Известия НАН Армении, Математика, том 59, н. 1, 2024, стр. 54 – 63.

ON THE EUCLIDEAN DISTANCE BETWEEN TWO GAUSSIAN POINTS AND THE NORMAL COVARIOGRAM OF $\mathbb{R}^d$

D. M. MARTIROSYAN, V. K. OHANYAN

Yerevan State University[1]
E-mails: *davit.martirosyan@ysu.am*; *victoohanyan@ysu.am*

Abstract. The concept of covariogram is extended from bounded convex bodies in $\mathbb{R}^d$ to the entire space $\mathbb{R}^d$ by obtaining integral representations for the distribution and density functions of the Euclidean distance between two d-dimensional Gaussian points that have correlated coordinates governed by a covariance matrix. When $d = 2$, a closed-form expression for the density function is obtained. Precise bounds for the moments of the considered distance are found in terms of the extreme eigenvalues of the covariance matrix.

MSC2020 numbers: 60D05; 52A22; 53C65.

Keywords: covariogram; multivariate normal distribution; covariance matrix; integral representation; moment estimation.

1. Introduction

Consider two fundamental characteristics of a bounded body $\mathbb{D} \subset \mathbb{R}^d$. Let the first be the covariogram of $\mathbb{D}$ which has a geometric nature: for any vector $\boldsymbol{t} \in \mathbb{R}^d$, it represents the d-dimensional Lebesgue measure of the region shared between $\mathbb{D}$ and its translated copy by vector $\boldsymbol{t}$. We denote the covariogram of $\mathbb{D}$ by $C_{\mathbb{D}}(\boldsymbol{t})$.

Let the second characteristic be the Euclidean distance between two random points chosen independently and uniformly from $\mathbb{D} \subset \mathbb{R}^d$. This is a well-known random variable studied in geometric probability (see, for example [1]). We denote it by $D_d(\mathbb{D})$. Extensive research has been conducted on this random variable for various bounded bodies $\mathbb{D}$, including computation of the average distance within a cube [2], on the surface of a cube [3], within a hyperball [4], as well as bounding the average distance within a hypercube [5] or furthermore, within compact subsets of $\mathbb{R}^d$ with unit diameter [4]. In dimensions $d \leq 3$, closed-form expressions are obtained for the density function of $D_d(\mathbb{D})$ in [6]-[11] for numerous geometric shapes of $\mathbb{D}$. A unified approach for determining the density function of $D_d(\mathbb{D})$ for typical compact sets is suggested in [12]. It also provides a good list of references for related results of theoretical and applied character.

[1]The research of the first author is supported by the Science Committee of the Ministry of Science, Education, Culture and Sports RA: Grant 21AA-1A024. The research of the second author is partially supported by the Mathematical Studies Center at Yerevan State University.

When $\mathbb{D}$ is a bounded convex body with a non-empty interior in $\mathbb{R}^d$, then the two considered characteristics of $\mathbb{D}$ are interrelated as follows:

$$f_{D_d(\mathbb{D})}(h) = \frac{h^{d-1}}{L_d^2(\mathbb{D})} \int_{S_{d-1}} C_{\mathbb{D}}(h\boldsymbol{u})d\boldsymbol{u}, \ h > 0, \tag{1.1}$$

where S_{d-1} is the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$, centered at the origin, and $L_d(\mathbb{D})$ is Lebesgue d-measure of $\mathbb{D}$.

In this paper, we aim to extend the concepts of covariogram $C_{\mathbb{D}}(t)$ and interpoint distance $D_d(\mathbb{D})$ from bounded convex bodies to the entire space $\mathbb{R}^d$ and establish a relation between them.

The first problem that arises in our way is the nature of randomness of choosing a point from $\mathbb{D} = \mathbb{R}^d$. The uniform distribution is no longer applicable to this case and therefore we will naturally replace it with a multivariate normal distribution.

The second obstacle lies in the challenge of applying the language and sense of geometry to define the covariogram of $\mathbb{R}^d$. We will define it analytically based on the following observation. If $\mathbb{D}$ is a convex body and $\mathcal{P}_1$, $\mathcal{P}_2$ are chosen uniformly and independently from $\mathbb{D}$, then it is easy to check (see, for example, [11]) that

$$f_{\mathcal{P}_1-\mathcal{P}_2}(\boldsymbol{t}) = \frac{C_{\mathbb{D}}(\boldsymbol{t})}{L_d^2(\mathbb{D})},$$

which can be equivalently written as

$$f_{\mathcal{P}_1-\mathcal{P}_2}(\boldsymbol{t}) = \frac{C_{\mathbb{D}}(\boldsymbol{t})}{C_{\mathbb{D}}^2(\boldsymbol{0})}. \tag{1.2}$$

Thus, the covariogram should be a positive function defined on the entire space that satisfies (1.2).

We have defined the normal covariogram of $\mathbb{R}^d$ and established an analogous relationship to (1.1) in section 4, with the foundational basis of the proof presented in the preceding section. Notably, section 3 unveils novel findings, including integral representations for the distribution and density functions of the Euclidean distance between two d-dimensional Gaussian points, characterized by correlated coordinates through a covariance matrix. Precise bounds for the moments of the considered distance in terms of the extreme eigenvalues of the covariance matrix are found. When $d = 2$, an expression for the density function in terms of a modified Bessel function is obtained. In section 2, we independently address the scenario of uncorrelated coordinates and deduce the density and moments of the interpoint distance, drawing upon the results by Mathai and Provost [13].

In the upcoming text, a d-dimensional vector $\mathbf{v} \in \mathbb{R}^d$ will be assumed to be a column vector, or, equivalently, a $d \times 1$ matrix. The transpose of matrix $\mathbf{A}$ will be denoted by $\mathbf{A}^T$. $\mathbf{0}$ will stand for the vector with all zero coordinates, $\mathbf{1}$ for the vector whose all coordinates are equal to 1. $\mathbf{I}_d$ will represent the identity $d \times d$ matrix, $\|\cdot\|_d$ the Euclidean norm in $\mathbb{R}^d$, and $|\mathbf{A}|$ the determinant of matrix $\mathbf{A}$.

If $\mathbf{X}$ is a d-variate normal random vector having mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$ then we denote this condition by $\boldsymbol{X} \sim N_d(\boldsymbol{\mu}, \mathbf{\Sigma})$. We denote $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, ..., \lambda_d]^T$, where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d > 0$ are the eigenvalues of $\mathbf{\Sigma}$.

From now onwards, we assume $\boldsymbol{\mu} = \mathbf{0}$ and the diagonal of $\mathbf{\Sigma}$ consisting of 1s. If $\boldsymbol{X_1}, \boldsymbol{X_2} \sim N_d(\mathbf{0}, \mathbf{\Sigma})$ are independent, we denote

$$D_d = \|\boldsymbol{X_1} - \boldsymbol{X_2}\|_d.$$

2. The density of D_d

Let $\boldsymbol{U} = \boldsymbol{X_1} - \boldsymbol{X_2}$. Since $\boldsymbol{U} \sim N_d(\mathbf{0}, 2\mathbf{\Sigma})$ and $D_d^2 = \boldsymbol{U}^T\boldsymbol{U}$, then the distribution function of D_d^2 can be written in the following form (see [13], page 95):

Theorem 2.1.

$$\mathbb{P}(D_d^2 \leq y) \stackrel{def}{=} F_{D_d^2}(\mathbf{\Sigma}, y) = \sum_{k=0}^{\infty} (-1)^k c_k \frac{y^{\frac{d}{2}+k}}{\Gamma\left(\frac{d}{2}+k+1\right)}, \; y > 0, \tag{2.1}$$

where

$$c_0 = \frac{1}{2^d\sqrt{|\mathbf{\Sigma}|}}, \; c_k = \frac{1}{k}\sum_{r=0}^{k-1} \delta_{k-r} c_r, \; k \geq 1, \tag{2.2}$$

$$\delta_k = \frac{1}{2^{2k+1}} \sum_{i=1}^{d} \frac{1}{\lambda_i^k}, \tag{2.3}$$

and Γ *is the Gamma function*

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt, \quad x > 0.$$

When the coordinates of the Gaussian points are uncorrelated univariate standard normal variables, then $\mathbf{\Sigma} = \boldsymbol{I}_d$ is the identity $d \times d$ matrix and, consequently, $\boldsymbol{\lambda} = \mathbf{1}$. In this case, one can obtain from Theorem 2.1 that D_d follows $GG(a, d, p)$, a generalized Gamma distribution, introduced by E. W. Stacy [14], which has the probability density function

$$f(x; a, d, p) = \frac{\left(p/a^d\right) x^{d-1} e^{-(x/a)^p}}{\Gamma(d/p)}, \; x > 0,$$

where $d > 0$ and $p > 0$ are the shape parameters, and a is a scale parameter. The result is formulated below.

Let $f_{D_d}(\mathbf{\Sigma}, \cdot)$ be the density function of D_d.

Theorem 2.2.

$$f_{D_d}(\boldsymbol{I}_d,\, R) = \frac{R^{d-1} e^{-\frac{R^2}{4}}}{2^{d-1}\Gamma\left(\frac{d}{2}\right)},\ \ R > 0, \tag{2.4}$$

that is, if $\boldsymbol{\Sigma} = \boldsymbol{I}_d$ *then* $D_d \sim GG(2, d, 2)$.

Proof. Since $\boldsymbol{\lambda} = \mathbf{1}$, (2.3) and (2.2) imply $c_0 = 2^{-d}$ and

$$c_k = \frac{d}{k2^{2k+1}} \sum_{r=0}^{k-1} 4^r c_r,\ \ k \geq 1.$$

It is easy to verify by mathematical induction that

$$c_k = \frac{1}{2^d k! 4^k} \prod_{j=0}^{k-1} \left(\frac{d}{2} + j\right),\ k \geq 1,$$

which, based on the identity $x\Gamma(x) = \Gamma(x+1)$, $x > 0$, can be rewritten as

$$c_k = \frac{\Gamma(\frac{d}{2} + k)}{2^d k! 4^k \Gamma(\frac{d}{2})},\ k \geq 1. \tag{2.5}$$

By substituting (2.5) in (2.1) and using $f_{D_d}(\boldsymbol{I}_d,\, R) = 2R\frac{\partial}{\partial R} F_{D_d^2}(\boldsymbol{I}_d,\, R^2)$, we obtain

$$f_{D_d}(\boldsymbol{I}_d,\, R) = 2R \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\frac{d}{2} + k)(R^2)^{\frac{d}{2}+k-1}(\frac{d}{2} + k)}{2^d k! 4^k \Gamma(\frac{d}{2})\Gamma(\frac{d}{2} + k + 1)} =$$

$$= 2R \cdot \frac{R^{d-2}}{2^d \Gamma(\frac{d}{2})} \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{R^2}{4}\right)^k = \frac{R^{d-1} e^{-\frac{R^2}{4}}}{2^{d-1}\Gamma\left(\frac{d}{2}\right)}.$$

□

The moments of the generalized Gamma distribution are well known. If $X \sim GG(a, d, p)$, then (see, for example [15], section 17.8.7)

$$\mathbb{E}\left(X^r\right) = a^r \frac{\Gamma\left(\frac{d+r}{p}\right)}{\Gamma\left(\frac{d}{p}\right)},\ r = 0, 1, 2, \ldots\ .$$

As a result, from Theorem 2.2 we immediately obtain the corresponding formula for the moments of D_d.

Corollary 2.1. *If* $\boldsymbol{\Sigma} = \boldsymbol{I}_d$, *then*

$$\mathbb{E}\left(D_d^r\right) = 2^r \frac{\Gamma\left(\frac{d+r}{2}\right)}{\Gamma\left(\frac{d}{2}\right)},\ r = 0, 1, 2, \ldots\ . \tag{2.6}$$

In general, when $\boldsymbol{\Sigma} \neq \boldsymbol{I}_d$, even when $d = 2$, it is hard to compute the coefficients c_k from the recursive formulas (2.2) and evaluate the infinite sum (2.1).

3. An integral representation of the distribution function of D_d

As usual, we denote by $F_{D_d}(\boldsymbol{\Sigma}, \cdot)$ the distribution function of D_d.

Theorem 3.1. *Let* $\mathcal{E}_d(\boldsymbol{\lambda}, R)$ *be the ellipsoid*

$$\{\boldsymbol{y} = [y_1, y_2, ..., y_d]^T : \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_d y_d^2 \le R^2\}.$$

Then

$$F_{D_d}(\boldsymbol{\Sigma}, R) = \frac{1}{(2\sqrt{\pi})^d} \int_{\mathcal{E}(\boldsymbol{\lambda}, R)} \exp\left(-\frac{1}{4}\boldsymbol{y}^T\boldsymbol{y}\right) d\boldsymbol{y}, \; R > 0. \tag{3.1}$$

Proof. Consider the probability density function of $\boldsymbol{U} = \boldsymbol{X_1} - \boldsymbol{X_2}$:

$$f_{\boldsymbol{U}}(\boldsymbol{u}) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \exp\bigg(-\frac{1}{4}\boldsymbol{u}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{u}\bigg), \; \boldsymbol{u} \in \mathbb{R}^d. \tag{3.2}$$

We denote

$$\mathbf{diag}(\boldsymbol{\lambda}) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix}, \; \mathbf{diag}(\boldsymbol{\lambda}^{-1}) = \begin{bmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_d^{-1} \end{bmatrix}.$$

Due to orthogonal diagonalization theorem for symmetric matrices, there exists an orthogonal matrix $\boldsymbol{Q} = [q_{ij}]_{d\times d}$ such that $\boldsymbol{\Sigma} = \boldsymbol{Q}\,\mathbf{diag}(\boldsymbol{\lambda})\boldsymbol{Q}^T$, and therefore, $\boldsymbol{\Sigma}^{-1} = \boldsymbol{Q}\,\mathbf{diag}(\boldsymbol{\lambda}^{-1})\boldsymbol{Q}^T$. Denoting the i-th column of $\boldsymbol{Q}$ by $\boldsymbol{q}_i$, we obtain

$$\boldsymbol{u}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{u} = [\boldsymbol{u}^T\boldsymbol{q}_1, \, \boldsymbol{u}^T\boldsymbol{q}_2, \, ..., \, \boldsymbol{u}^T\boldsymbol{q}_d\,]\,\mathbf{diag}(\boldsymbol{\lambda}^{-1}) \begin{bmatrix} \boldsymbol{q}_1^T\boldsymbol{u} \\ \boldsymbol{q}_2^T\boldsymbol{u} \\ \vdots \\ \boldsymbol{q}_d^T\boldsymbol{u} \end{bmatrix} = \sum_{i=1}^{d} \frac{(\boldsymbol{q}_i^T\boldsymbol{u})^2}{\lambda_i},$$

and therefore,

$$f_{\boldsymbol{U}}(\boldsymbol{u}) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \exp\bigg(-\frac{1}{4}\sum_{i=1}^{d} \frac{(\boldsymbol{q}_i^T\boldsymbol{u})^2}{\lambda_i}\bigg), \; \boldsymbol{u} \in \mathbb{R}^d. \tag{3.3}$$

Let $x_1, \, x_2, \, ..., \, x_d > 0$, $\boldsymbol{U} = [U_1, \, U_2, \, ..., \, U_d]^T$, $\boldsymbol{U}^* = [|U_1|, \, |U_2|, \, ..., \, |U_d|]^T$ and $\boldsymbol{u} = [u_1, \, u_2, \, ..., \, u_d]^T$. Then, due to (3.3),

$$\mathbb{P}(|U_1| \le x_1, |U_2| \le x_2, ..., |U_d| \le x_d)) \overset{\text{def}}{=} F_{\boldsymbol{U}^*}(x_1, x_2, ..., x_d) =$$

$$= \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \int_{-x_1}^{x_1} du_1 \int_{-x_2}^{x_2} du_2 ... \int_{-x_d}^{x_d} \exp\bigg(-\frac{1}{4}\sum_{i=1}^{d} \frac{(\boldsymbol{q}_i^T\boldsymbol{u})^2}{\lambda_i}\bigg) du_d.$$

The joint density function of the random variables $|U_1|$, $|U_2|$, ..., $|U_d|$ can be reached by partial differentiation of the last iterated integral, i.e.

$$f_{\boldsymbol{U}^*}(x_1, x_2, ..., x_d) = \frac{\partial^d}{\partial x_d \partial x_{d-1} ... \partial x_1} F_{\boldsymbol{U}^*}(x_1, x_2, ..., x_d).$$

Applying the Leibnitz's rule of differentiation d times, we conclude

$$f_{\boldsymbol{U}^*}(x_1, x_2, ..., x_d) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \sum_{\boldsymbol{w} \in \Omega(x_1, x_2, ..., x_d)} \exp\bigg(-\frac{1}{4}\sum_{i=1}^{d} \frac{(\boldsymbol{q}_i^T\boldsymbol{w})^2}{\lambda_i}\bigg), \tag{3.4}$$

where $\Omega(x_1, x_2, ..., x_d) = \{\boldsymbol{w} = [w_1, w_2, ..., w_d]^T \, : \, |w_i| = x_i, \, i = 1, 2, ..., d\}$.

We now aim to replace the summing index in (3.4) and run it over all the binary strings of length d. For any $\boldsymbol{s} = (s_1, s_2, ..., s_d) \in \{0, 1\}^d$ consider the unique vector $\boldsymbol{w_s} = [w_1^{(\boldsymbol{s})}, w_2^{(\boldsymbol{s})}, ..., w_d^{(\boldsymbol{s})}]^T \in \Omega(x_1, x_2, ..., x_d)$ such that

$$w_i^{(\boldsymbol{s})} = \begin{cases} x_i & \text{if } s_i = 0 \\ -x_i & \text{if } s_i = 1 \end{cases}.$$

Formula (3.4) can be equivalently written in the following form:

$$f_{\boldsymbol{U}^*}(x_1, x_2, ..., x_d) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \sum_{\boldsymbol{s} \in \{0,\,1\}^d} \exp\left(-\frac{1}{4} \sum_{i=1}^{d} \frac{(\boldsymbol{q}_i^T \boldsymbol{w_s})^2}{\lambda_i} \right). \tag{3.5}$$

Since $F_{D_d}(\boldsymbol{\Sigma}, R) = \mathbb{P}(\|U^*\| \le R)$, $R > 0$, the formula (3.4) implies

$$F_{D_d}(\boldsymbol{\Sigma}, R) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \int_{B_d^{\boldsymbol{0}}(R)} \sum_{\boldsymbol{s} \in \{0,\,1\}^d} \exp\left(-\frac{1}{4} \sum_{i=1}^{d} \frac{(\boldsymbol{q}_i^T \boldsymbol{w_s})^2}{\lambda_i} \right) d\boldsymbol{x}, \tag{3.6}$$

where $B_d^{\boldsymbol{0}}(R) = \{(x_1, x_2, ..., x_d) \, : \, x_1^2 + x_2^2 + ... + x_d^2 \le R^2, \, x_i > 0, \, i = 1, 2, ..., d\}$ is an 2^d-quadrant of the d-dimensional ball $B_d(R)$ of radius R centered at the origin, and $d\boldsymbol{x} = dx_1 dx_2 ... dx_d$. Hereinafter, for any $\boldsymbol{s} = (s_1, s_2, ..., s_d) \in \{0, 1\}^d$, the symbol $B_d^{\boldsymbol{s}}(R)$ will stand for the 2^d-quadrant of $B_d(R)$ consisting of the points $(x_1, x_2, ..., x_d)$ such that $x_i > 0$, if $s_i = 0$ and $x_i < 0$, if $s_i = 1$.

By interchanging the sum with the integral in (3.6) and denoting

$$g_{\boldsymbol{s}}(x_1, x_2, ..., x_n) = \exp\left(-\frac{1}{4} \sum_{i=1}^{d} \frac{(\boldsymbol{q}_i^T \boldsymbol{w_s})^2}{\lambda_i} \right),$$

we receive

$$F_{D_d}(\boldsymbol{\Sigma}, R) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \sum_{\boldsymbol{s} \in \{0,\,1\}^d} \int_{B_d^{\boldsymbol{0}}(R)} g_{\boldsymbol{s}}(x_1, x_2, ..., x_d) dx_1 dx_2 ... dx_d. \tag{3.7}$$

Let us perform the following change of variable in the integral of $g_{\boldsymbol{s}}$ over $B_d^{\boldsymbol{0}}(R)$:

$$t_i = x_i, \text{ if } s_i = 0,$$

$$t_i = -x_i, \text{ if } s_i = 1.$$

The Jacobian $\frac{D(x_1, x_2, ..., x_d)}{D(t_1, t_2, ..., t_d)}$ is either 1 or -1, therefore after this change of variable we obtain

$$\int_{B_d^{\boldsymbol{0}}(R)} g_{\boldsymbol{s}}(x_1, x_2, ..., x_d) dx_1 dx_2 ... dx_d = \int_{B_d^{\boldsymbol{s}}(R)} g_{\boldsymbol{0}}(x_1, x_2, ..., x_d) dx_1 dx_2 ... dx_d. \tag{3.8}$$

Since the sets $B_d^{\boldsymbol{s}}(R)$, $\boldsymbol{s} \in \{0, 1\}^d$ are pairwise disjoint and the union of their closures is exactly equal to $B_d(R)$, from (3.7) and (3.8) we establish

$$F_{D_d}(\boldsymbol{\Sigma}, R) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \int_{B_d(R)} g_{\boldsymbol{0}}(x_1, x_2, ..., x_d) dx_1 dx_2 ... dx_d. \tag{3.9}$$

We have $\boldsymbol{w_0} = \boldsymbol{x} = [x_1, x_2, ..., x_d]^T$, which means that

$$g_{\boldsymbol{0}}(x_1, x_2, ..., x_d) = \exp\bigg(-\frac{1}{4}\sum_{i=1}^{d}\frac{(\boldsymbol{q}_i^T\boldsymbol{x})^2}{\lambda_i}\bigg). \tag{3.10}$$

To finish the proof, we make one more change of variable in the integral of $g_{\boldsymbol{0}}$ over the ball $B_d(R)$. Consider a new variable $\boldsymbol{y} = [y_1, y_2, ..., y_d]$, where

$$y_i = \frac{\boldsymbol{q}_i^T\boldsymbol{x}}{\sqrt{\lambda_i}}, \;\; i = 1, 2, ..., d. \tag{3.11}$$

Using orthogonality of $\boldsymbol{Q}$, we will have

$$\frac{D(x_1, x_2, ..., x_d)}{D(y_1, y_2, ..., y_d)} = \sqrt{\lambda_1}\sqrt{\lambda_2}...\sqrt{\lambda_d}\,|\boldsymbol{Q}| = |\boldsymbol{\Sigma}|^{1/2} \tag{3.12}$$

and

$$\sum_{i=1}^{d} x_i^2 = \sum_{i=1}^{d}(\sqrt{\lambda_1}q_{i1}y_1 + \sqrt{\lambda_2}q_{i2}y_2 + ... + \sqrt{\lambda_d}q_{id}y_d)^2 = \sum_{i=1}^{d}\lambda_i y_i^2. \tag{3.13}$$

Now (3.1) follows from (3.9)-(3.13).

Corollary 3.1. *The probability density function of D_d is representable as follows:*

$$f_{D_d}(\boldsymbol{\Sigma}, R) = \frac{R^{d-1}}{(2\sqrt{\pi})^d|\boldsymbol{\Sigma}|^{1/2}}\int_{S_{d-1}}\exp\bigg(-\frac{R^2}{4}\boldsymbol{u}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{u}\bigg)d\boldsymbol{u}. \tag{3.14}$$

Proof. As we saw in the last part of the proof of Theorem 3.1, the formula (3.1) is equivalent to

$$F_{D_d}(\boldsymbol{\Sigma}, R) = \frac{1}{(2\sqrt{\pi})^d|\boldsymbol{\Sigma}|^{1/2}}\int_{B_d(R)}\exp\bigg(-\frac{1}{4}\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\bigg)d\boldsymbol{x}. \tag{3.15}$$

The change of variable $\boldsymbol{x} = r\boldsymbol{u}$, $\boldsymbol{u} \in S_{d-1}$, $d\boldsymbol{x} = r^{d-1}drd\boldsymbol{u}$ in (3.15) produces

$$F_{D_d}(\boldsymbol{\Sigma}, R) = \frac{1}{(2\sqrt{\pi})^d|\boldsymbol{\Sigma}|^{1/2}}\int_{S_{d-1}}d\boldsymbol{u}\int_0^R\exp\bigg(-\frac{r^2}{4}\boldsymbol{u}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{u}\bigg)r^{d-1}dr.$$

By taking the derivatives of both sides in the last equation, we establish (3.14). □

As an application of the obtained integral representations, we easily found the probability density function of the Euclidean distance between two bivariate Gaussian points in the case when there is an intercoordinate correlation ρ.

Theorem 3.2. *If* $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, *then*

$$f_{D_2}(\boldsymbol{\Sigma}, R) = \frac{Re^{-\frac{R^2}{4|\boldsymbol{\Sigma}|}}}{2\sqrt{|\boldsymbol{\Sigma}|}}I_0\bigg(\frac{\rho R^2}{4|\boldsymbol{\Sigma}|}\bigg),$$

where

$$I_0(x) = \sum_{k=0}^{\infty}\frac{x^{2k}}{((2k)!!)^2}$$

is the modified Bessel function of the first kind of order zero.

Proof. It is easy to see that $\lambda_1 = 1+\rho$, $\lambda_2 = 1-\rho$. By (3.14), we have

$$f_{D_2}(\boldsymbol{\Sigma}, R) = \frac{R}{4\pi\sqrt{1-\rho^2}} \int_0^{2\pi} \exp\left(-\frac{R^2\cos^2\varphi}{4+4\rho} - \frac{R^2\sin^2\varphi}{4-4\rho} \right) d\varphi =$$

$$= \frac{Re^{-\frac{R^2}{4(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} \int_0^{\pi} e^{a\cos 2\varphi} d\varphi,$$

where

$$a = \frac{\rho R^2}{4(1-\rho^2)}.$$

Since $|\boldsymbol{\Sigma}| = 1-\rho^2$, to complete the proof it remains to show that

$$\frac{1}{\pi}\int_0^{\pi} e^{a\cos 2\varphi} d\varphi = I_0(a).$$

Indeed, Taylor's expansion for e^x solves this problem:

$$\frac{1}{\pi}\int_0^{\pi} e^{a\cos 2\varphi} d\varphi = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{a^k}{k!}\int_0^{\pi}\cos^k 2\varphi d\varphi = \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{a^k}{k!}\int_0^{2\pi}\cos^k \psi d\psi =$$

$$= \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{a^{2k}}{(2k)!}\int_0^{\pi/2}\cos^{2k}\psi d\psi = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{a^{2k}}{(2k)!}\frac{(2k-1)!!}{(2k)!!}\frac{\pi}{2} = \sum_{k=0}^{\infty}\frac{a^{2k}}{((2k)!!)^2} = I_0(a).$$

□

As another application, we established lower and upper bounds for the moments of D_d in terms of the largest and the smallest eigenvalues of the covariance matrix.

Theorem 3.3. *Let $\mathbb{E}(D_d^r)$ be the r-th moment of D_d. Then*

$$\frac{2^r\Gamma\left(\frac{d+r}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\frac{\lambda_d^{\frac{d+r}{2}}}{|\boldsymbol{\Sigma}|^{1/2}} \le \mathbb{E}(D_d^r) \le \frac{2^r\Gamma\left(\frac{d+r}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\frac{\lambda_1^{\frac{d+r}{2}}}{|\boldsymbol{\Sigma}|^{1/2}},\ r = 0, 1, 2, \dots . \tag{3.16}$$

Proof. As $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_d > 0$ then we have

$$\frac{1}{\lambda_1}\sum_{i=1}^{d}(\boldsymbol{q}_i^T\boldsymbol{u})^2 \le \boldsymbol{u}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{u} \le \frac{1}{\lambda_d}\sum_{i=1}^{d}(\boldsymbol{q}_i^T\boldsymbol{u})^2. \tag{3.17}$$

Due to orthogonality of $\boldsymbol{Q}$,

$$\sum_{i=1}^{d}(\boldsymbol{q}_i^T\boldsymbol{u})^2 = \|\boldsymbol{u}\|_d^2 = 1, \text{ if } \boldsymbol{u} \in S_{d-1},$$

therefore, the integral representation (3.14) and inequalities (3.17) yield

$$\frac{e^{-\frac{R^2}{4\lambda_d}}R^{d-1}}{(2\sqrt{\pi})^d|\boldsymbol{\Sigma}|^{1/2}}\int_{S_{d-1}} d\boldsymbol{u} \le f_{D_d}(\boldsymbol{\Sigma}_d, R) \le \frac{e^{-\frac{R^2}{4\lambda_1}}R^{d-1}}{(2\sqrt{\pi})^d|\boldsymbol{\Sigma}|^{1/2}}\int_{S_{d-1}} d\boldsymbol{u}.$$

The surface area of S_{d-1} is well-known and equal to $\frac{2(\sqrt{\pi})^d}{\Gamma(\frac{d}{2})}$, so we obtain

$$\frac{R^{d-1}e^{-\frac{R^2}{4\lambda_d}}}{2^{d-1}\Gamma(\frac{d}{2})|\boldsymbol{\Sigma}|^{1/2}} \le f_{D_d}(\boldsymbol{\Sigma}_d, R) \le \frac{R^{d-1}e^{-\frac{R^2}{4\lambda_1}}}{2^{d-1}\Gamma(\frac{d}{2})|\boldsymbol{\Sigma}|^{1/2}}. \tag{3.18}$$

Multiplying all sides of (3.18) by R^r and applying integral over $(0, +\infty)$ to all sides leads to

$$\frac{\lambda_d^{\frac{d+r}{2}}}{|\mathbf{\Sigma}|^{1/2}} I(d) \le \mathbb{E}\left(D_d^r\right) \le \frac{\lambda_1^{\frac{d+r}{2}}}{|\mathbf{\Sigma}|^{1/2}} I(d),$$

where

$$I(d) = \frac{1}{2^{d-1}\Gamma(\frac{d}{2})} \int_0^{+\infty} R^{d+r-1} e^{-\frac{R^2}{4}} dR.$$

It remains to apply (2.4) and (2.6) to see that

$$I(d) = 2^r \frac{\Gamma\left(\frac{d+r}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}.$$

4. The normal covariogram of $\mathbb{R}^d$

The covariogram of a bounded domain $\mathbb{D} \subset \mathbb{R}^d$ is known to be the function

$$C_{\mathbb{D}}(\boldsymbol{t}) = L_d(\mathbb{D} \cap \{\mathbb{D} + \boldsymbol{t}\}), \;\; \boldsymbol{t} \in \mathbb{R}^d,$$

where $\mathbb{D} + \boldsymbol{t} = \{\mathcal{P} + \boldsymbol{t} \,:\, \mathcal{P} \in \mathbb{D}\}$ and $L_d(\cdot)$ is the d-dimensional Lebesgue measure in $\mathbb{R}^d$. If $\mathbb{D}$ is a convex body and $\mathcal{P}_1$, $\mathcal{P}_2$ are chosen uniformly and independently from $\mathbb{D}$, then the probability density function of $\mathcal{P}_1 - \mathcal{P}_2$ can be expressed by the covariogaram of $\mathbb{D}$ as shown in (1.2). This motivates us to extend the concept of the covariogram for $\mathbb{D} = \mathbb{R}^d$.

Definition 4.1. *Let $\mathcal{P}_1$, $\mathcal{P}_2 \sim N_d(\mathbf{0}, \mathbf{\Sigma})$ be independent and $f_{\mathcal{P}_1 - \mathcal{P}_2}$ be the probability density function of $\mathcal{P}_1 - \mathcal{P}_2$. The function $C_{\mathbf{\Sigma}} : \mathbb{R}^d \to (0, +\infty)$ that satisfies*

$$f_{\mathcal{P}_1 - \mathcal{P}_2}(\boldsymbol{t}) = \frac{C_{\mathbf{\Sigma}}(\boldsymbol{t})}{C^2_{\mathbf{\Sigma}}(\mathbf{0})},$$

is called the normal covariogram of $\mathbb{R}^d$ associated with $\mathbf{\Sigma}$.

By taking $\boldsymbol{t} = \mathbf{0}$ in this definition and using (3.2) we immediately obtain

$$C_{\mathbf{\Sigma}}(\mathbf{0}) = (2\sqrt{\pi})^d |\mathbf{\Sigma}|^{1/2},$$

and then

$$C_{\mathbf{\Sigma}}(\boldsymbol{t}) = (2\sqrt{\pi})^d |\mathbf{\Sigma}|^{1/2} \exp\left(-\frac{1}{4} \boldsymbol{t}^T \mathbf{\Sigma}^{-1} \boldsymbol{t} \right), \; \boldsymbol{t} \in \mathbb{R}^d. \tag{4.1}$$

It is remarkable that $C_{\boldsymbol{I}_d}(\boldsymbol{t}) = (2\sqrt{\pi})^d \exp\left(-\frac{1}{4}\|\boldsymbol{t}\|_d^2\right)$. It illustrates that if $\mathbb{R}^d$ is considered as a space of points with uncorrelated coordinates then the covariogram of the space is naturally independent on the direction of translation.

Taking into account (1.1), the following identity provides a further argument to ensure that the normal covariogram naturally extends the concept of covariogram.

Theorem 4.1.

$$f_{D_d}(\boldsymbol{\Sigma}, R) = \frac{R^{d-1}}{C^2_{\boldsymbol{\Sigma}}(\mathbf{0})} \int_{S_{d-1}} C_{\boldsymbol{\Sigma}}(R\boldsymbol{u}) d\boldsymbol{u}, \;\; R > 0. \tag{4.2}$$

Proof. By (4.1),

$$\frac{R^{d-1}}{C^2_{\boldsymbol{\Sigma}}(\mathbf{0})} \int_{S_{d-1}} C_{\boldsymbol{\Sigma}}(R\boldsymbol{u}) d\boldsymbol{u} = \frac{R^{d-1}}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \int_{S_{d-1}} \exp\left(-\frac{R^2}{4} \boldsymbol{u}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{u} \right) d\boldsymbol{u}.$$

Now due to (3.14), the right-hand-side of the above equality coincides with the left-hand-side of (4.2).

Список литературы

[1] L. A. Santalo, Integral Geometry and Geometric Probability, Cambridge Univ. Press, Cambridge (2004).

[2] D. P. Robbins, T. S. Bolis, “Solution to problem E2629: Average Distance between Two Points in a Box”, American Math. Monthly, **85**, no. 4, 277 – 278 (1978).

[3] D. H. Bailey, J. M. Borwein, R. E. Crandall, “Box integrals”, Journal of Computational and Applied Mathematics, **206**, Issue 1, 196 – 208 (2007).

[4] B. Burgstaller, F. Pillichshammer, “The average distance between two points”, Bull. Aust. Math. Soc. **80**, 353 – 359 (2009).

[5] R. S. Anderssen, R. P. Brent, D. J. Daley, P. A. P. Moran, “Concerning $\int_0^1 \cdots \int_0^1 \left(x_1^2 + \cdots + x_k^2\right)^{1/2} dx_1 \cdots dx_k$ and Taylor series method”, SIAM Journal on Applied Mathematics, **30** (1), 22 – 30 (1976).

[6] A. M. Mathai, P. Moschopoulos, G. Pederzoli, “Random points associated with rectangles”, Rendiconti del Circolo Matematico Di Palermo, **48**, 162 – 190 (1999).

[7] A. M. Mathai, P. Moschopoulos, G. Pederzoli, “Distance between random points in a cube”, Statistica, **59** (1), 61 – 81 (1999).

[8] N. G. Aharonyan, “The distribution of the distance between two random points in a convex set”, Russian Journal of Mathematical Research, Series A, **1**, 4 – 8 (2015).

[9] N. G. Aharonyan, V. K. Ohanyan, “Moments of the distance between two random points”, Modeling of Artificial Intelligence, **10**, Is. 2, 64 – 70 (2016).

[10] N. G. Aharonyan, V. H. Khalatyan, “Distribution of the distance between two random points in a body from $\mathbb{R}^n$”, Journal of Contemporary Mathematical Analysis **55** (6), 329 – 334 (2020).

[11] V. K. Ohanyan, V. H. Khalatyan, “Relation between the covariogram and distribution function of the distance between two uniform and independent points”, Proceedings of the YSU, Physical and Mathematical Sciences, **56**, Is. 1, 33 – 42 (2022).

[12] S. Lellouche, M. Souris, “Distribution of distances between elements in a compact set”, Stats, MDPI, **3** (1), 1 – 15 (2019).

[13] A. M. Mathai, S. B. Provost, Quadratic Forms in Random Variables, Theory and applications, Marcel Dekker, New York (1992).

[14] E. W. Stacy, “A generalization of the Gamma distribution”, Annals of Mathematical Statistics **33** (3), 1187 – 1192 (1962).

[15] N. L. Johnson, S. Kotz, N. Balakrishnan, Continuous Univariate Distributions, **1**, 2nd Edition, Wiley (1994).

Поступила 01 августа 2023

После доработки 01 августа 2023

Принята к публикации 10 октября 2023