2\lambda^2}{\alpha\varepsilon}\int_0^\infty R(x-\tau)e^{-p_\varepsilon\tau}d\tau \ge e^{-p_\varepsilon x} = \varphi_0(x)$$

(см. доказательство двустороннего неравенства (2.14)). С другой стороны, в силу (2.14) получаем $\varphi_0(x) \le g(x) \le \lambda\eta - F(x) + f(x)$. Действительно, имеем

$$\lambda\eta - F(x) + f(x) = \lambda\eta - \int_0^\infty T(x-t)\,\bar{h}\left(\int_0^t e^{-\lambda(t-\tau)}(F(\tau) - f(\tau))d\tau\right)dt - Q(x) + f(x) =$$

$$= \eta\int_x^\infty T(\tau)d\tau + \int_0^\infty T(x-t)\left(\eta - \bar{h}\left(\int_0^t e^{-\lambda(t-\tau)}(F(\tau) - f(\tau))d\tau\right)\right)dt + g(x) \ge g(x),$$

поскольку

$$\bar{h}\left(\int_0^t e^{-\lambda(t-\tau)}(F(\tau) - f(\tau))d\tau\right) \le \bar{h}(\eta) = \eta.$$

Но из (2.14) следует $g(x) \geq e^{-p_\varepsilon x}$, следовательно $\varphi_0(x) \leq g(x) \leq \lambda\eta - F(x) + f(x)$, $x \in \mathbb{R}^+$. Теперь предполагая, что $\varphi_n(x) \geq \varphi_{n-1}(x)$ и $\varphi_n(x) \leq \lambda\eta - F(x) + f(x)$ при некотором $n \in \mathbb{N}$, в силу монотонности функции $H(t,u)$ с учетом условия c), будем иметь:

$$\varphi_{n+1}(x) \geq \int_0^\infty T(x-t)H\left(t, \int_0^t e^{-\lambda(t-\tau)}\varphi_{n-1}(\tau)d\tau\right)dt = \varphi_n(x),$$

$$\varphi_{n+1}(x) \leq \int_0^\infty T(x-t)H\left(t, \int_0^t e^{-\lambda(t-\tau)}(\lambda\eta - F(\tau) + f(\tau))d\tau\right)dt =$$

$$= \int_0^\infty T(x-t)H\left(t, \eta - \eta e^{-\lambda t} + \int_0^t e^{-\lambda(t-\tau)}(f(\tau) - F(\tau))d\tau\right)dt \leq$$

$$\leq \int_0^\infty T(x-t)\left(h\left(\eta - \eta e^{-\lambda t} + \int_0^t e^{-\lambda(t-\tau)}(f(\tau) - F(\tau))d\tau\right) + \beta(t)\right)dt =$$

$$= g(x) + \int_0^\infty T(x-t)\left(\eta - \bar{h}(\eta e^{-\lambda t} + \int_0^t e^{-\lambda(t-\tau)}(F(\tau) - f(\tau))d\tau)\right)dt \leq$$

$$\leq g(x) + \lambda\eta - \int_0^\infty T(x-t)\,\bar{h}\left(\int_0^t e^{-\lambda(t-\tau)}(F(\tau) - f(\tau))d\tau\right)dt =$$

$$= g(x) + \lambda\eta + Q(x) - F(x) = \lambda\eta - F(x) + f(x).$$

С использованием условия b), индукцией по n из (3.17), легко можно доказать, что каждая из функций $\varphi_n(x)$, $(n = 0, 1, 2, \ldots)$– измерима на $\mathbb{R}^+$. Следовательно, с учетом (3.18) заключаем, что последовательность измеримых функций $\{\varphi_n(x)\}_{n=0}^\infty$ имеет поточечный предел, когда $n \to +\infty$: $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$, причем $\varphi(x)$ также является измеримой функцией и удовлетворяет следующему двойному неравенству:

$$e^{-p_\varepsilon x} \leq \varphi(x) \leq \lambda\eta - F(x) + f(x), \quad x \in \mathbb{R}^+ \tag{3.19}$$

С учетом (3.19) из (3.16) получим $\varphi \in L_1^0(\mathbb{R}^+) \cap \mathrm{L}_\infty(\mathbb{R}^+)$.

Из предельной теоремы Б. Леви следует, что $\varphi(x)$ удовлетворяет уравнению (3.8).

IV шаг. Доказательство включения $\varphi \in W_1^2(\mathbb{R}^+)$. Сначала заметим, что

$$L(x) \equiv \frac{dT}{dx} = -K(x) - \lambda\int_x^\infty K(u)e^{-\lambda(u-x)}du, \quad x \in \mathbb{R}. \tag{3.20}$$

Так как функция K непрерывна на $\mathbb{R}$, то из (3.20) следует, что функция L также непрерывна и суммируема на $\mathbb{R}$.

С другой стороны,в силу (3.19) и свойства c) для функции $H(t,u)$ следующий интеграл

$$\int_0^\infty |L(x-t)|\cdot H\left(t,\int_0^t e^{-\lambda(t-\tau)}\varphi(\tau)d\tau\right)dt$$

равномерно сходится на $\mathbb{R}^+$. Следовательно, по известной теореме о дифференцировании под знаком интеграла можем утверждать

$$\varphi'(x)=\int_0^\infty L(x-t)H\left(t,\int_0^t e^{-\lambda(t-\tau)}\varphi(\tau)d\tau\right)dt,\quad x\in\mathbb{R}^+. \tag{3.21}$$

С учетом (3.20) и условия c), из (3.21), имеем:

$$|\varphi'(x)|\leq\int_0^\infty[K(x-t)+\lambda T(x-t)]h\left(\int_0^t e^{-\lambda(t-\tau)}\varphi(\tau)d\tau\right)dt+\int_0^\infty K(x-t)\beta(t)dt+\lambda g(x)\leq$$

$$\leq\int_0^\infty[K(x-t)+\lambda T(x-t)]\int_0^t e^{-\lambda(t-\tau)}\varphi(\tau)d\tau dt+\int_0^\infty K(x-t)\beta(t)dt+\lambda g(x)=$$

$$=\int_0^\infty \widetilde{T}(x-\tau)\varphi(\tau)d\tau+\lambda\int_0^\infty R(x-\tau)\varphi(\tau)d\tau+\int_0^\infty K(x-t)\beta(t)dt+\lambda g(x)\leq$$

$$\leq c_0(\sqrt{\mu}+1+\mu c_1),$$

где

$$c_0\equiv\sup_{x\in\mathbb{R}^+}\varphi(x),\, c_1\equiv\sup_{x\in\mathbb{R}^+}\beta(x),\quad \text{и}$$

$$\widetilde{T}(x)=\int_0^\infty K(x-z)e^{-\lambda z}dz,\quad x\in\mathbb{R}. \tag{3.22}$$

Отсюда следует, что $\varphi'\in L_\infty(\mathbb{R}^+)$.

Но с другой стороны так как $\widetilde{T}\in L_1(\mathbb{R})$, $R\in L_1(\mathbb{R})$, $\varphi\in L_1(\mathbb{R}^+)$, $\beta\in L_1(\mathbb{R}^+)$, $g\in L_1(\mathbb{R}^+)$, то из неравенства

$$|\varphi'(x)|\leq\int_0^\infty \widetilde{T}(x-\tau)\varphi(\tau)d\tau+\lambda\int_0^\infty R(x-\tau)\varphi(\tau)d\tau+\int_0^\infty K(x-t)\beta(t)dt+\lambda g(x),$$

окончательно получим, что $\varphi'\in L_1(\mathbb{R}^+)$. Таким образом, $\varphi\in W_1^1(\mathbb{R}^+)$. Из (3.7) следует, что $y\in W_1^2(\mathbb{R}^+)$. Теорема 3.1 доказана.

В конце работы приведем несколько примеров $H(t,u)$ для иллюстрации полученного результата.

Пример 3.1.

$$H(t,u) = \frac{u^q}{\eta^{q-1}} + \frac{2c_\varepsilon u \rho_\varepsilon(t)}{u + \rho_\varepsilon(t)}, \quad \text{где} \quad q > 1,$$

а в качестве β можно выбрать следующую функцию: $\beta(t) = 2c_\varepsilon \rho_\varepsilon(t)$.

Пример 3.2.

$$H(t,u) = u - \frac{\eta}{\pi k} sin^2 \frac{u\pi k}{\eta} + 4c_\varepsilon \rho_\varepsilon(t) \frac{u^2}{(u + \rho_\varepsilon(t))^2},$$

где в качестве β можно выбрать $\beta(t) = 4c_\varepsilon \rho_\varepsilon(t)$. Здесь число c_ε задается по формуле (3.3).

Abstract. In this paper we study an initial-boundary value problem for a second order nonlinear integro-differential equation with Hammerstein type noncompact operator on the positive semi-axis. This problem has a direct application in the nonlocal problems of physical kinetics. We prove the existence of a nonnegative (nontrivial) solution of this equation in the space $W_1^2(\mathbb{R}^+)$. Some special examples of these equations are also discussed, which represent independent theoretical and applied interest.

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A BONUS-MALUS SYSTEM WITH AGGREGATE CLAIM AMOUNT COMPONENT

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Abstract. The paper introduces development of a bonus-malus system, where, as opposed on to the current system, the class of a policyholder in a given year is determined not only the basis of the class of the previous year and on the number of claims reported in that year, but also on the aggregate claim severity of a policyholder in that year. The movement between classes is modeled as a discrete time Markov chain, which is dependent on the claims number and aggregate claim severity processes which are also assumed to be Markovian. The transition matrices of the claims number and aggregate claim severity processes are assumed time-dependent and random. In this paper, the recursive filters are proposed for related transition matrices. Also, claims number and claim severity joint probability distribution is provided.

MSC2010 numbers: 60G35, 60J10, 62P05, 65C40, 91B30, 91B70.

Keywords: insurance; policyholder; Markov chain, change of measure; hidden Markov model; Bonus-Malus system.

1. Model description

In this paper an example of construction of a new Bonus-Malus System (BMS) is considered, where the movement between classes depends on the current class of the policyholder, the number of claims and the reported aggregate claim of a policyholder in the previous year.

Consider a set of L policyholders. Each policyholder belongs to one of a finite number C of classes (tariff groups) sorted by order; class 1 being the one with lowest premiums etc. That is, each premium depends on the class to which a policyholder belongs. Each year the class of a policyholder is determined on the basis of the class of the previous year, on the number of claims and on the aggregate claim loss reported during that year. If no claim has been reported, then the policyholder gets a bonus expressed in the lowering to a class with a lower premium or stay at the lowest premium class. Otherwise the policyholder may stay in the same class or gets maluses (penalized) by being shifted to a higher class with possibly higher premium. New policyholders are assigned to a certain class.

Let x_n^i be the number of policyholders in class i at time n, where $i = 1, 2, \dots, C$. Then $X_n = (x_n^1, \dots, x_n^C)$ is the distribution of policyholders among classes at time n. It is obvious that the state space $S_X = \{X_n\}$ of X_n process is finite.

We split the positive half of the real line into a convenient set of disjoint intervals $I_1, I_2, \dots, I_K$ and discuss the aggregate claim of each policyholder on that intervals. We denote by H_n^i the number of policyholders whose aggregate claim in the n-th year falls in interval I_i, $i = 1, 2 \dots, K$. So, row vector $H_n = (H_n^1, \dots, H_n^K)$ shows the distribution of policyholders among the reported aggregate claim intervals.

Consider the number of reported claims. Its state space will be the space of natural numbers $\{0, 1, 2 \dots\}$. Without any distortion we can suppose that it is limited by some number N. We denote by N_n^i the number of policyholders who have reported i ($i = 0, 1, \dots, N$) claims during the n-th year. Then row vector $N_n = (N_n^0, \dots, N_n^N)$ is the distribution of policyholders among the reported claim numbers.

Assumptions underlying the model.

(1) The processes X_n; H_n and N_n are Markov chains which, for technical reasons (that becomes apparent later) and without loss of generality, accordingly live on the standard basis $\{e_1, \dots, e_{|S_X|}\}$, $\{f_1, \dots, f_{|S_H|}\}$ and $\{h_1, \dots, h_{|S_N|}\}$ in $\mathbb{R}^{|S_X|}$, $\mathbb{R}^{|S_H|}$ and respectively $\mathbb{R}^{|S_N|}$, where the i-th component of each vector e_i, f_i and h_i is 1 and others are 0 (see [1], page 5). $|S_X|$, $|S_H|$ and $|S_N|$ are the sizes of S_X, S_H and S_N sets accordingly.

(2) It is assumed that the movement between classes is based on the current class of the policyholder, on the number of claims and on the aggregate claim reported in the year. So the movement of process X_n between its states depends on levels of X_{n-1}, H_{n-1} and N_{n-1} and the transition matrix is not time-dependent.

(3) The next assumption refers to aggregate claim process. It is assumed that the aggregate claims are not independent, so the aggregate claim of a policyholder in any year depends on the aggregate claim and reported claims number of the previous year. We can conclude that the movement between states of process H_n is based on the values of H_{n-1} and N_{n-1} as well. The transition matrix is time-dependent and is not known in advance.

(4) The yearly reported claims numbers for a policyholder are also suggested dependent. It is assumed that reported claims number of a policyholder depends on the claims number reported last year and on the policyholder's

current class of BMS. In other words, the movement from one state for process N_n to another is founded on N_{n-1} and on X_{n-1}. For this process also the transition matrix is assumed a stochastic one.

The time-dependence of transition matrices can be explained by change of policyholders' behavior year by year. They can make conclusions based on their insurance history and be more professional. As an example in motor insurance, the driver can be more careful and make fewer claims if he has many claims in the previous year. On the other hand, he can prefer to cover some small claims himself if he is on the higher bonus class.

Let $\mathfrak{I}_n = \sigma\{X_k, H_{k-1}, N_{k-1}, k \leq n\}$ be the complete filteration generated by processes X, H and N up to the n-th year. The use of Markov property for process X gives: $P(X_n = e_j \mid \mathfrak{I}_{n-1}, H_{n-1}, N_{n-1}) = P(X_n = e_j \mid X_{n-1}, H_{n-1}, N_{n-1})$. Conditional on the event $(X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m)$, the above mentioned expression may be rewritten as:

$$P(X_n = e_j \mid X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m) \triangleq p_{j,ikm}$$

Let $B = (p_{j,ikm})\begin{matrix} |S_X| & |S_H| & |S_N| \\ j,i=1 & k=1 & m=1 \end{matrix}$ be stochastic matrix of tensor mapping $\mathbb{R}^{|S_X||S_H||S_N|}$ into $\mathbb{R}^{|S_X|}$ and has the form

$$B = \begin{pmatrix} p_{1,111} & \cdots & p_{1,|S_X||S_H||S_N|)} \\ \vdots & \ddots & \vdots \\ p_{|S_X|,111} & \cdots & p_{|S_X|,|S_X||S_H||S_N|} \end{pmatrix}$$

where the sum of elements of each column is 1:

$$\sum_{j=1}^{|S_X|} p_{j,ikm} = 1 \tag{1.1}$$

Recall that a process Y_n is an $\mathfrak{I}_n$ martingale increment, if $E(Y_n \mid \mathfrak{I}_{n-1}) = 0$.

Lemma 1.1. *The process X_n has the following semimartingale representation (or Doob decomposition) $X_n = BX_{n-1} \otimes H_{n-1} \otimes N_{n-1} + V_n$, where the $|\mathrm{S_X}| \times 1$ column vector $\mathrm{V_n}$ is an $\mathfrak{I}_n$ martingale increment and $\otimes$ denotes the tensor product.*

The proof can be done similarly as the proof of Lemma 1 in [2].

For the transition matrix of process H_n we introduce

$$q_{s,rl}(n) \triangleq P(H_n = f_s \mid H_{n-1} = f_r,\ N_{n-1} = h_l),$$

where index n indicates the time-dependence of H_n transition matrix.

The same analysis as for Markov chain X_n shows that the Markov chain H_n has representation $H_n = Q_n H_{n-1} \otimes N_{n-1} + W_n$, where transition matrix $Q_n = (q_{s,rl}(n))\begin{smallmatrix}|S_H| & |S_H| & |S_N| \\ s=1 & r=1 & l=1\end{smallmatrix}$ has the form:

$$Q_n = \begin{pmatrix} q_{1,11}(n) & q_{1,12}(n) & \cdots & q_{1,|S_H||S_N|}(n) \\ \vdots & \vdots & \ddots & \vdots \\ q_{|S_H|,11}(n) & q_{|S_H|,12}(n) & \cdots & q_{|S_H|,|S_H||S_N|}(n) \end{pmatrix}.$$

$$\sum_{s=1}^{|S_H|} q_{s,rl}(n) = 1 \tag{1.2}$$

and W_n is a $|S_H| \times 1$ column vector, which is an $\mathfrak{I}_n$ martingale increment.

For the process N_n we denote

$$a_{u,vw}(n) \triangleq P\left(N_n = h_u \mid N_{n-1} = h_v,\ X_{n-1} = e_w\right)$$

Similarly, for the process N_n Doob decomposition has the form

$$N_n = A_n N_{n-1} \otimes X_{n-1} + L_n,$$

where matrix $A_n = (a_{u,vw}(n))\begin{smallmatrix}|S_N| & |S_N| & |S_X| \\ u=1 & v=1 & w=1\end{smallmatrix}$ matrix has the form

$$A_n = \begin{pmatrix} a_{1,11}(n) & a_{1,12}(n) & \cdots & a_{1,|S_N||S_X|}(n) \\ \vdots & \vdots & \ddots & \vdots \\ a_{|S_N|,11}(n) & a_{|S_N|,12}(n) & \cdots & a_{|S_N|,|S_N||S_X|}(n) \end{pmatrix}$$

and

$$\sum_{u=1}^{|S_N|} a_{u,vw}(n) = 1 \tag{1.3}$$

L_n is a $|S_N| \times 1$ column vector and is a martingale increment.

To eliminate the time-dependence of transition matrices Q_n and A_n, we developed matrix-processes, for which transition matrices are not time-dependent.

Consider the simplices U and V with nonnegative column vectors $U = \{(u_1, \dots u_{|S_H|})^T\}$ and respectively $V = \{(v_1, \dots v_{|S_N|})^T\}$, where $u_i \geq 0$, $v_j \geq 0$ and $\sum_{i=1}^{|S_H|} u_i = 1$, $\sum_{j=1}^{|S_N|} v_j = 1$.

By definition, each column of Q_n and A_n is a point in U and respectively in V. We have the following partitions of sets U and V $U = \bigcup_{k=1}^{Q} U_k$, (respectively $V = \bigcup_{k=1}^{A} V_k$) where $U_i \bigcap_{i \neq j} U_j = \emptyset$ ($V_i \bigcap_{i \neq j} V_j = \emptyset$). Let $\Theta = (U_1, U_2, \dots, U_Q)^{|S_H||S_N|}$ and $\Delta = (V_1, V_2, \dots, V_A)^{|S_X||S_N|}$. That is Θ (accordingly Δ) is the Cartesian product of $|S_H|\,|S_N|$ ($|S_X|\,|S_N|$) copies of the ordered set $(U_1, U_2, \dots, U_Q)$ (accordingly $(V_1, V_2, \dots, V_A)$). In other words, it represents the set of $|S_H| \times |S_H||S_N|$ ($|S_N| \times |S_X||S_N|$) sized matrices generated by partition U_1, U_2, ..., $U_Q(V_1, V_2, \dots, V_A)$ of the set $U(V)$.

We define the following Markov chain $\check{Q}_n$ (respectively $\check{A}_n$) on the set Θ (Δ) as follows: if the first column of Q_n (A_n) is a point in U_{i_1} (accordingly in V_{i_1}), the second column in U_{i_2} (accordingly in V_{i_2}), ..., the last column in $U_{i_{|S_H||S_N|}}$ (correspondingly in $V_{i_{|S_X||S_N|}}$), then $\check{Q}_n = (U_{i_1}, \dots, U_{i_{|S_H||S_N|}})$ ($\check{A}_n = (V_{i_1}, \dots, V_{i_{|S_X||S_N|}})$), that is $\check{Q}_n$ ($\check{A}_n$) keeps track only of the location of the columns of Q_n (A_n) in $(U_1, U_2, \dots, U_Q)$ (accordingly in $(V_1, V_2, \dots, V_A)$). So, with the help of $Q_n(A_n)$ the matrix $\check{Q}_n$ ($\check{A}_n$) can be defined uniquely.

We identify the ordered set $\Theta = \{\theta_1, \dots, \theta_{Q|S_H||S_N|}\}$ (correspondingly $\Delta = \{\delta_1, \dots, \delta_{A|S_X||S_N|}\}$) with the standard basis $\{b_1, \dots, b_{Q|S_H||S_N|}\}$ ($\{m_1, \dots, m_{Q|S_H||S_N|}\}$) of $\mathbb{R}^{Q|S_H||S_N|}$ ($\mathbb{R}^{A|S_X||S_N|}$). So, for matrices Q_n and A_n we develop processes $\check{Q}_n$ and accordingly $\check{A}_n$ (for more detail see [2]), for which transition matrices are not time-dependent and have the following form:

$\check{Q}_n = D\check{Q}_{n-1} + R_n$, where $D = (d_{qp})_{q,p=1}^{Q|S_H||S_N|}$ with $d_{qp} \triangleq P\left(\check{Q}_n = b_q \mid \check{Q}_{n-1} = b_p\right)$ and $\{R_n\}$ is a sequence of martingale increments on σ-field $\sigma\left\{\check{Q}_0, \check{Q}_1, \dots, \check{Q}_n\right\}$.

Similarly, $\check{A}_n = K\check{A}_{n-1} + T_n$, where $K = (k_{qp})_{q,p=1}^{A|S_H||S_N|}$ with $k_{qp} \triangleq P\left(\check{A}_n = m_q \mid \check{A}_{n-1} = m_p\right)$ and $\{T_n\}$ is a sequence of martingale increments on σ-field $\sigma\left\{\check{A}_0, \check{A}_1, \dots, \check{A}_n\right\}$.

So, we have the following generalized BMS model:

$$\left\{\begin{array}{l} X_n = BX_{n-1} \otimes H_{n-1} \otimes N_{n-1} + V_n \\ H_n = Q_n H_{n-1} \otimes N_{n-1} + W_n \\ N_n = A_n N_{n-1} \otimes X_{n-1} + L_n \\ \check{Q}_n = D\check{Q}_{n-1} + R_n \\ \check{A}_n = K\check{A}_{n-1} + T_n \end{array}\right. \tag{1.4}$$

It must be noted that the constructed model is a revised and an extended one described in [2]. It has the following peculiarities:

(a) The model, developed in this paper, is "policyholder-oriented", that is the considered group of policyholders is divided into subgroups from 3 different points of view:

(1) partition by levels of BMS;

(2) partition by groups of aggregate claim amounts;

(3) partition by number of reported claims.

(b) In [2], the partition is applied to two different events: first of all the group of policyholders is divided on subgroups by BMS levels and the other partition applied to the set of reported claims, which are sub grouped by claim amount.

(c) In [2] process Z_n is the distribution of claim numbers by claim amount intervals, so $\sum_{j=1}^{K} Z_n^j$ represents the total number of claims and it means that a policyholder, which makes more than one claim during the entire year, can appear in different groups of claim amounts simultaneously. In the model presented in this paper, H_n is the distribution of policyholders among aggregate claim amount groups, so $\sum_{j=1}^{K} H_n^j$ represents the number of policyholders and it means that policyholder's location within the aggregate amount intervals can be identified uniquely.

(d) In comparison with [2], where the transition between BMS levels depends on the reported claims number, in the model, presented in this paper, the above-mentioned transition depends on the aggregate claim amount, reported by policyholder as well.

2. Hidden Markov models. Change of measure

Hidden Markov model is a general tool for representing probability distributions over sequences of observations. The hidden Markov model gets its name from two defining properties. Let us denote the observation at time n by random variable Y_n. First, it assumes that the observation at time n was generated by some process $\check{F}_n$, whose state is hidden from the observer. Second, it assumes that the state of this hidden process satisfies the Markov property. The outputs Y_n have the same property. For more details about hidden Markov models see [1] and [3]. The processes $\check{Q}_n$ and $\check{A}_n$, mentioned above, are hidden Markov models. To estimate their states recursively, we define a measure $\tilde{P}$, on the measurable space $(\Omega, \mathcal{F})$, under which processes X, H and N are sequences of statistically independent and identically distributed random variables. The existence of measure $\tilde{P}$ is provided by Radon-Nicodym theorem. The probability measure P is referred to as the "real world" measure, under which we have the relations (1.4).

Suppose that under the measure $\tilde{P}$ the processes X, H and N are i.i.d sequences with the following distributions:

$$\tilde{P}\left(X_n = e_j \mid X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\right) \triangleq \tilde{p}_{j,ikm}$$

$$\tilde{P}\left(H_n = f_s \mid H_{n-1} = f_r,\ N_{n-1} = h_l\right) \triangleq \tilde{q}_{s,rl}(n)$$

$$\tilde{P}\left(N_n = h_u \mid N_{n-1} = h_v,\ X_{n-1} = e_w\right) \triangleq \tilde{a}_{u,vw}(n),$$

where the corresponding matrices are $\tilde{B} = (\tilde{p}_{j,ikm})\begin{array}{ccc} |S_X| & |S_H| & |S_N| \\ j,i=1 & k=1 & m=1 \end{array}$,

$\tilde{Q}_n = (\tilde{q}_{s,rl}(n))\begin{array}{ccc} |S_H| & |S_H| & |S_N| \\ s=1 & r=1 & l=1 \end{array}$ and $\tilde{A}_n = (\tilde{a}_{u,v,w}(n))\begin{array}{ccc} |S_N| & |S_N| & |S_X| \\ u=1 & v=1 & w=1 \end{array}$.

Under the measure $\tilde{P}$ the dynamics of $\tilde{Q}_n$ and $\tilde{A}_n$ remain unchanged. Define

$$\Lambda_n = \prod_{t=1}^{n} \lambda_t \tag{2.1}$$

where

$$\lambda_t = \prod_{i,j=1}^{|S_X|}\prod_{k=1}^{|S_H|}\prod_{m=1}^{|S_N|}\left(\frac{p_{j,ikm}}{\tilde{p}_{j,ikm}}\right)^{\langle X_t,e_j\rangle\langle X_{t-1},e_i\rangle\langle H_{t-1},f_k\rangle\langle N_{t-1},h_m\rangle} \times$$

$$\times \prod_{r,s=1}^{|S_H|}\prod_{l=1}^{|S_N|}\left(\frac{q_{s,rl}(n)}{\tilde{q}_{s,rl}(n)}\right)^{\langle H_t,f_s\rangle\langle H_{t-1},f_r\rangle\langle N_{t-1},h_l\rangle} \times \tag{2.2}$$

$$\times \prod_{u,v=1}^{|S_N|}\prod_{w=1}^{|S_X|}\left(\frac{a_{v,vw}(n)}{\tilde{a}_{u,vw}(n)}\right)^{\langle N_t,h_u\rangle\langle N_{t-1},h_v\rangle\langle X_{t-1},e_w\rangle}$$

where $\langle \;\;,\;\; \rangle$ denotes the usual scalar product and $\Lambda_0 = 1$.

Let $\mathcal{G}_n$ be the complete filtration generated by $\left\{X_k,\ H_{k-1}, N_{k-1}, \tilde{Q}_k,\ \tilde{A}_k,\ k \le n\right\}$.

Lemma 2.1. *The sequence of random variables* $\{\Lambda_n\}_{n\ge0}$ *is a* $\left\{\mathcal{G}_n, \tilde{P}\right\}$ *martingale with expectation* $\tilde{E}(\Lambda_n) = 1$.

Proof. We have to show that $\tilde{E}(\Lambda_n \mid \mathcal{G}_{n-1}) = \Lambda_{n-1}$. From (2.1) we have $\Lambda_n = \Lambda_{n-1}\lambda_n$. As Λ_{n-1} is $\mathcal{G}_{n-1}$ measurable; it is sufficient to show that $\tilde{E}(\lambda_n \mid \mathcal{G}_{n-1}) = 1$. Using (2.2) and properties of conditional expectation, we get

$$\tilde{E}(\lambda_n \mid \mathcal{G}_{n-1}) = \tilde{E}\left(\tilde{E}(\lambda_n \mid \mathcal{G}_{n-1},\ X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m)\,\Big|\, \mathcal{G}_{n-1}\right) =$$

$$= \tilde{E}\left(\tilde{E}\left(\left(\frac{p_{j,ikm}}{\tilde{p}_{j,ikm}}\right)^{\langle X_n,e_j\rangle}\left(\frac{q_{s,rl}(n)}{\tilde{q}_{s,rl}(n)}\right)^{\langle H_n,f_s\rangle}\left(\frac{a_{u,vw}(n)}{\tilde{a}_{u,vw}(n)}\right)^{\langle N_n,h_u\rangle}\Big|\mathcal{G}_{n-1},\ X_{n-1} = e_i,\right.\right.$$

$$\left.\left. H_{n-1} = f_k, N_{n-1} = h_m\right)\Big|\mathcal{G}_{n-1}\right)$$

Under the measure $\tilde{P}$ all processes are independent, so the expectation of the product can be written as the product of expectations. The calculation is given only for the process X and for the others it is similar.

$$\tilde{E}\left(\tilde{E}\left(\frac{p_{j,ikm}}{\tilde{p}_{j,ikm}}\right)^{\langle X_n,e_j\rangle}\Big|\ \mathcal{G}_{n-1},\ X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\Big|\ \mathcal{G}_{n-1}\right) =$$

$$= \tilde{E}\left(\sum_{j=1}^{|S_X|}\left(\frac{p_{j,ikm}}{\tilde{p}_{j,ikm}}\right)\tilde{P}(X_n = e_j|X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m)\Big|\ \mathcal{G}_{n-1}\right) =$$

$$= \sum_{j=1}^{|S_X|} p_{j,ikm} = 1.$$ □

According to Radon-Nicodym's theorem and Kolmogorov's extension theorem (see [1], pages 26, 317), with the help of measure $\tilde{P}$ we can define the "real world" measure P as follows:

$$\left.\frac{dP}{d\tilde{P}}\right|_{\mathcal{G}_n} \triangleq \Lambda_n \tag{2.3}$$

Theorem 2.1. *Under probability measure P, as defined from $\tilde{P}$ via* (2.3) *the dynamics* (1.4) *hold.*

Proof. The proof is given only for process N. Using Lemma 2.1, the generalized version of Bayes' theorem (see [4], page 132) for the process N_n gives:

$$\begin{aligned} &P\left(N_n = h_u|\mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\right) = \\ &= \frac{\tilde{E}\left(\langle N_n, h_u\rangle \Lambda_n|\mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\right)}{\tilde{E}\left(\Lambda_n|\mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\right)} = \\ &= \tilde{E}\left(\langle N_n, h_u\rangle \lambda_n|\mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\right) \end{aligned} \tag{2.4}$$

Note that under the condition $(\mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m)$ and $\langle N_n, h_u\rangle$, formula (2.2) takes the form:

$$\lambda_n = \left(\frac{p_{j,ikm}}{\tilde{p}_{j,ikm}}\right)^{\langle X_n, e_j\rangle} \left(\frac{q_{s,km}(n)}{\tilde{q}_{s,km}(n)}\right)^{\langle H_n, f_s\rangle} \left(\frac{a_{u,mi}(n)}{\tilde{a}_{u,mi}(n)}\right)$$

so, (2.4) gives:

$$\begin{aligned} &\tilde{E}\left(\langle N_n, h_u\rangle \lambda_n|\mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\right) = \\ &= \sum_{j=1}^{|S_X|}\sum_{s=1}^{|S_H|} \left(\frac{p_{j,ikm}}{\tilde{p}_{j,ikm}}\right)\left(\frac{q_{s,km}(n)}{\tilde{q}_{s,km}(n)}\right)\left(\frac{a_{u,mi}(n)}{\tilde{a}_{u,mi}(n)}\right)\cdot \\ &\cdot \tilde{E}\left(\langle X_n, e_j\rangle \langle H_n, f_s\rangle \langle N_n, h_u\rangle \mid \mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\right) = \\ &= \sum_{j=1}^{|S_X|}\sum_{s=1}^{|S_H|} p_{j,ikm} q_{s,km}(n) a_{u,mi}(n) = a_{u,mi}(n), \end{aligned}$$

where the distributions of processes X_n, H_n, N_n under the measure $\tilde{P}$ and properties (1.1) and (1.2) have been used. □

Theorem 2.2. *Joint distribution of processes H_n and N_n expressed via their marginal distributions is as follows:*

$$\begin{aligned} P\left(H_n = f_s, N_n = h_u|\mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\right)& \\ = q_{s,km}(n) a_{u,mi}(n).& \end{aligned}$$

This means that under the measure P processes H_n and N_n are statistically independent.

Proof. Using again the generalized Bayes' theorem, for the joint distribution, we can write:

$$
\begin{aligned}
&P\left(H_n = f_s, N_n = h_u | \mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\right) = \\
(2.5)\quad &= \frac{\tilde{E}\left(\langle H_n, f_s\rangle \langle N_n, h_u\rangle \Lambda_n | \mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\right)}{\tilde{E}\left(\Lambda_n | \mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\right)} = \\
&= \tilde{E}\left(\langle H_n, f_s\rangle \langle N_n, h_u\rangle \lambda_n | \mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\right)
\end{aligned}
$$

In this case for (2.2) we have: $\lambda_n = \left(\frac{p_{j,ikm}}{\bar{p}_{j,ikm}}\right)^{\langle X_n, e_j\rangle} \left(\frac{q_{s,km}(n)}{\bar{q}_{s,km}(n)}\right) \left(\frac{a_{u,mi}(n)}{\bar{a}_{u,mi}(n)}\right)$, so (2.5) takes the form:

$\tilde{E}\left(\langle H_n, f_s\rangle \langle N_n, h_u\rangle \lambda_n | \mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\right) =$
$= \sum_{j=1}^{|S_X|} \left(\frac{p_{j,ikm}}{\bar{p}_{j,ikm}}\right) \left(\frac{q_{s,km}(n)}{\bar{q}_{s,km}(n)}\right) \left(\frac{a_{u,mi}(n)}{\bar{a}_{u,mi}(n)}\right) \cdot$
$\cdot \tilde{E}\left(\langle X_n, e_j\rangle \langle H_n, f_s\rangle \langle N_n, h_u\rangle \mid \mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m\right) =$
$= \sum_{j=1}^{|S_X|} p_{j,ikm} q_{s,km}(n) a_{u,mi}(n) = q_{s,km}(n) a_{u,mi}(n).$ □

Define the measure valued process $g_n(\omega, \varpi)$, which is unnormalized conditional probability under measure $\tilde{P}$

$$
(2.6)\qquad g_n(\omega, \varpi) = \tilde{E}\left(\Lambda_n \langle \check{Q}_n, b_\omega\rangle \langle \breve{A}_n, m_\varpi\rangle | \mathfrak{I}_n\right)
$$

Using the generalized Bayes' theorem, the normalized conditional probability takes the form:

$$
\begin{aligned}
P\left(\check{Q}_n = b_\omega, \breve{A}_n = m_\varpi \mid \mathfrak{I}_n\right) &= \frac{\tilde{E}\left(\Lambda_n \langle \check{Q}_n, b_\omega\rangle \langle \breve{A}_n, m_\varpi\rangle | \mathfrak{I}_n\right)}{\tilde{E}\left(\Lambda_n | \mathfrak{I}_n\right)} = \\
&= \frac{g_n(\omega, \varpi)}{\sum_{v=1}^{Q|S_H||S_N|} \sum_{\varpi=1}^{A|S_X||S_N|} g_n(v, \varpi)}
\end{aligned}
$$

Theorem 2.3. *The unnormalized probability $g_n(\omega, \varpi)$ satisfies the recursion:*

$$
\begin{aligned}
g_n(\omega, \varpi) = &\left\langle \frac{B}{\bar{B}} X_{n-1} \otimes H_{n-1} \otimes N_{n-1}, X_n \right\rangle \times \sum_{s=1}^{|S_H|} \langle \check{Q}_\omega H_{n-1} \otimes N_{n-1}, f_s\rangle \times \\
&\times \sum_{u=1}^{|S_N|} \langle \breve{A}_\varpi N_{n-1} \otimes X_{n-1}, h_u\rangle \times \sum_{v=1}^{Q|S_H||S_N|} \sum_{\vartheta=1}^{A|S_X||S_N|} d_{\omega v} k_{\varpi\vartheta} g_{n-1}(v, \vartheta),
\end{aligned}
$$

where $g_0(\omega, \varpi)$ is the initial joint probability of $\check{Q}_n$ and $\breve{A}_n$.

Proof. In view of (2.1) and (2.2) and using the distributions of H_n and N_n under measure $\tilde{P}$ for (2.6) we can write:

$$
g_n(\omega, \varpi) = \tilde{E}\left(\Lambda_n \langle \check{Q}_n, b_\omega\rangle \langle \breve{A}_n, m_\varpi\rangle | \mathfrak{I}_n\right) = \sum_{s=1}^{|S_H|} \sum_{u=1}^{|S_N|} \Big(\Lambda_{n-1} \langle \check{Q}_n, b_\omega\rangle \langle \breve{A}_n, m_\varpi\rangle \times
$$

$$\times \prod_{i,j=1}^{|S_X|}\prod_{k=1}^{|S_H|}\prod_{m=1}^{|S_N|}\left(\frac{p_{j,ikm}}{\bar{p}_{j,ikm}}\right)^{\langle X_n,e_j\rangle\langle X_{n-1},e_i\rangle\langle H_{n-1},f_k\rangle\langle N_{n-1},h_m\rangle}\times$$

$$\times \prod_{r,s=1}^{|S_H|}\prod_{l=1}^{|S_N|}\left(\frac{q_{s,rl}(\omega)}{\bar{q}_{s,rl}(\omega)}\right)^{\langle H_{n-1},f_r\rangle\langle N_{n-1},h_l\rangle}\times \tag{2.7}$$

$$\times \prod_{u,v=1}^{|S_N|}\prod_{w=1}^{|S_X|}\left(\frac{a_{u,vw}(\varpi)}{\bar{a}_{u,vw}(\varpi)}\right)^{\langle N_{n-1},h_v\rangle\langle X_{n-1},e_w\rangle}\times \bar{P}\left(H_n=f_s,N_n=h_u|\mathfrak{I}_n\right)\Big)$$

Now recall the dynamics (1.4) for the processes $\check{Q}_n$ and $\check{A}_n$, that is $\langle \check{Q}_n, b_\omega\rangle = d_{\omega v}\langle \check{Q}_{n-1}, b_v\rangle$, and $\langle \check{A}_n, m_\varpi\rangle = k_{\varpi\vartheta}\langle \check{A}_{n-1}, m_\vartheta\rangle$, for (2.7) we get

$$g_n(\omega,\varpi)=\sum_{s=1}^{|S_H|}\sum_{u=1}^{|S_N|}\Big(\prod_{i,j=1}^{|S_X|}\prod_{k=1}^{|S_H|}\prod_{m=1}^{|S_N|}\left(\frac{p_{j,ikm}}{\bar{p}_{j,ikm}}\right)^{\langle X_n,e_j\rangle\langle X_{n-1},e_i\rangle\langle H_{n-1},f_k\rangle\langle N_{n-1},h_m\rangle}\times$$

$$\times\prod_{r,s=1}^{|S_H|}\prod_{l=1}^{|S_N|}q_{s,rl}(\omega)^{\langle H_{n-1},f_r\rangle\langle N_{n-1},h_l\rangle}\times\prod_{u,v=1}^{|S_N|}\prod_{w=1}^{|S_X|}a_{u,vw}(\varpi)^{\langle N_{n-1},h_v\rangle\langle X_{n-1},e_w\rangle}\times$$

$$\times\bar{E}\left(\Lambda_{n-1}d_{\omega v}\langle \check{Q}_{n-1},b_v\rangle k_{\varpi\vartheta}\langle \check{A}_{n-1},m_\vartheta\rangle|\mathfrak{I}_{n-1}\right)=$$

$$=\prod_{i,j=1}^{|S_X|}\prod_{k=1}^{|S_H|}\prod_{m=1}^{|S_N|}\left(\frac{p_{j,ikm}}{\bar{p}_{j,ikm}}\right)^{\langle X_n,e_j\rangle\langle X_{n-1},e_i\rangle\langle H_{n-1},f_k\rangle\langle N_{n-1},h_m\rangle}\times$$

$$\times\sum_{s=1}^{|S_H|}\prod_{r,s=1}^{|S_H|}\prod_{l=1}^{|S_N|}q_{s,rl}(\omega)^{\langle H_{n-1},f_r\rangle\langle N_{n-1},h_l\rangle}\sum_{u=1}^{|S_N|}\left(\prod_{u,v=1}^{|S_N|}\prod_{w=1}^{|S_X|}a_{u,vw}(\varpi)^{\langle N_{n-1},h_v\rangle\langle X_{n-1},e_w\rangle}\right)\times$$

$$\times\sum_{v=1}^{Q|S_H||S_N|}\sum_{\vartheta=1}^{A|S_X||S_N|}d_{\omega v}k_{\varpi\vartheta}\bar{E}\left(\Lambda_{n-1}\langle \check{Q}_{n-1},b_v\rangle\langle \check{A}_{n-1},m_\vartheta\rangle|\mathfrak{I}_{n-1}\right)$$

It must be noted that the last equation holds due to distributions of $\check{Q}_n$ and $\check{A}_n$ under the measure $\bar{P}$. In view of (2.6) for $n-1$, we get the proof of the theorem. □

3. Predictions for Claims number and aggregate claim amounts

It is important for insurance companies to predict the possible number and amount of future claims m years into the future.

As a result of generalized Bayes' theorem, the joint probability distribution of the claims number and aggregate claim amount reported by a policyholder has the form:

$$P\left(H_n=f_s,N_n=h_m\mid\mathfrak{I}_n\right)=\frac{\bar{E}\left(\langle H_n,f_s\rangle\langle N_n,h_m\rangle\Lambda_n|\mathfrak{I}_n\right)}{\bar{E}\left(\Lambda_n|\mathfrak{I}_n\right)}$$

where the denominator is constant.

Having the information up to time n, the above-mentioned distribution's behavior at the end of current year is presented by the following Lemma

Lemma 3.1. *The unnormalized joint conditional probability distribution of reported claims number and aggregate claim amount has the form:*

$$\tilde{E}\left(H_n N_n \Lambda_n \mid \mathfrak{I}_n\right) = \left\langle \frac{B}{\bar{B}} X_{n-1} \otimes H_{n-1} \otimes N_{n-1}, X_n \right\rangle \times$$

$$\times \sum_{s=1}^{|S_H|} \sum_{u=1}^{|S_N|} \sum_{\omega=1}^{Q|S_H||S_N|} \sum_{\varpi=1}^{A|S_X||S_N|} f_s \left\langle \check{Q}_\omega H_{n-1} \otimes N_{n-1}, f_s \right\rangle h_u \left\langle \check{A}_\varpi N_{n-1} \otimes X_{n-1}, h_u \right\rangle g_n(\omega, \varpi)$$

Proof. Using (2.1) and (2.2) and the distributions of H_n and N_n under measure $\tilde{P}$, we get

$$\tilde{E}\left(H_n N_n \Lambda_n \mid \mathfrak{I}_n\right) = \sum_{s=1}^{|S_H|} f_s N_n \Lambda_n \tilde{P}\left(H_n = f_s \mid \mathfrak{I}_n\right) = \sum_{s=1}^{|S_H|} \sum_{u=1}^{|S_N|} f_s h_u \tilde{E}\left(\langle H_n, f_s\rangle \langle N_n, h_m\rangle \Lambda_n \mid \mathfrak{I}_n\right)$$

$$= \sum_{s=1}^{|S_H|} f_s \sum_{u=1}^{|S_N|} h_u \sum_{\omega=1}^{Q|S_H||S_N|} \sum_{\varpi=1}^{A|S_X||S_N|} \sum_{s=1}^{|S_H|} \sum_{u=1}^{|S_N|} \langle H_n, f_s\rangle \langle N_n, h_u\rangle \langle \check{Q}_n, b_\omega\rangle \langle \check{A}_n, m_\varpi\rangle \Lambda_{n-1} \times$$

$$\times \prod_{i,j=1}^{|S_X|} \prod_{k=1}^{|S_H|} \prod_{m=1}^{|S_N|} \left(\frac{p_{j,ikm}}{\tilde{p}_{j,ikm}}\right)^{\langle X_n, e_j\rangle \langle X_{n-1}, e_i\rangle \langle H_{n-1}, f_k\rangle \langle N_{n-1}, h_m\rangle} \times$$

$$\times \prod_{r,s=1}^{|S_H|} \prod_{l=1}^{|S_N|} \left(\frac{q_{s,rl}(\omega)}{\tilde{q}_{s,rl}(\omega)}\right)^{\langle H_{n-1}, f_r\rangle \langle N_{n-1}, h_l\rangle} \times$$

$$\times \prod_{u,v=1}^{|S_N|} \prod_{w=1}^{|S_X|} \left(\frac{a_{u,vw}(\varpi)}{\tilde{a}_{u,vw}(\varpi)}\right)^{\langle N_{n-1}, h_v\rangle \langle X_{n-1}, e_w\rangle} \tilde{P}\left(H_n = f_s, N_n = h_m \mid \mathfrak{I}_n\right) =$$

$$= \prod_{i,j=1}^{|S_X|} \prod_{k=1}^{|S_H|} \prod_{m=1}^{|S_N|} \left(\frac{p_{j,ikm}}{\tilde{p}_{j,ikm}}\right)^{\langle X_n, e_j\rangle \langle X_{n-1}, e_i\rangle \langle H_{n-1}, f_k\rangle \langle N_{n-1}, h_m\rangle} \times$$

$$\times \sum_{s=1}^{|S_H|} f_s \sum_{u=1}^{|S_N|} h_u \sum_{\omega=1}^{Q|S_H||S_N|} \prod_{r,s=1}^{|S_H|} \prod_{l=1}^{|S_N|} q_{s,rl}(\omega)^{\langle H_{n-1}, f_r\rangle \langle N_{n-1}, h_l\rangle} \times$$

$$\times \sum_{\varpi=1}^{A|S_X||S_N|} \prod_{u,v=1}^{|S_N|} \prod_{w=1}^{|S_X|} a_{u,vw}(\varpi)^{\langle N_{n-1}, h_v\rangle \langle X_{n-1}, e_w\rangle} \times$$

$$\times \sum_{s=1}^{|S_H|} \sum_{u=1}^{|S_N|} \tilde{E}\left(\langle H_n, f_s\rangle \langle N_n, h_u\rangle \langle \check{Q}_n, b_\omega\rangle \langle \check{A}_n, m_\varpi\rangle \Lambda_{n-1} \mid \mathfrak{I}_n\right)$$

We assume that the processes under consideration are sequences of independent random variables under measure $\tilde{P}$, so instead of condition $\mathfrak{I}_n$ we can write $\mathfrak{I}_{n-1}$. Then we get the following result:

$$\tilde{E}(H_n N_n \Lambda_n | \mathfrak{I}_n) = \left\langle \frac{B}{\bar{B}} X_{n-1} \otimes H_{n-1} \otimes N_{n-1}, X_n \right\rangle \times$$

$$\times \sum_{s=1}^{|S_H|} f_s \sum_{u=1}^{|S_N|} h_u \sum_{\omega=1}^{Q|S_H||S_N|} \prod_{r,s=1}^{|S_H|} \prod_{l=1}^{|S_N|} q_{s,rl}(\omega)^{\langle H_{n-1}, f_r\rangle \langle N_{n-1}, h_l\rangle} \times$$

$$\times \sum_{\varpi=1}^{A|S_X||S_N|} \prod_{u,v=1}^{|S_N|} \prod_{w=1}^{|S_X|} a_{u,vw}(\varpi)^{\langle N_{n-1}, h_v\rangle \langle X_{n-1}, e_w\rangle} \times$$

$$\times \sum_{s=1}^{|S_H|} \sum_{u=1}^{|S_N|} \frac{\tilde{E}\left(\langle H_n, f_s\rangle \langle N_n, h_u\rangle \langle \check{Q}_n, b_\omega\rangle \langle \check{A}_n, m_\varpi\rangle \Lambda_n \mid \mathfrak{I}_{n-1}\right)}{\tilde{E}(\lambda_n \mid \mathfrak{I}_{n-1})}$$

Using the proof of Lemma 2.1 and now changing $\mathfrak{I}_{n-1}$ by $\mathfrak{I}_n$ in the condition, we get the statement of the Lemma 3.1. □

To get a formula for the m step prediction of the joint distribution function of claims number and aggregate claim amounts given the history up to time n, first, we will find out it for 2 step prediction and then extend it to m steps.

Consider the unnormalized conditional joint distribution of $\langle H_n, f_{s_0}\rangle$,$\langle H_{n+1}, f_{s_1}\rangle$, $\langle N_n, h_{u_0}\rangle$ and $\langle N_{n+1}, h_{u_1}\rangle$:

$$\tilde{E}\left(\langle H_n, f_{s_0}\rangle \langle H_{n+1}, f_{s_1}\rangle \langle N_n, h_{u_0}\rangle \langle N_{n+1}, h_{u_1}\rangle \Lambda_{n+1} | \mathfrak{I}_n\right) =$$

$$= \sum_{\omega_0,\omega_1=1}^{Q|S_H||S_N|} \sum_{\varpi_0,\varpi_1=1}^{A|S_X||S_N|} \sum_{j_1=1}^{|S_X|} \tilde{E}\left(\langle H_n, f_{s_0}\rangle \langle H_{n+1}, f_{s_1}\rangle \langle N_n, h_{u_0}\rangle \langle N_{n+1}, h_{u_1}\rangle \langle X_{n+1}, e_{j_1}\rangle \cdot\right.$$

$$\left.\cdot \langle \check{Q}_n, b_{\omega_0}\rangle \langle \check{Q}_{n+1}, b_{\omega_1}\rangle \langle \check{A}_n, m_{\varpi_0}\rangle \langle \check{A}_{n+1}, m_{\varpi_1}\rangle \Lambda_n \lambda_{n+1} | \mathfrak{I}_n\right) =$$

$$= \sum_{\omega_0,\omega_1=1}^{Q|S_H||S_N|} \sum_{\varpi_0,\varpi_1=1}^{A|S_X||S_N|} \sum_{j_1=1}^{|S_X|} \prod_{i=1}^{|S_X|} \left(\frac{p_{j_1,is_0u_0}}{\bar{p}_{j_1,is_0u_0}}\right)^{\langle X_n, e_i\rangle} \left(\frac{q_{s_1,s_0u_0}(\omega_1)}{\bar{q}_{s_1,s_0u_0}(\omega_1)}\right) \prod_{w=1}^{|S_X|} \left(\frac{a_{u_1,u_0w}(\varpi_1)}{\bar{a}_{u_1,u_0w}(\varpi_1)}\right)^{\langle X_n, e_w\rangle} \cdot$$

$$\cdot \tilde{E}\left(\langle H_n, f_{s_0}\rangle \langle H_{n+1}, f_{s_1}\rangle \langle N_n, h_{u_0}\rangle \langle N_{n+1}, h_{u_1}\rangle \langle X_{n+1}, e_{j_1}\rangle \cdot\right.$$

$$\left.\cdot \langle \check{Q}_n, b_{\omega_0}\rangle \langle D\check{Q}_n, b_{\omega_1}\rangle \langle \check{A}_n, m_{\varpi_0}\rangle \langle K\check{A}_n, m_{\varpi_1}\rangle \Lambda_n | \mathfrak{I}_n\right)$$

Using the Markov property of $\check{Q}_n$ and properties of conditional expectation, we obtain

$$\tilde{E}\left(\langle \check{Q}_{n+1}, b_{\omega_1}\rangle | \mathfrak{I}_n\right) = \sum_{\omega_1=1}^{Q|S_H||S_N|} \langle \check{Q}_{n+1}, b_{\omega_1}\rangle \tilde{P}\left(\check{Q}_{n+1} = b_{\omega_1} \mid \check{Q}_n = b_{\omega_0}\right) = d_{\omega_1\omega_0}$$

A similar relationship is valid for the process $\breve{A}_n$. Note that $H_n, H_{n+1}, N_n, N_{n+1}$ and X_{n+1} are not included in $\mathfrak{I}_n$. Using their distributions under $\tilde{P}$, we get:

(3.1)

$$\tilde{E}\left(\langle H_n, f_{s_0}\rangle \langle H_{n+1}, f_{s_1}\rangle \langle N_n, h_{u_0}\rangle \langle N_{n+1}, h_{u_1}\rangle \Lambda_{n+1} | \mathfrak{I}_n\right) =$$

$$= \sum_{\omega_0,\omega_1=1}^{Q|S_H||S_N|} q_{s_1,s_0u_0}(\omega_1) \sum_{\varpi_0,\varpi_1=1}^{A|S_X||S_N|} \sum_{j_1=1}^{|S_X|} \prod_{i=1}^{|S_X|} (p_{j_1,is_0u_0} a_{u_1,u_0i}(\varpi_1))^{\langle X_n, e_i\rangle} d_{\omega_1\omega_0} k_{\varpi_1\varpi_0} \cdot$$

$$\cdot \tilde{E}\left(\langle H_n, f_{s_0}\rangle \langle N_n, h_{u_0}\rangle \mid \mathfrak{I}_n\right) g_n(\omega_0, \varpi_0)$$

Now applying the concept of mathematical induction to formula (3.1), for m steps we come to the following statement.

Theorem 3.1. *The m step prediction for joint probability distribution of the processes H_n and N_n given the information $\mathfrak{I}_n$ has the form*

$$\tilde{E}\left(\Lambda_{n+m} \prod_{t=0}^{m} \langle H_{n+t}, f_{s_t}\rangle \langle N_{n+t}, h_{u_t}\rangle \middle| \mathfrak{I}_n\right) = \sum_{\omega_0,\omega_1\ldots\omega_m=1}^{Q|S_H||S_N|} q_{s_1,s_0u_0}(\omega_1)$$

$$\sum_{\varpi_0,\varpi_1\ldots\varpi_m=1}^{A|S_X||S_N|} \sum_{j_1,j_2,\ldots j_m=1}^{|S_X|} \prod_{i=1}^{|S_X|} (p_{j_1,is_0u_0} \cdot a_{u_1,u_0i}(\varpi_1))^{\langle X_n, e_i\rangle} \times$$

$$\times \prod_{t=1}^{m} p_{j_{t+1},j_ts_tu_t} \cdot q_{s_{t+1},s_tu_t}(\omega_{t+1}) \cdot a_{u_{t+1},u_tj_t}(\varpi_{t+1}) d_{\omega_t\omega_{t-1}} k_{\varpi_t\varpi_{t-1}} \cdot$$

$$\cdot \tilde{E}\left(\langle H_n, f_{s_0}\rangle \langle N_n, h_{u_0}\rangle \mid \mathfrak{I}_n\right) g_n(\omega_0, \varpi_0)$$

4. An application of the model

In this section a practical example of the model described above is presented. Let take a BMS with "C" levels of premium where the first level presents the lowest premium and the last level corresponds to the highest possible premium. Let take $I_1 = [0, a), I_2 = [a, b)$ and $I_3 = [b, +\infty)$ intervals for possible aggregate claim amounts, where $0 < a < b < +\infty$. We also assume that a policyholder can make 0, 1 or 2 claims per year (claims number greater than 2 is considered as 2, so we write "2+", that is two or more claims). The transition of a policyholder between the premium levels can be given by the following table:

Number of claims	Aggregate Claim Interval	Level (+/-)
0		-1
1	I_1	+1
1	I_2	+3
1	I_3	+5
2+	I_1	+2
2+	I_2	+4
2+	I_3	+6

Table 1

As an example we explain the 6-th row of the table as follows: if the policyholder makes two or more claims during the year and the total loss of the company from that claims belongs to the second interval (that is to interval $[a, b)$), then the policyholder is penalized by 4 levels from his/her current level for the next year. It is obvious that in case of no claim the probability of having 0 amount of loss is 1.

If the policyholder is on the first level and makes no claim in the current year, then he/she stays at the same level on the next year. On the other side, if there is not sufficient number of levels to go up (for example if the policyholder is on the "C-1" level and makes "2+" claims, then he/she must be shifted by at least 2 levels up, but only 1 level remains to reach the maximum premium), then the policyholder moves to the highest level.

$$B = \begin{pmatrix} p_0 & p_0 & 0 & & 0 & 0 \\ p_1q_1 & 0 & p_0 & & 0 & 0 \\ p_2q_1 & p_1q_1 & 0 & & 0 & 0 \\ p_1q_2 & p_2q_1 & p_1q_1 & & 0 & 0 \\ p_2q_2 & p_1q_2 & p_2q_1 & & 0 & 0 \\ p_1q_3 & p_2q_2 & p_1q_2 & & 0 & 0 \\ p_2q_3 & p_1q_3 & p_2q_2 & & 0 & 0 \\ 0 & p_2q_3 & p_1q_3 & \ddots & 0 & 0 \\ 0 & 0 & p_2q_3 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & p_0 & 0 \\ 0 & 0 & 0 & & 0 & p_0 \\ 0 & 0 & 0 & & 1-p_0 & 1-p_0 \end{pmatrix}$$

From Table 1 we can conclude that the number of BMS levels must be about or even more than 20. Otherwise, soon after the start (in few years) the majority of policyholders will be concentrated about the highest level of premium. Considering for example 20 levels of BMS and 50 policyholders, we get the cardinality $|S_X| = 20^{50} \approx 1.13 * 10^{65}$. It is obvious that it would be too hard to work with a vector of that size in practice. When a policyholder changes his/her level of BMS, the process X_n changes its state, so instead of transition matrix of X_n it is enough to analyze the one step transition matrix of a policyholder between BMS levels which in the case of Table 1 is a $C \times C$ matrix like B presented above.

Note that the i-th column represents the probabilities of movements from the current level i to the level which corresponds to the number of row of the matrix. Here p_k, $(k = 0, 1, 2)$ is the probability of having k claims (in case of $k = 2$ it is the probability of having 2 or more claims) and q_m, $m = 1, 2, 3$ is the probability that the aggregate claim amount of a policyholder who made k claims belongs to the I_m interval.

In case of insurance event during the next year the policyholder decides: "to claim" or "not to claim" and this will the "Hidden Process".

To get a more flexible BMS the decision-makers have to think about the values of k, m and C.

5. Conclusion

The BMS model (1.4) presented in this paper, can be a tool for constructing more flexible systems to eliminate some of the disadvantages of the current ones. The model development is based on the Markov property of discussed processes. With the help of measure change techniques and hidden Markov models, the recursive formulas for transition matrices and m step predictions for claim numbers and aggregate claim amounts are derived.

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