UDC 517.9
Mathematics

Georgi SAHAKYAN
Ph.D., associate professor of the Department of Applied Mathematics and Computer Science, ArSU.
E-mail: ter_saak_george@mail.ru
A. OHANYAN

ON SOME PROPERTIES OF SOLUTIONS OF A THREE-DIMENTIONAL LINEAR HOMOGENEOUS SYSTEM OF DIFFERENTIAL EQUATIONS WITH A POSITIVE COEFFICIENTS

The paper considers the some properties of solutions of a three-dimentional linear homogeneous system of differential equations with a postive and continuous on the whole numerical line coefficients.

Keywords. *system of linear differential equations.*

Գ. Սահակյան, Ա.Օհանյան

ԴՐԱԿԱՆ ԳՈՐԾԱԿԻՑՆԵՐՈՎ ԴԻՖԵՐԵՆՑԻԱԼ ՀԱՎԱՍԱՐՈՒՄՆԵՐԻ ԵՌԱՉԱՓ ԳԾԱՅԻՆ ՀԱՄԱՍԵՌ ՀԱՄԱԿԱՐԳԻ ԼՈՒԾՈՒՄՆԵՐԻ ՈՐՈՇ ՀԱՏԿՈՒԹՅՈՒՆՆԵՐԻ ՄԱՍԻՆ

Աշխատանքում ամբողջ թվային առանցքի վրա անընդհատ և դրական գործակիցներով եռաչափ գծային համասեռ դիֆերենցիալ հավասարումների համակարգերի համար դիտարկվում են լուծումների որոշ հատկությունները:

Բանալի բառեր՝ *գծային դիֆերենցիալ հավասարումների համակարգ*

Г. Саакян, А. Оганян

О НЕКОТОРЫХ СВОЙСТВАХ РЕШЕНИЙ ТРЕХМЕРНОЙ ЛИНЕЙНОЙ ОДНОРОДНОЙ СИСТЕМЫ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ПОЛОЖИТЕЛЬНЫМИ КОЭФФИЦИЕНТАМИ

В работе рассматриваются свойства решений трехмерной линейной однородной системы дифференциальных уравнений с положительными и непрерывными на всей числовой оси коэффициентами.

Ключевые слова: *система линейных дифференциальных уравнений*

The following three-dimensional linear homogeneous system of ordinary differential equations with continuous coefficients on the whole numerical line is considered (see, for example, [1])

$$\boldsymbol{y}' = \boldsymbol{A}(t)\boldsymbol{y}, \tag{1}$$

where

$$\boldsymbol{A}(t)=\{a_{ij}(t)\},\ \ a_{ij}(t)\geq 0,\quad i,j=1,2,3,\ \ \boldsymbol{y}=\boldsymbol{y}(t)=\begin{pmatrix} y_1(t)\\ y_2(t)\\ y_3(t)\end{pmatrix}.$$

If we uing the transform ([2])

$$y_i(t)=z_i(t)\exp\left(\int_{t_0}^{t} a_{ii}(\tau)d\tau\right), \tag{2}$$

then system (1) is transformed into an equivalent system

$$\boldsymbol{z}'=\boldsymbol{B}(t)\boldsymbol{z}, \tag{3}$$

in which the coefficients of the matrix $\boldsymbol{B}(t)$ are determined by the relations:

$$b_{ij}(t)=a_{ij}(t)\exp\left(\int_{t_0}^{t}\left(a_{jj}(\tau)-a_{ii}(\tau)\right)d\tau\right), \tag{4}$$

and, in particular, the diagonal elements of the matrix $\boldsymbol{B}(t)$ will be equal to zero. On the other hand, it follows from formulas (4) that as a result of this transformation, the signs of the coefficients of the matrix $\boldsymbol{A}(t)$ will remain in the matrix $\boldsymbol{B}(t)$ Note also that if a component $z_i(t)$ of a solution $\boldsymbol{z}(\boldsymbol{t})$ of system (3) takes on only positive values, then an increase in this component will result in an increase in its corresponding component $y_i(t)$. The latter will follow from relations (2) with $y_i(t)>0$ and $z_i'(t)>0$, since in this case we will have

$$z_i'(t)=\left(y_i(t)\exp\left(-\int_{t_0}^{t}a_{ii}(\tau)d\tau\right)\right)'=y_i'(t)\exp\left(-\int_{t_0}^{t}a_{ii}(\tau)d\tau\right)-y_i(t)a_{ii}(\tau)\exp\left(-\int_{t_0}^{t}a_{ii}(\tau)d\tau\right)>0,$$

whence, dividing the inequality by the expression $\exp\left(-\int_{t_0}^{t}a_{ii}(\tau)d\tau\right)$, we obtain

that

$$y_i'(t)>y_i(t)a_{ii}(\tau)>0.$$

From the above considerations, it follows that when considering the monotonicity of the components of the solutions of system (1) in the case under consideration, without losing generality of reasoning, we can restrict ourselves to the case when the diagonal elements of the matrix $\boldsymbol{A}(t)$ are zero. Thus, in future discussions, we will assume that the coefficients $a_{ii}(\mathrm{t})\equiv 0,\ i=1,2,3.$

Theorem 1. *The components of every solution of system* (1) *with positive initial values are everywhere increasing functions.*

Proof. Suppose now that $\boldsymbol{y}(t)$ is a solution of the system (1) that satisfies the conditions

$$y_1(t_0)>0, y_2(t_0)>0,\ \ y_3(t_0)>0.$$

We write the system (1) in the expanded form, namely

$$\begin{cases} y_1' = a_{12}(t)y_2 + a_{13}(t)y_3, \\ y_2' = a_{21}(t)y_1 + a_{23}(t)y_3, \\ y_3' = a_{31}(t)y_1 + a_{32}(t)y_2. \end{cases} \quad (5)$$

From system (5), due to our assumptions, it follows that all the components of the solution $\boldsymbol{y}(t)$ at the point t_0 are increase. Without losing common reasoning, we now assume that one of the components of the solution of the system (5), for example, $y_1(t)$), does not preserve the nature of monotonicity. To do this, we assume that $y_1^{'}(t) > 0$ for $t \in [t_0, t_1)$) and $y_1^{'}(t) < 0$ for $t \in (t_1, t_2)$, and therefore, $y_1^{'}(t_1) = 0$ due to the continuity of $y_1(t)$. Then from the first equation of system (2) we will have

$$a_{12}(t_1)y_2(t_1) + a_{13}(t_1)y_3(t_1) = 0,$$

where do we find that

$$y_2(t_1)y_3(t_1) < 0.$$

Suppose now

$$y_2(t_1) > 0, y_3(t_1) < 0. \quad (6)$$

Then, since, according to our assumptions, $y_3(t_0) > 0$ and $y_3(t)$ is continuous on the whole number axis, there is $t_3 \in (t_0, t_1)$ so that

$$y_3^{'}(t_3) = 0$$

and for $t \in (t_3, t_1]$

$$y_3^{'}(t) < 0.$$

But then from the 3rd equation of system (5) we will have

$$a_{31}(t)y_1(t) + a_{32}(t)y_2(t) < 0$$

or

$$a_{32}(t)y_2(t) < -a_{31}(t)y_1(t),$$

whence it will follow that for $t \in (t_3, t_1]$ $y_2(t) < 0$, and, in particular,

$$y_2(t_1) < 0,$$

which contradicts conditions (6). We have come to a contradiction, which proves the theorem.

The assertion of Theorem 1 directly implies.

Corollary. *The components of any solution of system* (2) *with positive initial values at the point* t_0 *take on the interval* $[t_0, +\infty)$ *only positive values***.**

Theorem 2. *For every solution* $\boldsymbol{y}(t)$ *of the system* (1)

$$\|\boldsymbol{y}\| \leq C \exp\left(\frac{1}{8}\int_{t_0}^{t} \|\boldsymbol{A}\|\, d\tau\right), \quad (7)$$

where $C = \|y\||_{t=t_0}$

Proof. Suppose again $\mathbf{y}(t)$ is the solution of system (5). Multiplying the first equation of the system by y_1, the second by y_2, the third by y_3, and then adding the results, we arrive at the following equation

$$(y_1^2 + y_2^2 + y_3^2)' = \frac{1}{2}[(a_{12}(t) + a_{21}(t))y_1y_2 + (a_{13}(t) + a_{31}(t))y_1y_3 + (a_{23}(t) + a_{32}(t))y_2y_3]. \quad (8)$$

Next, in view of the obvious inequality

$$x(t)y(t) \leq \frac{1}{2}\left(x^2(t) + y^2(t)\right), \quad (9)$$

we will have

$$\left|(y_1^2+y_2^2+y_3^2)'\right| = \frac{1}{2}\left|(a_{12}(t)+a_{21}(t))y_1y_2+(a_{13}(t)+a_{31}(t))y_1y_3+(a_{23}(t)+a_{32}(t))y_2y_3)\right| \le$$

$$\le \frac{1}{4}\left|a_{12}(t)+a_{21}(t)|(y_1^2+y_2^2)+|a_{13}(t)+a_{31}(t)|(y_1^2+y_3^2)+|a_{23}(t)+a_{32}(t)|(y_2^2+y_3^2))\right| \le$$

$$\le \frac{1}{4}|a_{12}(t)+a_{21}(t)|(y_1^2+y_2^2+y_3^2)+|a_{13}(t)+a_{31}(t)|(y_1^2+y_2^2+y_3^2)+ +|a_{23}(t)+a_{32}(t)|(y_1^2+y_2^2+y_3^2)$$

$$\le \frac{1}{4}\sum_{\substack{i,j=1\\ i<j}}^{3}\left(|a_{ij}(t)|+|a_{ji}(t)|\right)(y_1^2+y_2^2+y_3^2) \qquad (10)$$

We introduce the notation

$$u(t)=y_1^2(t)+y_2^2(t)+y_3^2(t),$$

then inequality (10) is written as

$$\left|u'(t)\right| \le \frac{1}{4}\sum_{\substack{i,j=1\\ i<j}}^{3}\left(|a_{ij}(t)|+|a_{ji}(t)|\right)u(t).$$

Given the fact that

$$\|\boldsymbol{y}\|^2=y_1^2(t)+y_2^2(t)+y_3^2(t), \text{ and } \quad \|A\|=\sum_{\substack{i,j=1\\ i<j}}^{3}\left(|a_{ij}(t)|+|a_{ji}(t)|\right), \qquad (11)$$

will have

$$(\|\boldsymbol{y}\|^2)' \le \frac{1}{4}\|A\|\|y\|^2,$$

or

$$\frac{(\|\boldsymbol{y}\|^2)'}{\|\boldsymbol{y}\|^2} \le \frac{1}{4}\|A\|.$$

Integrating this inequality in the range from t_0 to t, we find that

$$\|y\| \le C\exp\left(\frac{1}{8}\int_{t_0}^{t}\|A\|\,d\tau\right),$$

where $C=\|y\||_{t=t_0}$. The theorem is proved.

Corollary. *For any solution $\boldsymbol{y}(t)$ of the system* (1) *with positive initial values at the point t_0 $\|\boldsymbol{y}\|$ is an increasing function on the interval* $[t_0, +\infty)$.

The proof follows from relations (8), (11) and from Theorem1.

Theorem 3. *The product of the components of every solution $\boldsymbol{y}(t)$ of system* (1) *with positive initial values at a point t_0 is an increasing function on the interval* $[t_0, +\infty)$ *and tends to infinity when $t \to \infty$.*

Proof. Suppose $\boldsymbol{y}(t)$ is a solution of system (1) with initial conditions

$$y_1(t_0)>0, y_2(t_0)>0, \quad y_3(t_0)>0\,.$$

Then, according to the corollary of Theorem 1, the components of the solution in question will be positive on the whole numerical line. Multiplying the first equation of the system by y_2y_3, the second equation by y_1y_3, the third equation by y_1y_2, and then adding the resulting equation, we will have

$$(y_1y_2y_3)' = (a_{12}(t) + a_{21}(t))y_3(y_1^2 + y_2^2) + (a_{13}(t) + a_{31}(t))y_2(y_1^2 + y_3^2) + \\ +(a_{23}(t) + a_{32}(t))y_1(y_2^2 + y_2^2)).$$

Using the inequality (9) again, and also taking into account the positiveness of the components of the solution, we obtain

$$(y_1y_2y_3)' \geq 2\left(\sqrt{a_{12}(t)a_{21}(t)}y_1y_2y_3 + \sqrt{a_{13}(t)a_{31}(t)}y_1y_2y_3 \\ + \sqrt{a_{12}(t)a_{21}(t)}y_1y_2y_3\right) = \\ = 2\sum_{\substack{i,j=1\\ i<j}}^{3} \sqrt{a_{12}(t)a_{21}(t)}\, y_1y_2y_3. \tag{12}$$

If we difine $v(t)$ by

$$v(t) = y_1(t)y_2(t)y_3(t),$$

then inequality (12) is written as

$$v' \geq 2\sum_{\substack{i,j=1\\ i<j}}^{3} \sqrt{a_{12}(t)a_{21}(t)}\, v,$$

integrating that in the range from t_0 to t, we get the inequality

$$v(t) \geq Ce^{a(t)}, \tag{13}$$

in which $C > 0$, and

$$a(t) = 2\int_{t_0}^{t} \sum_{\substack{i,j=1\\ i<j}}^{3} \sqrt{a_{12}(\tau)a_{21}(\tau)}\, d\tau.$$

Since the integrand in this relation is positive, the function $a(t)$ is increasing, and therefore, as $t \to \infty$ we have $a(t) \to \infty$. In this case, from inequality (13) (or from (12)) it will follow that $v(t)$ is also an increasing function and $v(t) \to \infty$ as $t \to \infty$, as required to be proved.

Below, in Figures 1-3, graphical interpretations of assertions of proved theorems and corollaris are given on the example of one particular solution of the system

$$\begin{cases} y_1' = 0.02ty_2 + 0.03t^2y_3, \\ y_2' = 0.003ty_1 + \frac{0.04}{t}y_3, \\ y_3' = 0.01(1+t)y_1 + \frac{0.05}{t^2}y_2, \end{cases} \tag{14}$$

with the initial conditions: $y_1(1) = 2,\ y_2(1) = 35, y_3(1) = 16$ $(t_0 = 1)$, in the segment $[1, 5]$, built in the Mathcad environment (in the figures y0 corresponds to the component y_1, y1 - to the component y_2, and y2 to the component y_3).

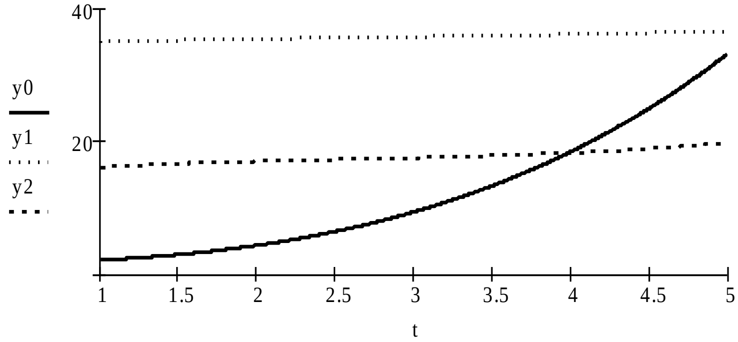

Fig. 1. The graphs of the components $\boldsymbol{y}(t)$

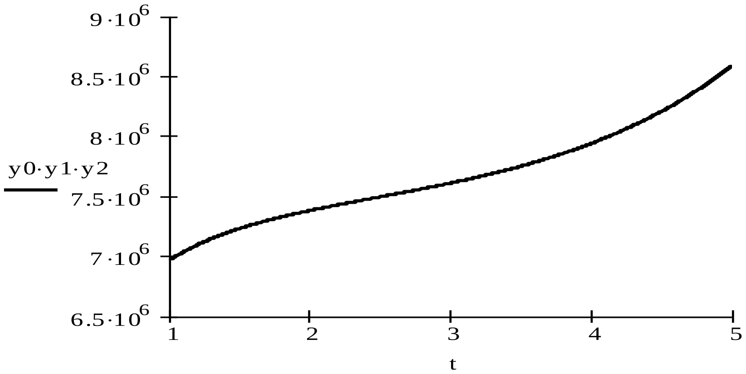

Fig. 2. The graph of the $v(t)$ - product of the components $\boldsymbol{y}(t)$

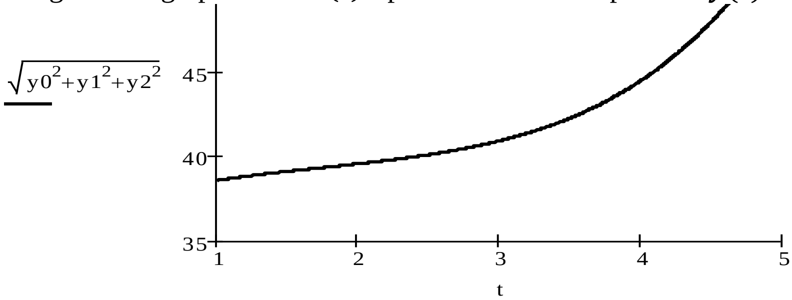

Fig. 3. The graph of the $\|\boldsymbol{y}\|$

Literature

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Հոդվածը տպագրության է երաշխավորել ԵՊՀ ֆունկցիոնալ ամբիոնի և դիֆերենցիալ հավասարումների ամբիոնի պրոֆեսոր, ֆ.մ.գ.դ. Տ.Ն.Հարությունյանը: