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SOME ESTIMATES FOR RIESZ TRANSFORMS ASSOCIATED WITH SCHRÖDINGER OPERATORS

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Abstract. Let $\mathcal{L} = -\Delta + V$ be the Schrödinger operator on $\mathbb{R}^n$, where $n \geq 3$, and nonnegative potential V belongs to the reverse Hölder class RH_q with $n/2 \leq q < n$. Let $H^p_{\mathcal{L}}(\mathbb{R}^n)$ denote the Hardy space related to $\mathcal{L}$ and $BMO_{\mathcal{L}}(\mathbb{R}^n)$ denote the dual space of $H^1_{\mathcal{L}}(\mathbb{R}^n)$. In this paper, we show that $T_{\alpha,\beta} = V^\alpha \nabla \mathcal{L}^{-\beta}$ is bounded from $H^{p_1}_{\mathcal{L}}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$ for $\frac{n}{n+\delta'} < p_1 \leq 1$ and $\frac{1}{p_2} = \frac{1}{p_1} - \frac{2(\beta-\alpha)}{n}$, where $\delta' = \min\{1, 2-n/q_0\}$, and q_0 is the reverse Hölder index of V. Moreover, we prove $T^*_{\alpha,\beta}$ is bounded on $BMO_{\mathcal{L}}(\mathbb{R}^n)$ when $\beta - \alpha = 1/2$.

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1. Introduction and results

The study of the theory of harmonic analysis related to Schrödinger operator is one of the most interesting topic, which has attracted a great deal of attention of many researchers; see [7][8],[12]-[16],[20]-[23] and references therein. The present paper investigate the boundedness of Riesz transform associated with Schrödinger operator on Hardy space and BMO space.

Let $\mathcal{L} = -\Delta + V$ be the Schrödinger operator on $\mathbb{R}^n$, where $n \geq 3$ and the nonnegative potential V belongs to reverse Hölder class RH_q for $q \geq n/2$. Recall that given $0 \leq V \in L^q_{loc}(\mathbb{R}^n)$ for $1 < q < \infty$, V is said to belong to the reverse Hölder class RH_q if there exists a constant $C = C(q, V) > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(y)^q dy\right)^{1/q} \leq \frac{C}{|B|}\int_B V(y)dy$$

holds for every ball $B \subset \mathbb{R}^n$.

Clearly, if V belongs to $RH_q, q > 1$, then V is a Muckenhoup A_∞ weight; see [17]. From weight theory we know that $V(x)dx$ is a doubling measure and the class RH_q has self-improvement property; that is, if $V \in RH_q$ for some $q > 1$, then there exists $\epsilon > 0$ such that $V \in RH_{q+\epsilon}$. We define the reverse Hölder index of V as $q_0 = \sup\{q : \ V \in RH_q\}$. From now on, we always use δ' to denote $\min\{1, 2-n/q_0\}$.

As in [18], for a given potential $V \in RH_q$ with $q \geq n/2$, the auxiliary function is defined as

$$\rho(x) = \frac{1}{m(x,V)} = \sup\left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq 1 \right\}, \quad x \in \mathbb{R}^n.$$

It is well known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

The Hardy space related to $\mathcal{L}$ had been studied by Dziubański and Zienkiewicz in [5, 6]. Because $0 \leq V \in L^q_{loc}(\mathbb{R}^n)$, the Schrödinger operator $\mathcal{L}$ generates a (C_0) contraction semigroup $\{T_s^{\mathcal{L}} : s > 0\} = \{e^{-s\mathcal{L}} : s > 0\}$. The maximal function associated with $\{T_s^{\mathcal{L}} : s > 0\}$ is defined by $M^{\mathcal{L}} f(x) = \sup_{s>0} |T_s^{\mathcal{L}} f(x)|$. The Hardy space $H^1_{\mathcal{L}}(\mathbb{R}^n)$ is defined as follows.

Definition 1.1. We say that f is an element of $H^1_{\mathcal{L}}(\mathbb{R}^n)$ if the maximal function $M^{\mathcal{L}} f$ belongs to $L^1(\mathbb{R}^n)$. The quasi-norm of f is defined by $\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)} = \|M^{\mathcal{L}} f\|_{L^1(\mathbb{R}^n)}$.

The dual space of $H^1_{\mathcal{L}}(\mathbb{R}^n)$ is the BMO type space $BMO_{\mathcal{L}}(\mathbb{R}^n)$ (see [1]).

Definition 1.2. Let f be a locally function on $\mathbb{R}^n$ and $B = B(x,r)$. Set $f_B = \frac{1}{|B|}\int_B f(y)dy$ and $f(B,V) = f_B$ if $r < \rho(x)$; $f(B,V) = 0$ if $r \geq \rho(x)$. We say $f \in BMO_{\mathcal{L}}(\mathbb{R}^n)$ if

$$\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} = \sup_B \frac{1}{|B|}\int_B |f(y) - f(B,V)|dy < \infty.$$

It follows from [1] that $\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}$ is actually a norm which makes $BMO_{\mathcal{L}}(\mathbb{R}^n)$ a Banach space. Since $H^1(\mathbb{R}^n) \subset H^1_{\mathcal{L}}(\mathbb{R}^n)$, it conclude by duality that $BMO_{\mathcal{L}}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$. Some papers have studied the boundedness of operators on $BMO_{\mathcal{L}}(\mathbb{R}^n)$ space; See[1, 3, 4].

To give the definition of Hardy space $H^p_{\mathcal{L}}(\mathbb{R}^n)$ for $\frac{n}{n+\delta'} < p < 1$, we introduce the Campanato type space.

Assume that $\frac{n}{n+\delta'} < p < 1$, and $1 \leq q' \leq \infty$. A locally integrable function f is said to be in the Campanato type space $\Lambda^L_{\frac{1}{p}-1,q'}$, if

$$\|f\|_{\Lambda^L_{\frac{1}{p}-1,q'}} = \sup_{B\subset\mathbb{R}^n}\left\{ |B|^{1-\frac{1}{p}} \left(\int_B |f(y) - f(B,V)|^{q'} \frac{dy}{|B|} \right)^{1/q'} \right\} < \infty.$$

For any $1 \leq q' \leq \infty$, the spaces $\Lambda^L_{\frac{1}{p}-1,q'}$ are mutually coincident with equivalent norms, it will be simply denoted by $\Lambda^L_{\frac{1}{p}-1}$. It can be proved that the maximal function $M^{\mathcal{L}} f$ is well defined for $f \in \left(\Lambda^L_{\frac{1}{p}-1}\right)^*$.

Definition 1.3. [2] For $\frac{n}{n+\delta'} < p < 1$, we say that $f \in \left(\Lambda^L_{\frac{1}{p}-1}\right)^*$ is an element of $H^p_{\mathcal{L}}(\mathbb{R}^n)$ if the maximal function $M^{\mathcal{L}} f$ belongs to $L^p(\mathbb{R}^n)$. The quasi-norm of f is defined by $\|f\|_{H^p_{\mathcal{L}}(\mathbb{R}^n)} = \|M^{\mathcal{L}} f\|_{L^p(\mathbb{R}^n)}$.

We consider the Riesz transform

$$T_{\alpha,\beta} = V^{\alpha}\nabla\mathcal{L}^{-\beta}, \quad 0 \le \alpha \le \frac{1}{2} \le \beta \le 1, \quad \beta - \alpha \ge \frac{1}{2}.$$

The boundedness of $T_{\alpha,\beta}$ has been studied under the condition $V \in RH_q$ for $n/2 \le q < n$. In [18], Shen showed that $T_{0,\frac{1}{2}}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < p_0$, $\frac{1}{p_0} = \frac{1}{s_0} - \frac{1}{n}$; he proved $T_{\frac{1}{2},1}$ is also bounded on $L^p(\mathbb{R}^n)$ for $1 < p < p_1$, $\frac{1}{p_1} = \frac{3}{2s_0} - \frac{1}{n}$. Li and Peng in [10] obtained that $T_{0,\frac{1}{2}}$ is bounded from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$, Hu and Wang in [9] established the boundedness of commutator for $T_{\alpha,\alpha}$, Liu [11] obtained the boundedness of $T_{0,\beta}$ on $H^1_{\mathcal{L}}(\mathbb{R}^n)$.

If $0 \le \alpha \le \frac{1}{2} < \beta \le 1, \beta - \alpha \ge \frac{1}{2}, V \in RH_q$ for $n/2 \le q < n$, Sugano in [19] given the L^p-estimates for $T_{\alpha,\beta}$ and its adjoint operator $T^*_{\alpha,\beta}$.

Proposition 1.1. *Suppose that $V \in RH_q$ with $\frac{n}{2} \le q < n$. Let $0 \le \alpha \le \frac{1}{2} < \beta \le 1, \beta - \alpha \ge \frac{1}{2}, \frac{1}{p_\alpha} = \frac{\alpha+1}{q} - \frac{1}{n}$.*

(i) *If $1 < p < \frac{1}{\frac{1}{p_\alpha} + \frac{2(\beta-\alpha)-1}{n}}$ and $\frac{1}{q} = \frac{1}{p} - \frac{2(\beta-\alpha)-1}{n}$, then*

$$\|T_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)};$$

(ii) *If $p'_\alpha < p < \frac{n}{2(\beta-\alpha)-1}$ and $\frac{1}{q} = \frac{1}{p} - \frac{2(\beta-\alpha)-1}{n}$, then*

$$\|T^*_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

In this paper, we investigate the boundedness of $T_{\alpha,\beta}$ and $T^*_{\alpha,\beta}$ on $H^p_{\mathcal{L}}$ space and $BMO_{\mathcal{L}}(\mathbb{R}^n)$ space respectively, and get the following results.

Theorem 1.1. *Suppose $V \in RH_q$ with $\frac{n}{2} \le q < n$. Let $0 \le \alpha \le \frac{1}{2} < \beta \le 1, \beta - \alpha \ge \frac{1}{2}$. If $\frac{n}{n+\delta'} < p_1 \le 1$, and $\frac{1}{p_2} = \frac{1}{p_1} - \frac{2(\beta-\alpha)-1}{n}$, then*

$$\|T_{\alpha,\beta}(f)\|_{L^{p_2}(\mathbb{R}^n)} \le C\|f\|_{H^{p_1}_{\mathcal{L}}(\mathbb{R}^n)}.$$

By Theorem 1.1 we get

Corollary 1.1. *Suppose $V \in RH_q$ with $\frac{n}{2} \le q < n$. Let $0 \le \alpha \le \frac{1}{2} < \beta \le 1, \beta - \alpha > \frac{1}{2}$. Then*

$$\|T^*_{\alpha,\beta}(f)\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \le C\|f\|_{L^{p_0}(\mathbb{R}^n)},$$

where $p_0 = \frac{n}{2(\beta-\alpha)-1}$.

When $\beta - \alpha = 1/2$, we have

Theorem 1.2. *Suppose $V \in RH_q$ with $\frac{n}{2} \le q < n$. Let $0 \le \alpha \le \frac{1}{2} < \beta \le 1$ and $\beta - \alpha = \frac{1}{2}$. Then*

$$\|T^*_{\alpha,\beta}(f)\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \le C\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \le CB$. $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Some preliminaries

Throughout this section we always assume $V \in RH_q$ with $\frac{n}{2} \le q < n$.

2.1 Some results concerning the auxiliary function

Lemma 2.1. [5] *There exists constant $l_0 > 0$ such that*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \lesssim \left(1 + \frac{r}{\rho(x)}\right)^{l_0}.$$

Lemma 2.2. [18] *For $0 < r < R < \infty$, we have*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \lesssim \left(\frac{R}{r}\right)^{n/s-2} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy.$$

Lemma 2.3. [18] *There exist C and $k_0 \ge 1$ such that*

$$C^{-1}\left(1 + \frac{|x-y|}{\rho(y)}\right)^{\frac{1}{1+k_0}} \le 1 + \frac{|x-y|}{\rho(x)} \le C\left(1 + \frac{|x-y|}{\rho(y)}\right)^{1+k_0}$$

for all $x, y \in \mathbb{R}^n$.

A ball $B(x, \rho(x))$ is called critical. Assume that $Q = B(x_0, \rho(x_0))$, for $x \in Q$, the inequality above tell us that $\rho(x) \sim \rho(y)$, if $|x - y| < C\rho(x)$.

2.2 Atomic decomposition of Hardy space $H^p_{\mathcal{L}}(\mathbb{R}^n)$

Let $\frac{n}{n+\delta'} < p \le 1 \le q \le \infty$ and $p \ne q$. A function $a \in L^2(\mathbb{R}^n)$ is called an $H^{p,q}_{\mathcal{L}}$-atom if $r < \rho(x_0)$ and the following conditions hold:

(i) supp $a \subset B(x_0, r)$,

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \le |B(x_0, r)|^{1/q-1/p}$,

(iii) if $r < \rho(x_0)/4$, then $\int_{B(x_0,r)} a(x)dx = 0$.

By [5] and [6], the Hardy space $H^p_{\mathcal{L}}(\mathbb{R}^n)$ admits the following atomic decomposition:

Lemma 2.4. *Let $\frac{n}{n+\delta'} < p \le 1 \le q \le \infty$. Then $f \in H^p_{\mathcal{L}}(\mathbb{R}^n)$ if and only if f can be written as $f = \sum_j \lambda_j a_j$, where a_j are $H^{p,q}_{\mathcal{L}}$- atoms , $\sum_j |\lambda_j|^p < \infty$, and the sum converges in the $H^p_{\mathcal{L}}(\mathbb{R}^n)$ quasi-norm. Moreover*

$$\|f\|_{H^p_{\mathcal{L}}(\mathbb{R}^n)} \sim \inf\left\{\left(\sum_j |\lambda_j|^p\right)^{1/p}\right\},$$

where the infimum is taken over all atomic decompositions of f into $H^{p,q}_{\mathcal{L}}$- atoms.

2.3 Estimates for the kernel functions

Suppose $\mathcal{W}_\beta = \nabla\mathcal{L}^{-\beta}$. Let $\mathcal{W}^*_\beta$ be the adjoint of $\mathcal{W}_\beta, K$ and K^* be the kernels of $\mathcal{W}_\beta$ and $\mathcal{W}^*_\beta$ respectively, then $K(x,z) = K^*(z,x)$ and we have the following estimates.

Lemma 2.5. [9] *Suppose* $1/2 < \beta \le 1$.

(i) *For every N there exists a constant C such that*

$$|K^*(x,z)| \le \frac{C_N}{\left(1+\frac{|x-z|}{\rho(x)}\right)^N} \frac{1}{|x-z|^{n-2\beta}} \left(\int_{B(z,|x-z|/4)} \frac{V(\xi)}{|\xi-z|^{n-1}} d\xi + \frac{1}{|x-z|}\right).$$

Moreover, the inequality above also holds with $\rho(x)$ replaced by $\rho(z)$.

(ii) *For every N, and $\delta = 2 - n/q$, there exists a constant C_N such that*

$$|K^*(x,z) - K^*(y,z)| \le \frac{C_N}{\left(1+\frac{|x-z|}{\rho(x)}\right)^N} \times \frac{|x-y|^\delta}{|x-z|^{n-2\beta+\delta}} \left(\int_{B(z,|x-z|/4)} \frac{V(\xi)}{|\xi-z|^{n-1}} d\xi + \frac{1}{|x-z|}\right)$$

whenever $|x-y| < \frac{1}{16}|x-z|$. Moreover, the inequality above also holds with $\rho(x)$ replaced by $\rho(z)$.

2.4 Characterization of space $BMO_{\mathcal{L}}(\mathbb{R}^n)$

Lemma 2.6. [1] *Let $1 \le p < \infty, B = B(x,r)$. If $f \in BMO_{\mathcal{L}}(\mathbb{R}^n)$, then*

$$\sup_B \left(\frac{1}{|B|}\int_B |f(y) - f(B,V)|^p dy\right)^{1/p} \le C\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

A function $f \in BMO_{\mathcal{L}}(\mathbb{R}^n)$ if and only if there exists a suitable constant c_B depending on B and satisfying $c_B = 0$ whenever $r \ge \rho(x)$ such that

$$\sup_B \left(\frac{1}{|B|}\int_B |f(y) - c_B|^p dy\right)^{1/p} < \infty$$

and

$$\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \le C \sup_B \left(\frac{1}{|B|}\int_B |f(y) - c_B|^p dy\right)^{1/p}.$$

3. Proof of main results

Proof of Theorem 1.1. By Lemma 2.4, we only need to prove

$$\|T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} \lesssim 1$$

holds for any $H^{p_1,q_1}_{\mathcal{L}}$-atom. Because $0 \le \alpha \le \frac{1}{2} < \beta \le 1, \beta - \alpha \ge \frac{1}{2}$, we can choose q_1 and q_2 such that

$$1 < q_1 < \frac{1}{\frac{1}{p_\alpha} + \frac{2(\beta-\alpha)-1}{n}}$$

and

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{2(\beta-\alpha)-1}{n}.$$

Assume that supp$a \subset B(x_0,r), r < \rho(x_0)$. Then

$$\|T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} \le \|\chi_{16B}T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} + \|\chi_{(16B)^c}T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} = I_1 + I_2.$$

By Hölder inequality, Proposition 1.1 and $\frac{1}{p_2}=\frac{1}{p_1}-\frac{2(\beta-\alpha)-1}{n}$, we have

$$\begin{aligned} I_1 &= \|\chi_{16B}T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} \\ &\lesssim |16B|^{\frac{1}{p_2}-\frac{1}{q_2}}\left(\int_{\mathbb{R}^n}|T_{\alpha,\beta}a(x)|^{q_2}dx\right)^{1/q_2} \\ &\lesssim |16B|^{\frac{1}{p_2}-\frac{1}{q_2}}\left(\int_B|a(x)|^{q_1}dx\right)^{1/q_1} \\ &\lesssim \ |16B|^{\frac{1}{p_2}-\frac{1}{q_2}}|B|^{\frac{1}{q_1}-\frac{1}{p_1}}\lesssim 1. \end{aligned}$$

We divided into two case for the estimate of $I_2:\ r\geq\rho(x_0)/4$ and $r<\rho(x_0)/4$.

Case I: $r\geq\rho(x_0)/4$. In this case, we have $r\sim\rho(x_0)$. It follows from Lemma 2.3 and Lemma 2.5 that

$$\begin{aligned} \int_B\big|K(x,z)a(z)\big|dz \lesssim & \int_B\frac{|a(z)|dz}{\left(1+\frac{|x-z|}{\rho(x_0)}\right)^N|x-z|^{n-2\beta+1}} \\ &+\int_B\frac{|a(z)|}{\left(1+\frac{|x-z|}{\rho(x_0)}\right)^N|x-z|^{n-2\beta}}\int_{B(x,|x-z|/4)}\frac{V(\xi)}{|\xi-x|^{n-1}}d\xi dz. \end{aligned}$$

For any $x\in C_k=\{x:2^kr<|x-z|\leq 2^{k+1}r\}, k\geq 4$, we have

$$\begin{aligned} \int_B\big|K(x,z)a(z)\big|dz \lesssim & \frac{1}{2^{kN}(2^kr)^{n-2\beta+1}}\int_B|a(z)|dz \\ &+\frac{1}{2^{kN}(2^kr)^{n-2\beta}}\mathcal{I}_1(V\chi_{B(x_0,2^{k+1}r)})(x)\int_B|a(z)|dz, \end{aligned}$$

where $\mathcal{I}_1(f)(x)=\int_{\mathbb{R}^n}\frac{f(y)dy}{|x-y|^{n-1}}$. Then

$$I_2\leq\left(\sum_{k\geq 4}\int_{C_k}V(x)^{\alpha p_2}\left(\int_B\big|K(x,z)a(z)\big|dz\right)^{p_2}dx\right)^{1/p_2}\leq I_{21}+I_{22},$$

where

$$I_{21}=\left(\sum_{k\geq 4}\frac{(2^kr)^{(2\beta-n-1)p_2+n}}{2^{kNp_2}}\frac{1}{|2^kB|}\int_{2^kB}V(x)^{\alpha p_2}dx\right)^{1/p_2}\int_B|a(z)|dz,$$

and

$$I_{22}=\left(\sum_{k\geq 4}\frac{(2^kr)^{(2\beta-n)p_2+n}}{2^{kNp_2}}\frac{1}{|2^kB|}\int_{2^kB}V(x)^{\alpha p_2}(\mathcal{I}_1(V\chi_{2^{k+1}B})(x))^{p_2}dx\right)^{1/p_2}\int_B|a(z)|dz.$$

Notice

$$p_2\leq\frac{n}{n-2(\beta-\alpha)+1}<\frac{n}{2\alpha}<\frac{s}{\alpha}.$$

By Hölder inequality, $V \in RH_q$ and Lemma 2.1 we get

$$\frac{1}{|2^kB|}\int_{2^kB} V(x)^{\alpha p_2}dx \lesssim \left(\frac{1}{|2^kB|}\int_{2^kB} V(x)^s dx\right)^{\alpha p_2/s}$$

$$\lesssim \left(\frac{1}{|2^kB|}\int_{2^kB} V(x) dx\right)^{\alpha p_2} \lesssim (2^kr)^{-2\alpha p_2}\left(1+\frac{2^kr}{\rho(x_0)}\right)^{l_0\alpha p_2}. \tag{3.1}$$

Then

$$\frac{1}{|2^kB|}\int_{2^kB} V(x)^{\alpha p_2}dx \lesssim (2^kr)^{-2\alpha p_2}2^{kl_0\alpha p_2}.$$

Because a is a $H^{p_1,q_1}_{\mathcal{L}}$- atom, so

$$\int_B |a(y)|dy \le |B|^{1-\frac{1}{p_1}}. \tag{3.2}$$

Note

$$2(\beta-\alpha)-1+n(\frac{1}{p_2}-\frac{1}{p_1})=0. \tag{3.3}$$

Then, by (3.1) and (3.2) we obtain

$$I_{21} \lesssim \left(\sum_{k\ge 4}\frac{1}{(2^k)^{\left(N-l_0\alpha-(2(\beta-\alpha)-1)+n\right)p_2-n}}\right)^{1/p_2}.$$

Taking N large enough such that $N > l_0\alpha+(2(\beta-\alpha)-1)-n+n/p_2$, we get $I_{21}\lesssim 1$.

It is easy to see $p_2 < p_\alpha$. Then

$$\frac{1}{p_2} > \frac{1}{p_\alpha} = \frac{\alpha}{q}+\frac{1}{t}, \quad \frac{1}{t}=\frac{1}{q}-\frac{1}{n}.$$

By Hölder inequality, the boundedness of fractional integral $\mathcal{I}_1 : L^q \to L^t$ with $\frac{1}{t}=\frac{1}{q}-\frac{1}{n}$ and $V \in RH_q$, we obtain

$$\frac{1}{|2^kB|}\int_{2^kB} V(x)^{\alpha p_2}(\mathcal{I}_1(V\chi_{2^{k+1}B})(x))^{p_2}dx$$

$$\lesssim \left(\frac{1}{|2^kB|}\int_{2^kB} V(x)^q dx\right)^{p_2\alpha/q}\left(\frac{1}{|2^kB|}\int_{2^kB} (\mathcal{I}_1(V\chi_{2^{k+1}B})(x))^t dx\right)^{p_2/t}$$

$$\lesssim \left(\frac{1}{|2^kB|}\int_{2^kB} V(x) dx\right)^{p_2\alpha}\left(\frac{1}{|2^kB|}\int_{2^{k+1}B} V(x)^q dx\right)^{p_2/q}|2^kB|^{p_2(\frac{1}{q}-\frac{1}{t})}$$

$$\lesssim \left(\frac{1}{|2^kB|}\int_{2^kB} V(x) dx\right)^{p_2(\alpha+1)}(2^kr)^{p_2}$$

$$\lesssim (2^kr)^{-(2\alpha+1)p_2}\left(1+\frac{2^kr}{\rho(x_0)}\right)^{p_2l_0(\alpha+1)}. \tag{3.4}$$

Then

$$\frac{1}{|2^kB|}\int_{2^kB} V(x)^{\alpha p_2}(\mathcal{I}_1(V\chi_{2^{k+1}B})(x))^{p_2}dx \lesssim (2^kr)^{-(2\alpha+1)p_2}(2^k)^{p_2l_0(\alpha+1)}.$$

Noting (3.3) and taking N large enough such that $N > l_0(\alpha+1)+\big(2(\beta-\alpha)-1\big)-n+n/p_2$, we get

$$I_{22} \lesssim \left(\sum_{k\geq 4} \frac{1}{(2^k)^{p_2\left(N-l_0(\alpha+1)-\left(2(\beta-\alpha)-1\right)+n-n/p_2\right)}}\right)^{1/p_2} \lesssim 1.$$

Case II: $r < \rho(x_0)/4$. When $p=1$, by Lemma 2.3 and Lemma 2.5, for $\delta = 2-n/q$,

$$|K(x,z)-K(x,x_0)|$$
$$\lesssim \frac{1}{\left(1+\frac{|x-z|}{\rho(x_0)}\right)^{N'}} \frac{|z-x_0|^\delta}{|x-z|^{n+\delta-2\beta}} \left(\int_{B(x_0,|x-z|/4)} \frac{V(\xi)}{|\xi-x|^{n-1}} d\xi + \frac{1}{|x-z|}\right).$$

Then, for $x \in C_k$, we have

$$|K(x,z)-K(x,x_0)| \leq \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^N} \frac{r^\delta}{(2^k r)^{n+\delta-2\beta+1}}$$
$$+ \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^N} \frac{r^\delta}{(2^k r)^{n+\delta-2\beta}} \int_{2^k B} \frac{V(\xi)}{|\xi-x|^{n-1}} d\xi. \tag{3.5}$$

It follows from the vanishing condition of a, (3.5) and (3.2) that

$$I_2 = \left(\int_{(16B)^c} V(x)^{\alpha p_2} \left(\int_B |(K(x,z)-K(x,x_0))a(z)|dz\right)^{p_2} dx\right)^{1/p_2} \lesssim I'_{21} + I'_{22},$$

where

$$I'_{21} \lesssim \left(\sum_{k\geq 4} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2}} \frac{r^{\delta p_2}}{(2^k r)^{(n+\delta-2\beta+1)p_2}} \int_{C_k} V(x)^{\alpha p_2} dx\right)^{1/p_2},$$

and

$$I'_{22} \lesssim \left(\sum_{k\geq 4} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2}} \frac{r^{\delta p_2}}{(2^k r)^{(n+\delta-2\beta)p_2}} \int_{C_k} V(x)^{\alpha p_2} \left(\mathcal{I}_1(V\chi_{2^{k+1}B})(x)\right)^{p_2} dx\right)^{1/q}.$$

By (3.1), noting $\frac{1}{p_2}-1+\frac{2(\beta-\alpha)-1}{n}=0$ and taking $N \geq l_0\alpha$ we have

$$I'_{21} \lesssim \left(\sum_{k\geq 4} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2-l_0\alpha p_2}} \frac{1}{2^{k\delta p_2}}\right)^{1/p_2} \lesssim \left(\sum_{k\geq 4} \frac{1}{2^{k\delta p_2}}\right)^{1/p_2} \lesssim 1.$$

By (3.4) and taking $N \geq l_0(\alpha+1)$ we have

$$I'_{22} \lesssim \left(\sum_{k\geq 4} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2-l_0(\alpha+1)p_2}} \frac{1}{2^{k\delta p_2}}\right)^{1/p_2} \lesssim \left(\sum_{k\geq 4} \frac{1}{2^{k\delta p_2}}\right)^{1/p_2} \lesssim 1.$$

We consider the case $\frac{n}{n+\delta'} < p_1 < 1$. By $n/2 \le q < n$ and the self-improvement property of the class RH_q, for some p_0 such that $\frac{n}{n+\delta'} < p_0 < p_1 < 1$, we have some $\delta_0 = 2 - n/q < 2 - n/q_0$ so that $p_0 = \frac{n}{n+\delta_0}$. Then

$$\frac{1}{p_2} = \frac{1}{p_1} - \frac{2(\beta-\alpha)-1}{n} < \frac{n+\delta_0-2(\beta-\alpha)+1}{n}.$$

So, we have $p_2 > \frac{n}{n+\delta_0-2(\beta-\alpha)+1}$.

By (3.5), for $x \in C_k$, we have

$$|K(x,z)-K(x,x_0)| \le \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^N} \frac{r^{\delta_0}}{(2^k r)^{n+\delta_0-2\beta+1}} + \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^N} \frac{r^{\delta_0}}{(2^k r)^{n+\delta_0-2\beta}} \int_{2^k B} \frac{V(\xi)}{|\xi-x|^{n-1}} d\xi.$$

Thus,

$$I_2 = \left(\int_{(4B)^c} \left|T_{\alpha,\beta}(a)(x)\right|^{p_2} dx\right)^{1/p_2} \lesssim I_{21}^{''} + I_{22}^{''},$$

where

$$I_{21}^{''} = \left(\sum_{k\ge 4} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2}} \frac{r^{\delta_0 p_2}}{(2^k r)^{(n+\delta_0-2\beta+1)p_2}} \int_{2^k B} V(x)^{\alpha p_2} dx\right)^{1/p_2} \int_B |a(y)|dy,$$

and

$$I_{22}^{''} \lesssim \left(\sum_{k\ge 4} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2}} \frac{r^{\delta_0 p_2}}{(2^k r)^{(n+\delta_0-2\beta)p_2}} \int_{C_k} V(x)^{\alpha p_2} \left(\mathcal{I}_1(V\chi_{2^{k+1}B})(x)\right)^{p_2} dx\right)^{1/q} \times \int_B |a(y)|dy.$$

Note $p_2 > \frac{n}{n+\delta_0-2(\beta-\alpha)+1}$. Then, by (3.1),(3.2) and (3.3) and taking $N \ge l_0\alpha$, we get

$$I_{21}^{''} \lesssim \left(\sum_{k\ge 4} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{(N-l_0\alpha)p_2}} \frac{1}{\left(2^k\right)^{p_2(n+\delta_0-2(\beta-\alpha)+1)-n}}\right)^{1/p_2} \lesssim 1.$$

By (3.3), (3.4) and taking $N > l_0(\alpha+1)$, we get

$$I_2 \lesssim \left(\sum_{k\ge 3} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{(N-l_0(\alpha+1))p_2}} \frac{1}{\left(2^k\right)^{p_2(n+\delta_0-2(\beta-\alpha)+1)-n}}\right)^{1/p_2} \lesssim 1.$$

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. For $\beta - \alpha = 1/2$, it follows from Proposition 1.1 that

$$\|T^*_{\alpha,\beta} f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

for $p'_\alpha < p < \infty$.

Let $f \in BMO_{\mathcal{L}}(\mathbb{R}^n)$ and fix a ball $B = B(x_0, r)$. We consider two cases: $r \geq \rho(x_0)$ and $r < \rho(x_0)$. For the case $r \geq \rho(x_0)$, we write

$$f = f\chi_{B^*} + f\chi_{(B^*)^c} = f_1 + f_2,$$

where $B^* = 2B$. Owing to the fact that $T^*_{\alpha,\beta}$ is bounded on $L^p(\mathbb{R}^n)$, we have

$$\frac{1}{|B|}\int_B |T^*_{\alpha,\beta}(f_1)(x)|dx \lesssim \left(\frac{1}{|B|}\int_B |T^*_{\alpha,\beta}(f_1)(x)|^p dx\right)^{1/p} \lesssim \left(\frac{1}{|B^*|}\int_{B^*} |f(x)|^p dx\right)^{1/p}.$$

By Lemma 2.6 we get

$$\frac{1}{|B|}\int_B |T^*_{\alpha,\beta}(f_1)(x)|dx \lesssim C\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

Let $C_k = \{z : 2^k r < |x - z| \leq 2^{k+1} r\}, k \geq 1$. Then by Lemma 2.3 and Lemma 2.5,

$$\begin{aligned}
&|T^*_{\alpha,\beta}(f_2)(x)| \leq \int_{(B^*)^c} |K^*(x,z)|V(z)^\alpha |f(z)|dz \\
&\lesssim \int_{(B^*)^c} \frac{V(z)^\alpha}{\left(1 + \frac{|x-z|}{\rho(x_0)}\right)^N} \frac{|f(z)|}{|x-z|^{n-2\beta+1}} dz \\
&\quad + \int_{(B^*)^c} \frac{V(z)^\alpha}{\left(1 + \frac{|x-z|}{\rho(x_0)}\right)^N} \frac{|f(z)|}{|x-z|^{n-2\beta}} \int_{B(z,|x-z|/4)} \frac{V(\xi)}{|\xi - z|^{n-1}} d\xi dz \\
&\lesssim \sum_{k=1}^\infty \frac{1}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^N} \frac{1}{(2^k r)^{n-2\beta+1}} \int_{2^k B} V(z)^\alpha |f(z)|dz \\
&\quad + \sum_{k=1}^\infty \frac{1}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^N} \frac{1}{(2^k r)^{n-2\beta}} \int_{2^k B} V(z)^\alpha |f(z)|\mathcal{I}_1(V\chi_{2^{k+1}B})(z)dz \\
&= J_1 + J_2.
\end{aligned}$$

Observe that $\frac{1}{p'_\alpha} + \frac{\alpha}{q} + \frac{1}{t} = 1, \frac{1}{t} = \frac{1}{q} - \frac{1}{n}$, by Hölder inequality and the boundedness of fractional integral $\mathcal{I}_1 : L^q \to L^t$ we get

$$\begin{aligned}
&\frac{1}{(2^k r)^n}\int_{2^k B} V(z)^\alpha |f(z)|\mathcal{I}_1(V\chi_{2^{k+1}B})(z)dz \\
&\lesssim \left(\frac{1}{(2^k r)^n}\int_{2^k B} V(z)^q dz\right)^{\alpha/q} \left(\frac{1}{(2^k r)^n}\int_{2^k B} |f(z)|^{p'_\alpha} dz\right)^{1/p'_\alpha} \\
&\quad \times \left(\frac{1}{(2^k r)^n}\int_{2^k B} |\mathcal{I}_1(V\chi_{2^{k+1}B})(z)|^t dz\right)^{1/t} \\
&\lesssim \left(\frac{1}{(2^k r)^n}\int_{2^k B} V(z)dz\right)^{\alpha} \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \\
&\quad \times \left(\frac{1}{(2^k r)^n}\int_{2^k B} V(z)^q dz\right)^{1/q} (2^k r)^{\frac{n}{q} - \frac{n}{t}} \\
&\lesssim (2^k r)^{\frac{n}{q} - \frac{n}{t} - 2(\alpha+1)} \left(1 + \frac{2^k r}{\rho(x_0)}\right)^{l_0(\alpha+1)} \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.
\end{aligned}$$

Due to $2\beta - 2(\alpha+1) + \frac{n}{s} - \frac{n}{t} = 0$, taking $N > l_0(\alpha+1)$ we get

$$J_2 \lesssim \sum_{k=1}^{\infty} \frac{1}{(2^k)^{N-l_0(\alpha+1)}} \|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

By Hölder inequality,

$$\begin{aligned}
&\frac{1}{(2^k r)^n} \int_{2^k B} V(z)^{\alpha} |f(z)| dz \\
\lesssim &\left(\frac{1}{(2^k r)^n} \int_{2^k B} V(z)^q dz \right)^{\alpha/q} \left(\frac{1}{(2^k r)^n} \int_{2^k B} |f(z)|^{p'_\alpha} dz \right)^{1/p'_\alpha} \\
\lesssim &\left(\frac{1}{(2^k r)^n} \int_{2^k B} V(z) dz \right)^{\alpha} \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \\
\lesssim &(2^k r)^{-2\alpha} \left(1 + \frac{2^k r}{\rho(x_0)} \right)^{\alpha l_0} \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.
\end{aligned}$$

Then

$$\begin{aligned}
J_1 &= \sum_{k=1}^{\infty} \frac{(2^k r)^{2\beta-1}}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^N} \frac{1}{(2^k r)^n} \int_{2^k B} V(z)^{\alpha} |f(z)| dz \\
&\lesssim \sum_{k=1}^{\infty} \frac{(2^k r)^{2\beta-1-2\alpha}}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^{N-\alpha l_0}} \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{2^{k(N-l_0\alpha)}} \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.
\end{aligned}$$

Thus, for any $x \in B(x_0, r)$, we get

$$|T^*_{\alpha,\beta}(f_2)(x)| \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

Consequently,

$$\frac{1}{|B|} \int_B |T^*_{\alpha,\beta}(f_2)(x)| dx \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

Let us consider the case $r < \rho(x_0)$. We set $B^\sharp = B(x_0, 2\rho(x_0))$ and write

$$f = f\chi_{B^\sharp} + f\chi_{(B^\sharp)^c} = f_1^\sharp + f_2^\sharp.$$

Similar to the estimates for $|T^*_{\alpha,\beta} f_2(x)|$, we have

$$|T^*_{\alpha,\beta}(f_2^\sharp)(x)| \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

Then

$$\frac{1}{|B|} \int_B |T^*_{\alpha,\beta}(f_2^\sharp)(x) - (T^*_{\alpha,\beta}(f_2^\sharp))_B| dx \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

For any $x \in B(x_0, r)$, let $B_{x,k} = B(x, 2^{2-k}\rho(x_0))$. It is obvious that

$$f(B_{x,k}, V) = 0$$

for $k=0,1,2$. Notice

$$|f(B_{x,3},V)-f(B_{x,2},V)|=|f(B_{x,3},V)|$$
$$\lesssim \frac{1}{|B(x,\rho(x_0))|}\int_{B(x,\rho(x_0))}|f(z)|dz \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

So for $k=3,4,\cdots$, we have

$$|f(B_{x,k},V)-f(B_{x,k-1},V)|\lesssim \|f\|_{BMO(\mathbb{R}^n)}\lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

Then, for $k=3,4,\cdots$, we get

$$|f(B_{x,k},V)|\lesssim k\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

Hence, by Lemma 2.6, for any $p\geq 1$ and $k=0,1,2,\cdots$, we get

$$\left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}}|f(z)|^p dz\right)^{1/p}$$
$$\lesssim \left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}}|f(z)-f(B_{x,k},V)|^p dz\right)^{1/p}+|f(B_{x,k},V)|$$
$$\lesssim k\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

For any $x\in B$,

$$|T^*_{\alpha,\beta}(f_1^\sharp)(x)|\lesssim \int_{B^\sharp}|K^*(x,z)|V(z)^\alpha|f(z)|dz$$
$$\lesssim \sum_{k=0}^\infty \int_{B_{x,k}\setminus B_{x,k+1}}\frac{V(z)^\alpha}{\left(1+\frac{|x-z|}{\rho(x_0)}\right)^N}\frac{|f(z)|}{|x-z|^{n-2\beta+1}}dz$$
$$+\sum_{k=0}^\infty \int_{B_{x,k}\setminus B_{x,k+1}}\frac{V(z)^\alpha}{\left(1+\frac{|x-z|}{\rho(x_0)}\right)^N}\frac{|f(z)|}{|x-z|^{n-2\beta}}\int_{B(z,|x-z|/4)}\frac{V(\xi)}{|\xi-z|^{n-1}}d\xi dz$$
$$\lesssim \sum_{k=0}^\infty \frac{(2^{2-k}\rho(x_0))^{2\beta-1}}{|B_{x,k}|}\int_{B_{x,k}}V(z)^\alpha|f(z)|dz$$
$$+\sum_{k=0}^\infty \frac{(2^{2-k}\rho(x_0))^{2\beta}}{|B_{x,k}|}\int_{B_{x,k}}V(z)^\alpha|f(z)|\mathcal{I}_1(V\chi_{B_{x,k-1}})(z)dz$$
$$=K_1+K_2.$$

Notice $0\leq\alpha\leq 1<q$. By Hölder inequality and $\beta-\alpha=1/2$, we get

$$K_1\lesssim \sum_{k=0}^\infty (2^{2-k}\rho(x_0))^{2\alpha}\frac{1}{|B_{x,k}|}\int_{B_{x,k}}V(y)^\alpha|f(y)|dy$$
$$\lesssim \sum_{k=0}^\infty (2^{2-k}\rho(x_0))^{2\alpha}\left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}}V(y)^q dy\right)^{\alpha/q}\left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}}|f(y)|^{(\frac{s}{\alpha})'}dy\right)^{1/(\frac{s}{\alpha})'}$$
$$\lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}\sum_{k=0}^\infty (k+1)\left(\frac{1}{(2^{2-k}\rho(x_0))^{n-2}}\int_{B_{x,k}}V(y)dy\right)^\alpha.$$

It follows from Lemma 2.2 that

$$\frac{1}{(2^{2-k}\rho(x_0))^{n-2}}\int_{B_{x,k}} V(y)dy \lesssim 2^{-k\delta}$$

for $k=3,\cdots$, where $\delta = 2-n/q>0$, and from Lemma 2.1 that

$$\frac{1}{(2^{2-k}\rho(x_0))^{n-2}}\int_{B_{x,k}} V(y)dy \lesssim 1$$

for $k=0,1,2$. Then $K_1 \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}$.

Pay attention to $\frac{1}{p'_\alpha}+\frac{\alpha}{q}+\frac{1}{t}=1, \frac{1}{t}=\frac{1}{q}-\frac{1}{n}$. By Hölder inequality and the boundedness of $\mathcal{I}_1$ we get

$$\begin{aligned}
&\frac{1}{|B_{x,k}|}\int_{B_{x,k}} V(z)^\alpha |f(z)| \mathcal{I}_1(V\chi_{B_{x,k-1}})(z)dz \\
&\lesssim \left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}} V(z)^q dz\right)^{\alpha/q}\left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}} |f(z)|^{p'_\alpha} dz\right)^{1/p'_\alpha} \\
&\quad\times\left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}} |\mathcal{I}_1(V\chi_{B_{x,k-1}})(z)|^t dz\right)^{1/t} \\
&\lesssim k\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}\left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}} V(z)dz\right)^{\alpha+1}|B_{x,k}|^{\frac{1}{s}-\frac{1}{t}} \\
&\lesssim k\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}(2^{2-k}\rho(x_0))^{-2(\alpha+1)}\left(\frac{1}{(2^{2-k}\rho(x_0))^{n-2}}\int_{B_{x,k}} V(z)dz\right)^{\alpha+1}
\end{aligned}$$

Then

$$K_2 \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}\sum_{k=0}^{\infty}(k+1)2^{-k\delta 2(\alpha+1)} \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

Combining the estimates for K_1 and K_2, we have proved the inequality

$$|T^*_{\alpha,\beta}(f_1^\sharp)(x)| \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}$$

for any $x\in B(x_0,r), r<\rho(x_0)$. Thence

$$\frac{1}{|B|}\int_B |T^*_{\alpha,\beta}(f_1^\sharp)(x) - (T^*_{\alpha,\beta}(f_1^\sharp))_B| dx \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 1.2.

Список литературы

[1] J. Dziubański, G. Garrigos, T Martinez, JL Torrea and J. Zienkiewicz, “BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality”, Math. Z. **249**, no.2, 329 – 352 (2005).

[2] J. F. Dong, J. Z. Huang and Y. Liu, “Boundedness of singular integrals on Hardy type spaces Associated with Schrödinger operators”, J. Funct. Spaces **2015,** Article ID 409215.

[3] J. F. Dong and H. P. Liu, “The BMO_L space and Riesz transforms associated with Schrödinger operators”, Acta Math. Sinica (Engl. Ser.), **26**, 659 – 668 (2010).

[4] J. F. Dong and Y. Liu, “The higher order Riesze transform and BMO type space associated to Schrödinger operators”, Math. Nachr., **285**, no. 4, 486 – 496 (2012).

[5] J. Dziubański and J. Zienkiewicz, “Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality”, Rev. Mat. Iberoamericana, **15**, no. 2, 279 – 296 (1999).
[6] J. Dziubański and J. Zienkiewicz, “A characterization of Hardy spaces associated with certain Schrödinger operators”, Potential Anal., **41**, 917 – 930 (2014).
[7] Z. H. Guo, P. T Li and L. Z. Peng, “L^p boundedness of commutators of Riesz transforms associated to Schrödinger operator”, J. Math. Anal. Appl., **341**, no. 1, 421 – 432 (2008).
[8] Y. Hu and Y. S. Wang, “Hardy type estimates for Riesz transforms associated with Schrödinger operators”, Anal. Math. Phys., **9**, no. 1, 275 – 287 (2019).
[9] Y. Hu and Y. S. Wang, “Estimates for the commutators of operator $V^\alpha\nabla(-\Delta+V)^{-\beta}$”, J. Inequal. Appl., no. 126, 18 pp. (2019).
[10] P. T. Li and L. Z. Peng, “Endpoint estimates for commutators of Riesz transform associated with Schrödinger operators”, Bull. Aust. Math. Soc., **82**, no. 3, 367 – 389 (2010).
[11] Y. Liu, “Some estimates for the Schrödinger type operators with nonnegative potentials”, Arch. Math., **98**, 467 – 475 (2012).
[12] Y. Liu and S. Qi, “Endpoint estimates of Riesz transforms associated with generalized Schrödinger operators”, Can. Math. Bull., **61**, no.4, 787 – 801 (2018).
[13] Y. Liu and J. L. Sheng, “Some estimates for commutators of Riesz transforms associated with Schrödinger operators”, J. Math. Anal. Appl., **419**, no. 1, 298 – 328 (2014).
[14] Y. Liu, J. L. Sheng, and L. J. Wang, “Weighted endpoint estimates for commutators of Riesz transforms associated with Schrödinger operators”, Abstr. Appl. Anal., Paper No. 281562, 9 pp. (2013).
[15] P. T. Li and X. Wan, “The boundedness of commutators associated with Schrödinger operators on Herz spaces”, J. Inequal. Appl., Paper No. 172, 27 pp. (2016).
[16] P. T. Li, X. Wan and C. Y. Zhang, “Schrödinger type operators on generalized Morrey spaces”, J. Inequal. Appl., no. 229, 21 pp. (2015).
[17] B. Munckenhoupt, “Weighted norm inequality for the Hardy maximal function”, Trans. Amer. Math. Soc., **165**, 207 – 226 (1972).
[18] Z. W. Shen, “L^p estimates for Schrödinger operators with certain potentials”, Ann. Inst. Fourier (Grenoble), **45**, no. 2, 513 – 546 (1995).
[19] S. Sugano, “Estimates for the operators $V^\alpha(-\Delta+V)^{-\beta}$ and $V^\alpha\nabla(-\Delta+V)^{-\beta}$ with certain non-negative potentials V”, Tokyo J. Math., **21**, no. 2, 441 – 452 (1998).
[20] Y. H. Wang, “Estimates for the higher order Riesz transforms related to Schrödinger type operators”, Bull. Korean Math. Soc., **58**, no. 1, 235 – 251 (2021).
[21] Y. H. Wang, “Some estimates of Riesz transforms related to Schrödinger-type operators”, Bull. Malays. Math. Sci. Soc. **44**, no.5, 3161 – 3174 (2021).
[22] Y. H. Wang, “Endpoint estimates for commutators of Riesz transforms related to Schrödinger operators”, Indian J. Pure Appl. Math. **52**, no.2, 449 – 458 (2021).
[23] Y. S. Wang, “Estimates for the operators $V^\alpha(-\Delta+V)^{-\beta}$ with nonnegative potentials V,”, Anal. Math. Phys., **11**, 101 (2021).

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