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ON SHARED-VALUE PROPERTIES OF $f'(z) = f(z+c)$

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Abstract. This research is a continuation of the recent papers [20, 21]. In this paper, we deal with the uniqueness problems on the derivative of $f(z)$ with its shift $f(z+c)$, and give a new perspective on discussing the complex differential-difference equation $f'(z) = f(z+c)$.

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1. Introduction

It is well known that Nevanlinna theory has a wide range of applications in considering the value distribution of meromorphic solutions of complex differential equations. In addition, with the difference correspondence of the logarithmic derivative lemma obtained by Chiang-Feng [3], and Halburd-Korhonen [7] respectively, the complex domain differences and the complex difference equations also developed rapidly. The related results, readers can refer to [2].

Although the research of complex differential-difference equations can be traced back to Naftalevich's work in [5, 16, 17], the investigations on complex differential-difference field using Nevanlinna theory are still very few. Therefore, the relevant results are very limited, the reader is invited to see [11, 12, 14, 15, 19, 22].

In comparison, in real analysis, the researches on differential-difference equations are too numerous to enumerate. For example, there are extensive studies on the delay equations $f'(x) = f(x-k)$, $(k > 0)$ in real analysis. The related results can be found in [1]. Inspired by such results, Liu and Dong [13] discussed the properties of the solutions of complex differential-difference equations $f'(z) = f(z+c)$. Recently, we looked at this equation from another point of view, that is, "under what sharing value conditions, does $f'(z) = f(z+c)$ hold?" And in [20], we obtained

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Theorem A. *Let $f(z)$ be a transcendental entire function of finite order, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share 0, a CM, then $f'(z) = f(z+c)$.*

Here, we pose a list of questions related to Theorem A. These questions will be considered in the following.

1. If the condition "$f'(z)$ and $f(z+c)$ share 0, a CM" is changed to "$f'(z)$ and $f(z+c)$ share two distinct values a, b CM", is Theorem A still true?

2. Can value sharing condition or the restriction on the order of $f(z)$ be improved in Theorem A?

Remark. In fact, the solutions of $f'(z) = f(z+c)$ must be transcendental entire functions. Otherwise, suppose that z_0 is a pole of $f(z)$, then from $f'(z) = f(z+c)$, we know $z_0 + nc$ are poles of $f(z)$ also. Hence, $f(z)$ must have infinitely many poles. If m is the minimum order of all poles of $f(z)$, then m is the minimum order of all poles of $f(z+c)$ as well. However, the minimum order of all poles of $f'(z)$ is $1+m$, which contradicts $f'(z) = f(z+c)$. Hence, we just need to consider the condition that $f(z)$ is a transcendental entire function in the following.

In this paper, we will continue to consider the uniqueness problem for the derivative of $f(z)$ with its shift $f(z+c)$. The reminder of this paper is organized as follows: In Section 2, for Question 1, we will give a positive answer by giving Theorem 2.1. In Section 3, we will give two uniqueness results for $f'(z)$ sharing one value with $f(z+c)$, under some appropriate deficiency assumptions.

2. Functions share two values CM

Theorem 2.1. *Let $f(z)$ be a transcendental entire function of hyper-order strictly less than 1. If $f'(z)$ and $f(z+c)$ share two distinct values a, b CM, then $f'(z) = f(z+c)$.*

The following lemma plays a key role in proving Theorem 2.1.

Lemma 2.1. [10, Theorem 1] *Suppose that $f(z)$ and $g(z)$ are two distinct non-constant entire functions. If $f(z)$ and $g(z)$ share the values 0 and 1 CM, then they assume one of the following cases:*

(1) $f(z) = d(1-e^{A(z)})$, $g(z) = (1-d)(1-e^{-A(z)})$;
(2) $f(z) = e^{-nA(z)} \sum_{j=0}^{n} e^{jA(z)}$, $g(z) = \sum_{j=0}^{n} e^{jA(z)}$, $n = 1, 2, \ldots$;
(3) $f(z) = -e^{-(n+1)A(z)} \sum_{j=0}^{n} e^{jA(z)}$, $g(z) = -e^{A(z)} \sum_{j=0}^{n} e^{jA(z)}$, $n = 0, 1, 2, \ldots$,
where $d(\neq 0, 1)$ is a constant, and $A(z)$ is a non-constant entire function.

Lemma 2.2. [23, Theorem 1.51] *Suppose that $f_j(z)$ $(j = 1, \dots n)$ $(n \geq 2)$ are meromorphic functions and $g_j(z)$ $(j = 1, \dots, n)$ are entire functions satisfying the following conditions.*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} = 0$.

(2) $1 \leq j < k \leq n$, *$g_j(z) - g_k(z)$ are not constants for* $1 \leq j < k \leq n$.

(3) *For* $1 \leq j \leq n$, $1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\}, \quad r \to \infty, r \notin E,$$

where $E \subset (1, \infty)$ is of finite linear measure.

Then $f_j(z) = 0$.

Lemma 2.3. [2, Theorem 1.3] *Let $h_2(z) \not\equiv 0$, $h_1(z), F(z)$ be polynomials, $c_2, c_1(\neq c_2)$ be constants. Suppose that $f(z)$ is a transcendental meromorphic solution of difference equation*

$$h_2(z)f(z + c_2) + h_1(z)f(z + c_1) = F(z).$$

Then, $\rho(f) \geq 1$, where $\rho(f)$ is the order of $f(z)$.

Lemma 2.4. [23, Lemma 5.1] *Let $f(z)$ be a non-constant periodic meromorphic function. Then, $\rho(f) \geq 1$.*

Proof of Theorem 2.1. Suppose that $f'(z) \not\equiv f(z + c)$. Set

$$F(z) = \frac{f'(z) - a}{b - a}, \quad G(z) = \frac{f(z + c) - a}{b - a}. \tag{2.1}$$

Then, from the value sharing assumption and Lemma 2.1, one of the following cases holds:

Case 1. If

$$f'(z) = (b - a)d(1 - e^{A(z)}) + a \tag{2.2}$$

and

$$f(z + c) = (b - a)(1 - d)(1 - e^{-A(z)}) + a. \tag{2.3}$$

Here and below, $A(z)$ is a non-constant entire function of order less than 1. Then, (2.2) and (2.3) give

$$de^{A(z+c)} + (1 - d)A'e^{-A(z)} - (\frac{a}{b - a} + d) = 0. \tag{2.4}$$

Subcase 1.1. If $A(z)$ is a non-constant polynomial, then we have $A(z)$, $A(z+c)$ and $A(z + c) + A(z)$ are non-constant polynomials. Applying Lemma 2.2 to (2.4), we have a contradiction.

Subcase 1.2. If $A(z)$ is a transcendental entire function of order less than 1. Then, we confirm that $A(z+c)+A(z)$ must be transcendental. Otherwise, we suppose that

$A(z+c)+A(z)$ is a polynomial, then from Lemma 2.3, we deduce that $\rho(A)\geq 1$, which is a contradiction. Further, applying Lemma 2.2 to (2.4) again, we obtain a contradiction as well.

Case 2. If

$$f'(z)=(b-a)(1+e^{-A}+e^{-2A}+\cdots+e^{-nA})+a \tag{2.5}$$

and

$$f(z+c)=(b-a)(1+e^{A}+e^{2A}+\cdots+e^{nA})+a. \tag{2.6}$$

Then, combining (2.5) and (2.6), we have

$$nA'e^{nA}+\cdots 2A'e^{2A}+A'e^{A}-\frac{b}{b-a}-e^{-A(z+c)}-e^{-2A(z+c)}-\cdots-e^{-nA(z+c)}=0. \tag{2.7}$$

Subcase 2.1. If $A(z)$ is a non-constant polynomial, then we obtain that $sA(z)+tA(z+c)$ is a non-constant polynomial, where s, $t(\neq -s)$ are two integers such that $s^2+t^2\neq 0$. Hence, by Lemma 2.2 and (2.7), we have a contradiction.

Subcase 2.2. If $A(z)$ is a transcendental entire function of order less than 1. Then, using the same way of Subcase 1.2, we have $\lambda A(z)+\mu A(z+c)$ must be transcendental, where λ, μ are two integers such that $\lambda^2+\mu^2\neq 0$. Hence, applying Lemma 2.2 to (2.7), we obtian a contradiction.

Case 3. If

$$f'(z)=(a-b)(e^{-A}+e^{-2A}+\cdots+e^{-(n+1)A})+a, \tag{2.8}$$

and

$$f(z+c)=(a-b)(e^{A}+e^{2A}+\cdots+e^{(n+1)A})+a. \tag{2.9}$$

Then, by (2.8) and (2.9), it follows that

$$\begin{aligned}(n+1)A'e^{(n+1)A}+\cdots+2A'e^{2A}+A'e^{A}+\frac{a}{b-a}\\ -e^{-A(z+c)}-e^{-2A(z+c)}+\cdots-e^{-(n+1)A(z+c)}=0,\end{aligned} \tag{2.10}$$

and as in Case 2, we get a contradiction. Therefore, $f'(z)=f(z+c)$.

Remark. From the proof of the Theorem 2.1, we can find that Lemma 2.1 can make our proof of Theorem 2.1 very simple. However, without the application of Lemma 2.1, our proof will be very cumbersome. In fact, we have already given a complicated proof before. In addition, using Lemma 2.1, we can not only give a very simple proof of Theorem B [25, Theorem 1.1], but also improve Theorem B.

Theorem B. *Let $f(z)$ be a transcendental entire function of finite order and a, b be two distinct constants. If $\triangle f(z)=f(z+c)-f(z)(\not\equiv 0)$ and $f(z)$ share a, b CM, then $\triangle f(z)=f(z)$.*

In fact, we have

Theorem 2.2. *Let $f(z)$ be a transcendental entire function of hyper-order strictly less than 1, and let a, b be two distinct constants. If $\triangle f(z)(\not\equiv 0)$ and $f(z)$ share a, b CM, then $\triangle f(z) = f(z)$.*

The proof of Theorem 2.2 is similar to the proof of Theorem 2.1. For the convenience of the reader, we will give a brief proof here.

Proof of Theorem 2.2. Similarly as in Theorem 2.1, if $f(z) \not\equiv \triangle f(z)$, then we have three possibilities:

Case 1.

$$(1-d)e^{A(z+c)} - de^{A} - (1-d)e^{-A} + d + \frac{a}{b-a} = 0. \tag{2.11}$$

Case 2.

$$\begin{aligned} &e^{nA} + e^{(n-1)A} + \cdots e^{A} + \frac{b}{b-a} + e^{-A} + e^{-2A} + \cdots + e^{-nA} \\ &- e^{-A(z+c)} + e^{-2A(z+c)} + \cdots + e^{-nA(z+c)} = 0. \end{aligned} \tag{2.12}$$

Case 3.

$$\begin{aligned} &e^{(n+1)A} + e^{nA} + \cdots e^{A} - \frac{a}{b-a} + e^{-A} + e^{-2A} + \cdots + e^{-(n+1)A} \\ &- e^{-A(z+c)} + e^{-2A(z+c)} + \cdots + e^{-(n+1)A(z+c)} = 0. \end{aligned} \tag{2.13}$$

The only difference the proof of Theorem 2.1 is that, we need to prove one more case: $A(z+c) - A(z)$ **is not a constant, when** $A(z)$ **is a non-constant polynomial**. Here, we only prove the Case 1, as for the Cases 2 and 3, we can prove similarly.

Otherwise, we suppose $A(z+1) - A(z) = \alpha$, where α is a constant. Then, from Lemma 2.4, we have $\alpha \neq 0$. Further, we have

$$A(z) = \alpha z + \beta, \tag{2.14}$$

where β is a constant. Substituting (2.14) into (2.11), it follows that

$$((1-d)e^{\alpha c+\beta} - de^{\beta})e^{\alpha z} + d + \frac{a}{b-a} - (1-d)e^{-\beta}e^{-\alpha z} = 0. \tag{2.15}$$

Applying Lemma 2.2 to (2.15), we get a contradiction. Thus, $A(z+c) - A(z)$ is not a constant.

3. Functions share one value CM or IM

First of all, let's give the definitions that we need in the following proof.

Definitions. Suppose that z is a zero of $F-1$ with multiplicity m, meanwhile, a zero of $G-1$ with multiplicity n. Then, we denote by $N_L(r, \frac{1}{F-1})$ the reduced counting function of those 0-points of $F-1$ when $m > n$; by $N_E^{(2}(r, \frac{1}{F-1})$ the reduced counting function of those 0-points of $F-1$ when $m = n \geq 2$. In addition,

$\overline{N}_{(2}(r, \frac{1}{F})$ is the counting function of zeros of F whose multiplicities are greater than 2, $N_0(r, \frac{1}{F'})$ is the counting function of zeros of F' but not the zeros of F and $F-1$. Notations $N_L(r, \frac{1}{G-1})$, $N_E^{(2}(r, \frac{1}{G-1})$, $\overline{N}_{(2}(r, \frac{1}{G})$ and $N_0(r, \frac{1}{G'})$ can be similarly defined. Moreover, we define $\delta(0, f)$ as following

$$\delta(0, f) = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f})}{T(r, f)}.$$

Since in [21], we have given partial results for cases "1 CM + 1IM"and "2 IM". Hence, in the following, we just give the result of $f'(z)$ share one value with $f(z+c)$, under the deficiency assumption.

Theorem 3.1. *Let $f(z)$ be a transcendental entire function of hyper-order strictly less than 1, and let $a(\not\equiv 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a CM and $\delta(0, f) > \frac{1}{2}$. Then, $f'(z) = f(z+c)$.*

For the sharing assumption "1 IM", we obtain

Theorem 3.2. *Let $f(z)$ be a transcendental entire function of hyper-order strictly less than 1, and let $a(\not\equiv 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a IM and $\delta(0, f) > \frac{4}{5}$. Then, $f'(z) = f(z+c)$.*

In order to prove Theorems 3.1-3.2, we need the following lemmas. From Theorem 5.1 in [8], we can immediately obtain the following result:

Lemma 3.1. *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1. Then,*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Remark. Here and below, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. Meanwhile, by $S_1(r, f)$ we denote any quantity satisfying $S_1(r, f) = o(T(r, f))$ for all r outside of a possible exceptional set of finite linear measure.

From Lemma 8.3 in [8] and Lemma 3.1, we have the following lemma:

Lemma 3.2. [3, Lemma 5.1] *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, then we have*

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

The following result is just a simple modification of the result of meromorphic functions with finite order in Lemma 2.5 [18]:

Lemma 3.3. *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, then*

$$N\left(r, \frac{1}{f(z+c)}\right) \leq N(r, \frac{1}{f}) + S(r,f).$$

Lemma 3.4. [23, Theorem 1.24] *Suppose $f(z)$ is a non-zero meromorphic function in the complex plane and k is a positive integer. Then,*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N(r, \frac{1}{f}) + k\overline{N}(r,f) + S_1(r,f).$$

Lemma 3.5. [24, Lemma 3] *Let $F(z)$ and $G(z)$ be two non-constant meromorphic functions, and let*

$$\Phi(z) = \left(\frac{F''(z)}{F'(z)} - \frac{2F'(z)}{F(z)-1}\right) - \left(\frac{G''(z)}{G'(z)} - \frac{2G'(z)}{G(z)-1}\right). \tag{3.1}$$

If $F(z)$ and $G(z)$ share 1 IM and $\Phi(z) \not\equiv 0$. Then,

$$N_E^{1)}(r, \frac{1}{F-1}) \leq N(r,\Phi) + S_1(r,F) + S_1(r,G), \tag{3.2}$$

where $N_E^{1)}(r, \frac{1}{F-1})$ is the reduced counting function of the common simple zeros of $F-1$ and $G-1$.

Proof of Theorem 3.2. Set

$$F(z) = \frac{f'(z)}{a}, \quad G(z) = \frac{f(z+c)}{a}. \tag{3.3}$$

Then, by the sharing values assumption, we get $F(z)$ and $G(z)$ share 1 IM. Moreover,

$$T(r,F) = T(r,f') + S(r,f) \leq T(r,f) + S(r,f).$$

And Lemma 3.2 gives

$$T(r,G) = T(r, f(z+c)) + S(r,f) = T(r,f) + S(r,f).$$

Hence,

$$S(r,F) = S(r,f), \quad S(r,G) = S(r,f).$$

Further, from Lemma 3.4, it follows that

$$N(r, \frac{1}{F}) \leq N(r, \frac{1}{f'}) + S(r,f) \leq N(r, \frac{1}{f}) + S(r,f). \tag{3.4}$$

And Lemma 3.3 leads to

$$N(r, \frac{1}{G}) \leq N\left(r, \frac{1}{f(z+c)}\right) + S(r,f) \leq N(r, \frac{1}{f}) + S(r,f). \tag{3.5}$$

Let $\Phi(z)$ be given by (3.1). Then, we will discuss two cases as follows.

Case 1. Suppose $\Phi(z) \not\equiv 0$. Then, from (3.1) and the sharing values assumption, we have

$$N(r,\Phi) \le \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + N_L(r,\frac{1}{F-1}) + N_L(r,\frac{1}{G-1}) + N_0(r,\frac{1}{F'}) + N_0(r,\frac{1}{G'}) + S(r,f). \tag{3.6}$$

Moreover, we have

$$\overline{N}(r,\frac{1}{F-1}) = N_E^{1)}(r,\frac{1}{F-1}) + N_E^{(2}(r,\frac{1}{F-1}) + N_L(r,\frac{1}{F-1}) + N_L(r,\frac{1}{G-1}).$$

Noting $F(z)$ and $G(z)$ share 1 IM, and so

$$\overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) = 2N_E^{1)}(r,\frac{1}{F-1}) + 2N_E^{(2}(r,\frac{1}{F-1}) + 2N_L(r,\frac{1}{F-1}) + 2N_L(r,\frac{1}{G-1}). \tag{3.7}$$

Thus, combining (3.2), (3.6) and (3.7) yields

$$\begin{aligned} &\overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) \\ &\le N(r,\Phi) + N_E^{1)}(r,\frac{1}{F-1}) + 2N_E^{(2}(r,\frac{1}{F-1}) \\ &+ 2N_L(r,\frac{1}{F-1}) + 2N_L(r,\frac{1}{G-1}) + S(r,f) \\ &\le \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + 3N_L(r,\frac{1}{F-1}) + 3N_L(r,\frac{1}{G-1}) \\ &+ N_E^{1)}(r,\frac{1}{F-1}) + 2N_E^{(2}(r,\frac{1}{F-1}) + N_0(r,\frac{1}{F'}) + N_0(r,\frac{1}{G'}) + S(r,f). \end{aligned} \tag{3.8}$$

Obviously,

$$\begin{aligned} &N_L(r,\frac{1}{G-1}) + 2N_L(r,\frac{1}{F-1}) + N_E^{1)}(r,\frac{1}{F-1}) + 2N_E^{(2}(r,\frac{1}{F-1}) \\ &\le N(r,\frac{1}{F-1}) \le T(r,F) + S(r,f). \end{aligned}$$

Substituting the above inequality into (3.8) yields

$$\begin{aligned} &\overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) \\ &\le \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + 2N_L(r,\frac{1}{G-1}) + N_L(r,\frac{1}{F-1}) \\ &+ N_0(r,\frac{1}{F'}) + N_0(r,\frac{1}{G'}) + T(r,F) + S(r,f), \end{aligned} \tag{3.9}$$

On the other hand, applying the second main theorem, we derive that

$$\begin{aligned} &T(r,F) + T(r,G) \\ &< \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{G-1}) \\ &- N_0(r,\frac{1}{F'}) - N_0(r,\frac{1}{G'}) + S(r,f). \end{aligned} \tag{3.10}$$

It is easy to see

$$\overline{N}(r, \frac{1}{F}) + \overline{N}_{(2}(r, \frac{1}{F}) \le N(r, \frac{1}{F}), \quad \overline{N}(r, \frac{1}{G}) + \overline{N}_{(2}(r, \frac{1}{G}) \le N(r, \frac{1}{G}). \tag{3.11}$$

Hence, it follows from (3.4), (3.5) and (3.9)–(3.11), that

$$\begin{aligned} T(r, f) &= T(r, G) + S(r, f) \\ &\le N(r, \frac{1}{F}) + N(r, \frac{1}{G}) + 2N_L(r, \frac{1}{G-1}) + N_L(r, \frac{1}{F-1}) + S(r, f) \\ &\le 2N(r, \frac{1}{f}) + 2N_L(r, \frac{1}{G-1}) + N_L(r, \frac{1}{F-1}) + S(r, f). \end{aligned} \tag{3.12}$$

Furthermore, by Lemma 3.4 and (3.4), we obtain

$$N_L(r, \frac{1}{F-1}) \le N(r, \frac{1}{F'}) \le N(r, \frac{1}{F}) + S(r, f) \le N(r, \frac{1}{f}) + S(r, f). \tag{3.13}$$

Similarly, we have

$$N_L(r, \frac{1}{G-1}) \le N(r, \frac{1}{f}) + S(r, f). \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12) yields that

$$T(r, f) \le 5N(r, \frac{1}{f}) + S(r, f),$$

which contradicts the assumption $\delta(0, f) > \frac{4}{5}$.

Case 2. Suppose $\Phi(z) = 0$. Then, integrating twice, it follows from (3.1) that

$$\frac{1}{G-1} = \frac{\alpha}{F-1} + \beta, \tag{3.15}$$

where $\alpha(\neq 0)$ and β are constants. Rewrite (3.15) as

$$F = \frac{(\beta - \alpha)G + (\alpha - \beta - 1)}{\beta G - (\beta + 1)}. \tag{3.16}$$

Subcase 2.1. If $\beta \neq 0, -1$. Then, by (3.16), we have

$$\overline{N}\left(r, \frac{1}{G - \frac{\beta+1}{\beta}}\right) = \overline{N}(r, F).$$

From the second main theorem and (3.5), we obtain

$$\begin{aligned} T(r, f) &= T(r, G) + S(r, f) \\ &\le \overline{N}(r, \frac{1}{G}) + \overline{N}\left(r, \frac{1}{G - \frac{B+1}{B}}\right) + S(r, f) \\ &\le N(r, \frac{1}{G}) + \overline{N}(r, F) + S(r, f) \le N(r, \frac{1}{f}) + S(r, f), \end{aligned} \tag{3.17}$$

which contradicts the assumption $\delta(0, f) > \frac{4}{5}$.

Subcase 2.2. If $\beta = 0$. Then, we rewrite (3.16) as

$$F = \alpha G - (\alpha - 1). \tag{3.18}$$

If $\alpha \neq 1$, then by (3.18), we have

$$\overline{N}\left(r, \frac{1}{G - \frac{\alpha-1}{\alpha}}\right) = \overline{N}(r, \frac{1}{F}).$$

Similarly as Subcase 2.1, we get a contradiction as well.

If $\alpha = 1$, then by (3.18), we have $F = G$. That is, $f'(z) = f(z+c)$.

Subcase 2.3. If $\beta = -1$. Then, (3.16) can be rewritten as

$$F = \frac{(\alpha+1)G - \alpha}{G}. \tag{3.19}$$

If $\alpha \neq -1$, then by (3.19), it follows that

$$\overline{N}\left(r, \frac{1}{G - \frac{\alpha}{\alpha+1}}\right) = \overline{N}(r, \frac{1}{F}),$$

Using the same reasoning as in Subcase 2.1, we also get a contradiction.

If $\alpha = -1$. then (3.19) leads to $FG = 1$, which means that

$$f' f(z+c) = a^2. \tag{3.20}$$

By $f'(z)$ and $f(z+c)$ share ∞ CM and (3.20), we deduce that

$$N\left(r, \frac{1}{f(z+c)}\right) = S(r, f).$$

Moreover, from Lemma 3.1, Lemma on the logarithmic derivative and (3.20), it follows that

$$\begin{aligned} m\left(r, \frac{1}{f(z+c)}\right) &= \frac{1}{2} m\left(r, \frac{1}{f(z+c)^2}\right) + S(r,f) \\ &\leq m\left(r, \frac{f' f(z+c)}{f(z+c)^2}\right) + m\left(r, \frac{1}{f' f(z+c)}\right) + S(r,f) \\ &\leq m\left(r, \frac{f'}{f(z+c)} \frac{f}{f}\right) + m(r, \frac{1}{a^2}) + S(r,f) \\ &\leq m(r, \frac{1}{a^2}) + S(r,f) = S(r,f). \end{aligned}$$

Therefore, by Lemma 3.2, we have

$$T(r,f) = T(r, f(z+c)) + S(r,f) = S(r,f),$$

which is a contradiction.

Proof of Theorem 3.1. Using the same way of Theorem 3.2, we also obtain (3.12), i.e.,

$$T(r,f) \leq 2N(r, \frac{1}{f}) + 2N_L(r, \frac{1}{G-1}) + N_L(r, \frac{1}{F-1}) + S(r,f).$$

From the assumption that $f(z)$ and $f(z+c)$ share a CM, we know that $F(z)$ and $G(z)$ share 1 CM. Thus,

$$2N_L(r, \frac{1}{F-1}) + N_L(r, \frac{1}{G-1}) = 0.$$

And so,

$$T(r,f) \leq 2N(r,\frac{1}{f}) + S(r,f),$$

which contradicts the assumption that $\delta(0,f) > \frac{1}{2}$.

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