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# CAUSALITY BETWEEN STOPPED FILTRATIONS AND SOME APPLICATIONS

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Abstract. In this paper we consider the concept of statistical causality in continuous time between filtrations associated with stopping times, which is based on Granger's definition of causality. Especially, we consider a generalization of a causality relationship "**H** is a cause of **E** within **F**" from fixed to stopping time. Then we apply the given concept of causality to strongly orthogonal stopped martingales. We show the equivalence between the given concept of causality and orthogonality of stopped local martingales, too.

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## 1. Introduction

We consider causality in continuous time which unifies the nonlinear Granger's causality with some related concepts. Here, the concept of causality is analyzed using the tool of conditional independence among the $\sigma$-fields.

The Granger causality is focused on discrete time stochastic processes (time series). But, in many cases, for example in economy and finance, it may be difficult to capture relations of causality in discrete-time model and it may depend on the length of interval between each sampling. So, continuous time models become more and more frequent in econometrics (see, for example, [1] - [6]). In this paper we will consider the continuous time processes. The continuous time framework is fruitful, not only for the internal consistency of economic theories but also for the statistical approach to causality analysis between stochastic processes that rapidly evolve (see [7]).

The paper is organized as follows. After Introduction, in the Section 2 we present a generalization of a causality concept "**H** is a cause of **E** within **F**", which involves prediction in any horizon in continuous time. This concept is based on Granger's definition of causality (see [3]). The concept of causality in continuous time associated with stopping times with some basic properties is introduced in [8]. In this paper

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we give some new properties of causality concept between stopped filtrations and between stopped processes.

The given concept of causality can be connected with the stable subspaces of $H^p$ (see [9]) and with the orthogonality of martingales (see [10]). Also, weak solutions and local weak solutions of the stochastic differential equations driven with semimartingales, as well as solutions of martingale problem can be expressed using the given concept of causality (see [6, 11]). The preservation of the martingale property is directly connected with the concept of causality (see [12]).

The Section 3 and Section 4 contain our main results. The Section 4 relates the given concept of statistical causality in continuous time to the orthogonality of stopped martingales and stopped local martingales. Also, we investigate the case when the processes are stopped by the different stopping times.

Some applications in finance are given in the Section 5. More specifically, we showed that the given concept of causality is strongly connected with the question of locally risk minimization strategy for defaultable claims.

## 2. Preliminaries and notation

Causality is, in any case, a prediction property and the central question is: is it possible to reduce available information in order to predict a given filtration?
A probabilistic model for a time-dependent system is described by $(\Omega, \mathcal{F}, \mathbf{F}, P)$ where $(\Omega, \mathcal{F}, P)$ is a probability space and $\mathbf{F} = \{\mathcal{F}_t,\ t \in I, I \subseteq R^+\}$ is a "framework" filtration that satisfies the usual conditions of right continuity and completeness. $\mathcal{F}_\infty = \bigvee_{t\in I} \mathcal{F}_t$ is the smallest $\sigma$-algebra containing all the $\{\mathcal{F}_t\}$. An analogous notation will be used for filtrations $\mathbf{H} = \{\mathcal{H}_\mathbf{t}\}$, $\mathbf{G} = \{\mathcal{G}_\mathbf{t}\}$ and $\mathbf{E} = \{\mathcal{E}_\mathbf{t}\}$. It is said that the filtration $\mathbf{G}$ is a subfiltration of $\mathbf{H}$ and written as $\mathbf{G} \subseteq \mathbf{H}$, if $\mathcal{G}_t \subseteq \mathcal{H}_t$ for each $t$. Given a stochastic process $X$ we denote by $\{\mathcal{F}_t^X\}$ the smallest $\sigma$-algebra for which all $X_s$ with $s \leqslant t$, are measurable and $\mathbf{F}^X = \{\mathcal{F}_t^X, t \in I\}$ is the natural filtration of $X$. The natural filtration $\mathbf{F}^X$ is the smallest filtration that makes $X$ to be adapted.

The intuitive notion of causality in continuous time formulated in terms of Hilbert spaces is given in [5]. We consider the analogous notion of causality for filtrations using the conditional independence between sub-$\sigma$-algebras of $\mathcal{F}$ (see [13] and [14]).

**Definition 2.1.** (see [2] and [5]) It is said that $\mathbf{H}$ is a cause of $\mathbf{E}$ within $\mathbf{F}$ relative to $P$ (and written as $\mathbf{E} \mid\!\!< \mathbf{H}; \mathbf{F}; P$) if $\mathcal{E}_\infty \subseteq \mathcal{F}_\infty$, $\mathbf{H} \subseteq \mathbf{F}$ and if $\mathcal{E}_\infty$ is conditionally independent of $\{\mathcal{F}_t\}$ given $\{\mathcal{H}_t\}$ for each $t$, i.e. $\mathcal{E}_\infty \perp \mathcal{F}_t | \mathcal{H}_t$ (i.e. $\mathcal{E}_u \perp \mathcal{F}_t | \mathcal{H}_t$ holds for each $t$ and each $u$), or

$$(\forall A \in \mathcal{E}_\infty) \quad P(A|\mathcal{F}_t) = P(A|\mathcal{H}_t). \tag{2.1}$$

Intuitively, $\mathbf{E} \mid\!< \mathbf{H}; \mathbf{F}; P$ means that all information about $\mathcal{E}_\infty$ that gives $\{\mathcal{F}_t\}$ comes via $\{\mathcal{H}_t\}$ for arbitrary $t$; equivalently, $\{\mathcal{H}_t\}$ contains all the information from the $\{\mathcal{F}_t\}$ needed for predicting $\mathcal{E}_\infty$. We can consider subfiltration $\mathbf{H} \subseteq \mathbf{F}$ as a reduced information.

The definition similar to Definition 2.1 was first given in [4]: "It is said that $\mathbf{H}$ entirely causes $\mathbf{E}$ within $\mathbf{F}$ relative to $P$ (and written as $\mathbf{E} \mid\!< \mathbf{H}; \mathbf{F}$;P) if $\mathbf{E} \subseteq \mathbf{F}$, $\mathbf{H} \subseteq \mathbf{F}$ and if $\mathcal{E}_\infty \perp \mathcal{F}_t | \mathcal{H}_t$ for each $t$". Instead of $\mathcal{E}_\infty \subseteq \mathcal{F}_\infty$ this definition contains the condition $\mathbf{E} \subseteq \mathbf{F}$, or equivalently $\mathcal{E}_t \subseteq \mathcal{F}_t$ for each $t$, which does not have intuitive justification. Since the Definition 2.1 is a more general than the definition given in [4], all results related to causality in the sense of the Definition 2.1 will also be true in the sense of the Definition from [4] (pg.3), when we add the condition $\mathbf{E} \subseteq \mathbf{F}$.

It should be mentioned that the definition of causality from [4] is equivalent to definition of strong global noncausality as given in [1]. So, the Definition 2.1 is a generalization of the notion of strong global noncausality. The equivalence between the statistical causality concept and the concept of adapted distribution given by Hoover and Keisler in [15] is proven in [16].

If $\mathbf{H}$ and $\mathbf{F}$ are such that $\mathbf{H} \mid\!< \mathbf{H}; \mathbf{F}; P$ we shall say that $\mathbf{H}$ is its own cause (or, self caused) within $\mathbf{F}$ (compare with [4]). It should be noted that the statement "**H** is its own cause" sometimes occurs as a useful assumption in the theory of martingales and stochastic integration (see [12]). The concept of being "its own cause" is equivalent to the hypothesis $(\mathcal{H})$ introduced in [12]. It also, should be mentioned that the notion of subordination (as introduced in [17]) is equivalent to the notion of being "its own cause"as defined here.

If $\mathbf{H}$ and $\mathbf{F}$ are such that $\mathbf{H} \mid\!< \mathbf{H}; \mathbf{H} \bigvee \mathbf{F}$ (where $\mathbf{H} \bigvee \mathbf{F}$ is a family determined by $(H \bigvee F)_t = H_t \bigvee F_t$), we shall say that $\mathbf{F}$ does not cause $\mathbf{H}$. Now, it is clear that the interpretation of Granger–causality is that $\mathbf{F}$ does not cause $\mathbf{H}$ if $\mathbf{H} \mid\!< \mathbf{H}; \mathbf{H} \bigvee \mathbf{F}$ holds (see [4]). Without difficulty, it can be shown that this term and the term "**F** does not anticipate **H**"(as introduced in [17]) are identical.

These definitions can be applied to stochastic processes if we consider corresponding induced filtrations. For example, $\{\mathcal{F}_t\}$-adapted stochastic process $X_t$ is its own cause if $\{\mathcal{F}_t^X\}$ is its own cause within $\{\mathcal{F}_t\}$ i.e. if

$$\mathbf{F}^X \mid\!< \mathbf{F}^X; \mathbf{F}; P.$$

Process $X$ which is its own cause is completely described by its behavior with respect to its natural filtration $\mathbf{F}^X$ (see [10] ). For example, process $X = \{X_t, t \in I\}$ is a Markov process with respect to the filtration $\mathbf{F} = \{\mathcal{F}_t, t \in I\}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ if and only if $X$ is a Markov process with respect to $\mathbf{F}^X$ and if it is its own cause within $\mathbf{F}$ relative to $P$. As a consequence, Brownian motion $W = \{W_t, t \in I\}$ with respect to the filtration $\mathbf{F} = \{\mathcal{F}_t, t \in I\}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ is its own cause within $\mathbf{F} = \{\mathcal{F}_t, t \in I\}$ relative to probability $P$.

In many situations we observe certain systems up to some random time, for example up to time when something happens for the first time. So, it is natural to consider causality in continuous time which involves stopping times, a class of random variables that plays essential role in the theory of martingales (for details see [18] and [19]).

If $\tau$ is a stopping time with respect to the filtration $\mathbf{F} = \{\mathcal{F}_t\}$, the associated $\sigma$-algebra $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leqslant t\} \in \mathcal{F}_t \text{ for all } t \in \mathbb{R}_+\}$ is a set of events that occur up to time $\tau$. For a process $X$, we set $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$, whenever $\tau(\omega) < +\infty$. We define the stopped process $X^\tau = \{X_{t\wedge\tau}, t \in I\}$ with

$$X_t^\tau(\omega) = X_{t\wedge\tau(\omega)}(\omega) = X_t \chi_{\{t<\tau\}} + X_\tau \chi_{\{t\geqslant\tau\}}.$$

Note that if $X$ is adapted and cadlag and if $\tau$ is a stopping time, then the stopped process $X^\tau$ is adapted, too. The family of $\sigma$-fields $\mathbf{F}^\tau = \{\mathcal{F}_{t\wedge\tau}\}$ is a stopped filtration (for details, see [13]).

The generalization of the Definition 2.1 from fixed to stopping time is introduced in [8].

**Definition 2.2.** ([8]) Let $\mathbf{F} = \{\mathcal{F}_t\}$, $\mathbf{H} = \{\mathcal{H}_t\}$ and $\mathbf{E} = \{\mathcal{E}_t\}, t \in I$, be given filtrations on the probability space $(\Omega, \mathcal{F}, P)$ and let $\tau$ be a stopping time with respect to filtration $\mathbf{E}$. The filtration $\mathbf{H}^\tau$ entirely causes $\mathbf{E}^\tau$ within $\mathbf{F}^\tau$ relative to $P$ (and written as $\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P$) if $\mathbf{E}^\tau \subseteq \mathbf{F}^\tau$, $\mathbf{H}^\tau \subseteq \mathbf{F}^\tau$ and if $\mathcal{E}_\tau$ is conditionally independent of $\{\mathcal{F}_{t\wedge\tau}\}$ given $\{\mathcal{H}_{t\wedge\tau}\}$ for each $t$, i.e. $\mathcal{E}_\tau \perp \mathcal{F}_{t\wedge\tau} \mid \mathcal{H}_{t\wedge\tau}$ for all $t$, or

$$(\forall t \in I)(\forall A \in \mathcal{E}_\tau) \qquad P(A \mid \mathcal{F}_{t\wedge\tau}) = P(A \mid \mathcal{H}_{t\wedge\tau}). \tag{2.2}$$

The concept of causality given in the Definition 2.2 is defined up to some specified stopping time $\tau$. It includes the stopped filtrations. The relation (2.2) does not consider the causality up to infinite horizon, so it does not imply (2.1).

Compared to the Definition 2.1, in the Definition 2.2 we have reduced the amount of information needed for predicting some other filtration.

## 3. Some properties of the stopped causality

Some basic properties of the concept of causality characterized with the stopping time are given in ([8]). We now prove that some new properties holds.

**Theorem 3.1.** *Let* $\mathbf{F} = \{\mathcal{F}_t\}$, $\mathbf{H} = \{\mathcal{H}_t\}$ *and* $\mathbf{E} = \{\mathcal{E}_t\}$ *be given filtrations on the measurable space* $(\Omega, \mathcal{F})$ *and let* $\tau$ *be a stopping time with respect to* $\mathbf{E}$. *Let* $P$ *and* $Q$ *be a probability measures on* $\mathcal{F}$ *satisfying* $Q \ll P$ *with* $\dfrac{dQ}{dP}$ *as* $(\mathcal{E}_\tau)$*-measurable. Then*

$$\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P \quad \textit{implies} \quad \mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; Q.$$

**Proof.** Let $E_P$ and $E_Q$ be a conditional expectation for the measures $P$ and $Q$ respectively, and $I$ parametric set. Since $Q \ll P$, the right regular version of the density process $L_{t\wedge\tau}$, where

$$L_{t\wedge\tau} = L_t \chi_{\{t<\tau\}} + L_\tau \chi_{\{t\geqslant\tau\}}$$

is the cadlag modification of $E_P(L_\infty \mid \mathcal{F}_{t\wedge\tau})$ where $L_\infty = \dfrac{dQ}{dP}$ is the Radon-Nykodim derivative. Obviously, since $L_\infty$ is $\mathcal{E}_\tau$-measurable, we have

$$L_{t\wedge\tau} = E_P(L_\infty \mid \mathcal{F}_{t\wedge\tau}) = E_P(L_\infty \mid \mathcal{H}_{t\wedge\tau}). \tag{3.1}$$

Let $\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P$ holds. Because of causality, process $M_{t\wedge\tau} = P(A \mid \mathcal{F}_{t\wedge\tau})$, for all $A \in \mathcal{E}_\tau$ is $(\mathcal{H}_{t\wedge\tau}, P)$-martingale. Now, we proove that $(ML)^\tau$ is a $(\mathcal{F}_{t\wedge\tau}, P)$-martingale. For $s < t < \tau$ and $F \in \mathcal{F}_{s\wedge\tau}$ we have

$$\begin{aligned}\int_F M_{t\wedge\tau} L_{t\wedge\tau} dP &= \int_F M_{t\wedge\tau} \frac{dQ_t}{dP} dP = \int_F M_{t\wedge\tau} dQ_t = \int_F M_{s\wedge\tau} dQ_s \\ &= \int_F M_{s\wedge\tau} \frac{dQ_s}{dP} dP = \int_F M_{s\wedge\tau} L_{s\wedge\tau} dP,\end{aligned} \tag{3.2}$$

and for $\tau \leqslant t \leqslant r$, by the Optional Sampling Theorem for $F \in \mathcal{F}_{t\wedge\tau}$ and due to (3.2), we have

$$\begin{aligned}\int_F M_{t\wedge\tau} L_{t\wedge\tau} dP &= \int_F M_{t\wedge\tau} L_\tau dP = \int_F M_{t\wedge\tau} L_t dP = \int_F M_{t\wedge\tau} \frac{dQ_t}{dP} dP \\ &= \int_F M_{t\wedge\tau} dQ = \int_F M_{r\wedge\tau} dQ = \int_F M_{r\wedge\tau} L_{r\wedge\tau} dP.\end{aligned}$$

Therefore,

$$M_{t\wedge\tau} L_{t\wedge\tau} = E(M^\tau_\infty L^\tau_\infty \mid \mathcal{F}_{t\wedge\tau}). \tag{3.3}$$

According to Lemma 6.6 in [18], equalities (3.1) and (3.3), $(\forall A \in \mathcal{E}_\tau)$ $(\forall t \in I)$ it follows that

$$\begin{aligned} M_{t\wedge\tau}L_{t\wedge\tau} &= P(A \mid \mathcal{F}_{t\wedge\tau})L_{t\wedge\tau} = E_P(\chi_A \mid \mathcal{F}_{t\wedge\tau})E_P(L_\infty \mid \mathcal{F}_{t\wedge\tau}) \\ &= E_P(\chi_A L_\infty \mid \mathcal{F}_{t\wedge\tau}) = E_P(\chi_A \frac{dQ}{dP} \mid \mathcal{F}_{t\wedge\tau}) = E_Q(\chi_A \mid \mathcal{F}_{t\wedge\tau}) \\ &= Q(A \mid \mathcal{F}_{t\wedge\tau}). \end{aligned}$$

Due to (3.1), for all $A \in \mathcal{E}_\tau$ and $L_\infty = \frac{dQ}{dP}$ we have

$$\begin{aligned} Q(A \mid \mathcal{F}_{t\wedge\tau}) &= E_Q(\chi_A \mid \mathcal{F}_{t\wedge\tau}) = E_P(\chi_A \frac{dQ}{dP} \mid \mathcal{F}_{t\wedge\tau}) \\ &= E_P(\chi_A \mid \mathcal{H}_{t\wedge\tau})E_P(L_\infty \mid \mathcal{H}_{t\wedge\tau}) = E_P(\chi_A L_\infty \mid \mathcal{H}_{t\wedge\tau}) \\ &= E_Q(\chi_A \mid \mathcal{H}_{t\wedge\tau}) = Q(A \mid \mathcal{H}_{t\wedge\tau}). \end{aligned}$$

The result is proved. □

**Theorem 3.2.** *Let* $\mathbf{F} = \{\mathcal{F}_t\}$, $\mathbf{H} = \{\mathcal{H}_t\}$, $\mathbf{E} = \{\mathcal{E}_t\}$ *and* $\mathbf{G} = \{\mathcal{G}_t\}$ *be given filtrations on the probability space* $(\Omega, \mathcal{F}, P)$ *and let* $\tau$ *be a stopping time with respect to* $\mathbf{E}$. *Assume that* $\mathbf{E}^\tau \subseteq \mathbf{F}^\tau$, $\mathbf{H}^\tau \subseteq \mathbf{F}^\tau$ *and* $\mathbf{G}^\tau \subseteq \mathbf{F}^\tau$. *Then, the following assertions hold:*

(i) $\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P \implies \mathbf{E}^\tau \subseteq \mathbf{H}^\tau$,

(ii) $\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P \ \wedge\ \mathbf{E}^\tau \mid\!< \mathbf{G}^\tau; \mathbf{F}^\tau; P \implies \mathbf{E}^\tau \mid\!< (\mathbf{H}^\tau \wedge \mathbf{G}^\tau); \mathbf{F}^\tau; P$.

**Proof.** *(i)* Let the $Y_{t\wedge\tau}$ be (cadlag process) $\{\mathcal{E}_{t\wedge\tau}\}$-measurable. Then $Y_{t\wedge\tau}$ is, also, $\mathcal{E}_\tau$-measurable since $\mathcal{E}_{t\wedge\tau} \subset \mathcal{E}_{\infty\wedge\tau} = \mathcal{E}_\tau$. According to $\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P$ follows that for all $Y_{t\wedge\tau}$

$$E(Y_{t\wedge\tau} \mid \mathcal{F}_{t\wedge\tau}) = E(Y_{t\wedge\tau} \mid \mathcal{H}_{t\wedge\tau}). \tag{3.4}$$

Since $\mathcal{E}_{t\wedge\tau} \subseteq \mathcal{F}_{t\wedge\tau}$, it follows that $Y_{t\wedge\tau}$ is $\{\mathcal{F}_{t\wedge\tau}\}$-measurable, too. According to (3.4) we have $Y_{t\wedge\tau} = E(Y_{t\wedge\tau} \mid \mathcal{H}_{t\wedge\tau})$, so $Y_{t\wedge\tau}$ is $\{\mathcal{H}_{t\wedge\tau}\}$-measuarble for all $t$, and the assertion holds.

*(ii)* Let $\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P$ and $\mathbf{E}^\tau \mid\!< \mathbf{G}^\tau; \mathbf{F}^\tau; P$ hold. From $\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P$ we have $\mathcal{E}_\tau \subseteq \mathcal{F}_\tau$, $\mathbf{H}^\tau \subseteq \mathbf{F}^\tau$ and $\mathcal{E}_\tau \perp \mathcal{F}_{t\wedge\tau} \mid \mathcal{H}_{t\wedge\tau}, t \in I$, i.e. $\forall A \in \mathcal{E}_\tau$

$$P(A \mid \mathcal{H}_{t\wedge\tau}) = P(A \mid \mathcal{F}_{t\wedge\tau}). \tag{3.5}$$

Also, from $\mathbf{E}^\tau \mid\!< \mathbf{G}^\tau; \mathbf{F}^\tau; P$ we have $\mathbf{G}^\tau \subseteq \mathbf{F}^\tau$ and $\mathcal{E}_\tau \perp \mathcal{F}_{t\wedge\tau} \mid \mathcal{G}_{t\wedge\tau}, t \in I$, i.e. for all $A \in \mathcal{E}_\tau$

$$P(A \mid \mathcal{G}_{t\wedge\tau}) = P(A \mid \mathcal{F}_{t\wedge\tau}). \tag{3.6}$$

The intersection of two $\sigma$-algebras is also $\sigma$-algebra, so $\mathcal{H}_{t\wedge\tau} \cap \mathcal{G}_{t\wedge\tau} = \mathcal{H}_{t\wedge\tau} \wedge \mathcal{G}_{t\wedge\tau}$. Therefore, because of (3.5) and (3.6), for all $A \in \mathcal{E}_\tau$ we have

$$\begin{aligned} P(A \mid \mathcal{H}_{t\wedge\tau} \cap \mathcal{G}_{t\wedge\tau}) &= E(\chi_A \mid \mathcal{H}_{t\wedge\tau} \cap \mathcal{G}_{t\wedge\tau}) = E(E(\chi_A \mid \mathcal{H}_{t\wedge\tau}) \mid \mathcal{G}_{t\wedge\tau}) \\ &= E(E(\chi_A \mid \mathcal{F}_{t\wedge\tau}) \mid \mathcal{G}_{t\wedge\tau}) = E(\chi_A \mid \mathcal{F}_{t\wedge\tau} \cap \mathcal{G}_{t\wedge\tau}) \\ &= E(\chi_A \mid \mathcal{G}_{t\wedge\tau}) = E(\chi_A \mid \mathcal{F}_{t\wedge\tau}) = P(A \mid \mathcal{F}_{t\wedge\tau}) \end{aligned}$$

so, it follows that $\mathbf{E}^\tau \,|\!\!< (\mathbf{H}^\tau \wedge \mathbf{G}^\tau); \mathbf{F}^\tau; P$ holds. □

**Lemma 3.1.** *If $\mathcal{E}_\tau \subseteq \mathcal{H}_\tau$ holds, then from $\mathbf{H}^\tau \,|\!\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P$ it follows that $\mathbf{E}^\tau \,|\!\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P$ holds.*

**Proof.** The result follows directly from $\mathcal{E}_\tau \subseteq \mathcal{H}_\tau$ and $\mathbf{H}^\tau \,|\!\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P$, since (for all $A \in \mathcal{E}_\tau$) $P(A \mid \mathcal{F}_{t\wedge\tau}) = P(A \mid H_{t\wedge\tau})$. □

## 4. Causality and orthogonality of stopped martingales

The orthogonality of local martingales is considered in [10]. We now consider the orthogonality of stopped martingales and stopped local martingales in the sense of the Definition 2.2. Also, we consider the case when the processes are stopped by the different stopping times.

Let us briefly recall some basics about orthogonal martingales and properties which will be used later (see [19, 20, 21]).

**Definition 4.1.** ([19]) Two martingales $X$ and $Y$ are said to be weakly orthogonal if $E(X_\infty Y_\infty) = 0$.

**Definition 4.2.** ([19]) Two martingales $X$ and $Y$ are said to be strongly orthogonal if $XY$ is a martingale.

If $X$ and $Y$ are strongly orthogonal martingales they are weakly orthogonal, too. However, the converse is not true.

The definition of orthogonal local martingales is slightly different.

**Definition 4.3.** ([21]) Two local martingales $X$ and $Y$ are called orthogonal if their product $XY$ is a local martingale.

The equivalence between the concept of causality from the Definition 2.2 and strongly orthogonal stopped martingales is given in the following theorem.

**Theorem 4.1.** *Let $\tau$ be $\{\mathcal{F}_t^X\}$ and $\{\mathcal{F}_t^Y\}$-stopping time and let $X^\tau = X_{t\wedge\tau}$ and $Y^\tau = Y_{t\wedge\tau}$ be two independent $\mathbf{F}^\tau = \{\mathcal{F}_{t\wedge\tau}\}$-stopped martingales. Processes $X^\tau$ and $Y^\tau$ are strongly orthogonal if and only if each of them is its own cause within $\{\mathcal{F}_{t\wedge\tau}\}$, i.e. if $\mathbf{F}^{X^\tau} \,|\!\!< \mathbf{F}^{X^\tau}; \mathbf{F}^\tau; P$ and $\mathbf{F}^{Y^\tau} \,|\!\!< \mathbf{F}^{Y^\tau}; \mathbf{F}^\tau; P$ hold.*

**Proof.** Let $X_{t\wedge\tau}$ and $Y_{t\wedge\tau}$ be two strongly orthogonal and independent $\{\mathcal{F}_{t\wedge\tau}\}$ stopped martingales. Then $(XY)^\tau = (XY)_{t\wedge\tau}$ is a stopped martingale, too.

According to Theorem 6 in [8], each of the processes $X^\tau = X_{t\wedge\tau}$ and $Y^\tau = Y_{t\wedge\tau}$ is its own cause within $\mathbf{F}^\tau = \{\mathcal{F}_{t\wedge\tau}\}$, i.e.

$$\mathbf{F}^{X^\tau} \ltimes \mathbf{F}^{X^\tau}; \mathbf{F}^\tau; P \quad \text{and} \quad \mathbf{F}^{Y^\tau} \ltimes \mathbf{F}^{Y^\tau}; \mathbf{F}^\tau; P$$

hold.

Conversely, let the processes $X^\tau$ and $Y^\tau$ be its own cause within $\{\mathcal{F}_{t\wedge\tau}\}$, i.e.

$$\forall A \in \mathcal{F}^X_\tau \qquad P(A \mid \mathcal{F}^X_{t\wedge\tau}) = P(A \mid \mathcal{F}_{t\wedge\tau}), \tag{4.1}$$

$$\forall B \in \mathcal{F}^Y_\tau \qquad P(B \mid \mathcal{F}^Y_{t\wedge\tau}) = P(B \mid \mathcal{F}_{t\wedge\tau}). \tag{4.2}$$

Now, from independence of $X^\tau$ and $Y^\tau$, we have

$$\begin{aligned} X_{t\wedge\tau}Y_{t\wedge\tau} &= P(A \mid \mathcal{F}^X_{t\wedge\tau})P(B \mid \mathcal{F}^Y_{t\wedge\tau}) \\ &= P(A \mid \mathcal{F}_{t\wedge\tau})P(B \mid \mathcal{F}_{t\wedge\tau}) = P(AB \mid \mathcal{F}_{t\wedge\tau}). \end{aligned}$$

Due to Theorem 3 in [12], from causality it follows that the filtration $\mathbf{F}^\tau = \{\mathcal{F}_{t\wedge\tau}\}$ is generated by processes $X^\tau = X_{t\wedge\tau} = P(A \mid \mathcal{F}^X_{t\wedge\tau})$. Therefore, since $\chi_A$ is $\mathcal{F}^X_\tau$-measurable indicator function of the set $A \in \mathcal{F}^X_\tau$ ($B \in \mathcal{F}^Y_\tau$, so $\chi_B$ is $\mathcal{F}^Y_\tau$-measurable) we have

$$\begin{aligned} E(X^\tau_\infty Y^\tau_\infty \mid \mathcal{F}_{t\wedge\tau}) &= E(P(A \mid \mathcal{F}^X_\tau)P(B \mid \mathcal{F}^Y_\tau) \mid \mathcal{F}_{t\wedge\tau}) \\ &= E(E(\chi_A \mid \mathcal{F}^X_\tau) \mid \mathcal{F}_{t\wedge\tau})E(E(\chi_B \mid \mathcal{F}^Y_\tau) \mid \mathcal{F}_{t\wedge\tau}) \\ &= E(\chi_A \mid \mathcal{F}_{t\wedge\tau})E(\chi_B \mid \mathcal{F}_{t\wedge\tau}) \end{aligned}$$

So, from causality it follows

$$\begin{aligned} E(X^\tau_\infty Y^\tau_\infty \mid \mathcal{F}_{t\wedge\tau}) &= E(\chi_A \mid \mathcal{F}^X_{t\wedge\tau})E(\chi_B \mid \mathcal{F}^Y_{t\wedge\tau}) = P(A \mid \mathcal{F}^X_{t\wedge\tau})P(B \mid \mathcal{F}^Y_{t\wedge\tau}) \\ &= X_{t\wedge\tau}Y_{t\wedge\tau}. \end{aligned}$$

Thus, $(XY)^\tau = (XY)_{t\wedge\tau}$ is a (stopped) martingale with respect to $\{\mathcal{F}_{t\wedge\tau}\}$ and $X^\tau$ and $Y^\tau$ are two strongly orthogonal stopped martingales. □

The concept of causality from Definition 2.2 can be applied to martingales which are stopped at a different stopping times $\tau$ and $\sigma$.

**Proposition 4.1.** *Let $\tau$ be a $\{\mathcal{F}^X_t\}$-stopping time, $\sigma$ be a $\{\mathcal{F}^Y_t\}$-stopping time, $\tau \vee \sigma = max(\tau, \sigma)$ and processes $X^\tau = (X_{t\wedge\tau})$ and $Y^\sigma = (Y_{t\wedge\sigma})$ be two independent $\mathbf{F}^{\tau\vee\sigma} = \{\mathcal{F}^{\tau\vee\sigma}\}$-stopped martingales. Processes $X^\tau$ and $Y^\sigma$ are strongly orthogonal if and only if each of them is its own cause within $\{\mathcal{F}^{\tau\vee\sigma}\}$, i.e. if $\mathbf{F}^{X^\tau} \ltimes \mathbf{F}^{X^\tau}; \mathbf{F}^{\tau\vee\sigma}; P$ and $\mathbf{F}^{Y^\sigma} \ltimes \mathbf{F}^{Y^\sigma}; \mathbf{F}^{\tau\vee\sigma}; P$ hold.*

**Proof.** Suppose that the processes $X^\tau$ and $Y^\sigma$ are two independent, strongly orthogonal $\{\mathcal{F}^{\tau\vee\sigma}\}$-stopped martingales. Here we have to consider the following two cases: $\sigma \leqslant \tau$ and $\tau < \sigma$. In the first case, $\sigma \leqslant \tau$, we have $\tau \vee \sigma = max(\sigma, \tau) = \tau$ and $\mathbf{F}^{\tau\vee\sigma} = \mathbf{F}^\tau$. So, the process $X^\tau$ is its own cause within $\mathbf{F}^\tau$, and according to Theorem 6 in [8] the causality relation $\mathbf{F}^{X^\tau} \mid\!< \mathbf{F}^{X^\tau}; \mathbf{F}^\tau; P$ holds. Similary we can prove that the process $Y^\sigma$ is its own cause within $\mathbf{F}^\sigma$, i.e. $\mathbf{F}^{Y^\sigma} \mid\!< \mathbf{F}^{Y^\sigma}; \mathbf{F}^\tau; P$ holds.

Conversely, let the processes $X^\tau$ and $Y^\sigma$ be two independent $\{\mathcal{F}^\tau\}$-stopped martingales, for which $\mathbf{F}^{X^\tau} \mid\!< \mathbf{F}^{X^\tau}; \mathbf{F}^\tau; P$ and $\mathbf{F}^{Y^\sigma} \mid\!< \mathbf{F}^{Y^\sigma}; \mathbf{F}^\tau; P$ holds, i.e

$$\forall A \in \mathcal{F}^X_\tau \qquad P(A \mid \mathcal{F}_{t\wedge\tau}) = P(A \mid \mathcal{F}^X_{t\wedge\tau})$$
$$\forall B \in \mathcal{F}^Y_\sigma \qquad P(B \mid \mathcal{F}_{t\wedge\tau}) = P(B \mid \mathcal{F}^Y_{t\wedge\sigma})$$

Then, for all $A \in \mathcal{F}^X_\tau$ and for all $B \in \mathcal{F}^Y_\tau$ we have

$$\begin{aligned} E(X_\tau Y_\sigma \mid \mathcal{F}_{t\wedge\tau}) &= E(P(A \mid \mathcal{F}^X_{t\wedge\tau})P(B \mid \mathcal{F}^Y_{t\wedge\sigma}) \mid \mathcal{F}_{t\wedge\tau}) = E(\chi_A\chi_B \mid \mathcal{F}_{t\wedge\tau}) \\ &= P(A \mid \mathcal{F}_{t\wedge\tau})P(B \mid \mathcal{F}_{t\wedge\tau}) = P(A \mid \mathcal{F}^X_{t\wedge\tau})P(B \mid \mathcal{F}^Y_{t\wedge\sigma}) \\ &= X_{t\wedge\tau}Y_{t\wedge\sigma}. \end{aligned}$$

So, the processes $X^\tau$ and $Y^\sigma$ are strongly orthogonal stopped martingales.

The proof is similar in the second case if $\tau < \sigma$. □

The Theorem 4.1 can be extended to a larger class of processes, the stopped local martingales.

**Theorem 4.2.** *Let $\tau$ be a $\{\mathcal{F}^X_t\}$ and $\{\mathcal{F}^Y_t\}$-stopping time and let $X^\tau = X_{t\wedge\tau}$ and $Y^\tau = Y_{t\wedge\tau}$ be two independent $\{\mathcal{F}_{t\wedge\tau}\}$-stopped local martingales. Processes $X^\tau$ and $Y^\tau$ are orthogonal if and only if each of them is its own cause within $\{\mathcal{F}_{t\wedge\tau}\}$, i.e. if $\mathbf{F}^{X^\tau} \mid\!< \mathbf{F}^{X^\tau}; \mathbf{F}^\tau; P$ and $\mathbf{F}^{Y^\tau} \mid\!< \mathbf{F}^{Y^\tau}; \mathbf{F}^\tau; P$ hold.*

**Proof.** Suppose that $X^\tau$ and $Y^\tau$ are two orthogonal, independent $\{\mathcal{F}_{t\wedge\tau}\}$-stopped local martingales. Then, there exists a sequence of $\{\mathcal{F}^X_{t\wedge\tau}\}$ stopping times $\{\tau_n\} \longrightarrow \infty$, such that process $X_{t\wedge\tau\wedge\tau_n}$ is $\{\mathcal{F}_{t\wedge\tau}\}$-martingale (every martingale is local martingale, but converse is not true). As a consequence, this process is $\{\mathcal{F}^X_{t\wedge\tau}\}$ martingale. Therefore, $E(X_{\tau\wedge\tau_n} \mid \mathcal{F}^X_{t\wedge\tau}) = E(X_{\tau\wedge\tau_n} \mid \mathcal{F}_{t\wedge\tau})$ for all $X_{\tau\wedge\tau_n}$ (which are $\{\mathcal{F}^X_{\tau\wedge\tau_n}\}$-measurable). According to previous equality, it follows that $\mathbf{F}^{X^{\tau\wedge\tau_n}} \mid\!< \mathbf{F}^{X^\tau}; \mathbf{F}^\tau; P$ holds. Due to invariance of causality under convergence (for details see Theorem 3.5 in [22]), and by Theorem 4.1, we have that $\mathbf{F}^{X^\tau} \mid\!< \mathbf{F}^{X^\tau}; \mathbf{F}^\tau; P$ holds.

Similarly, for a sequence of $\{\mathcal{F}^Y_{t\wedge\tau}\}$ stopping times $\{\sigma_n\} \longrightarrow \infty$, we obtain that $\mathbf{F}^{Y^\tau} \mid\!< \mathbf{F}^{Y^\tau}; \mathbf{F}^\tau; P$ holds.

Conversely, let the causality relations hold. Then, due to Theorem 6 in [8], processes $X^\tau$ and $Y^\tau$ are independent $\{\mathcal{F}^\tau\}$-stopped martingales. Then

$$E((XY)^\tau_\infty \mid \mathcal{F}_{t\wedge\tau}) \quad = \quad E(X^\tau_\infty \mid \mathcal{F}_{t\wedge\tau})E(Y^\tau_\infty \mid \mathcal{F}_{t\wedge\tau}) = X_{t\wedge\tau}Y_{t\wedge\tau} = (XY)_{t\wedge\tau}.$$

So, $XY$ is stopped martingale (at the same time it is a stopped local martingale) and according to Definition 4.3, $X^\tau$ and $Y^\tau$ are two orthogonal stopped local martingales. □

## 5. Example

The concept of causality can be applied to the problem of local risk-minimization, that has become a popular criterion for pricing and hedging in incomplete markets (see [23] - [26]).

The time horizon $T \in (0,\infty)$ is fixed. The random time of default is represented by a stopping time $\tau : \Omega \to [0,T] \cup \{+\infty\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. For default time $\tau$ is introduced the associated default process $H$, given by $H_t = 1_{\{\tau\leqslant t\}}$ and $(\mathcal{F}^H_t)$ is its natural filtration. Let $W$ and $B$ be two one-dimensional independent Brownian motions and $\mathcal{G}_t = \mathcal{F}^W_t \vee \mathcal{F}^B_t$, where $\{\mathcal{F}^W_t\}$ and $\{\mathcal{F}^B_t\}$ denotes the natural filtrations of the processes $W$ and $B$.

The risky asset price $S$ is represented by a stochastic process on $(\Omega, \mathcal{F}, P)$ whose dynamics is given by

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \quad S_0 = s_0 > 0$$

where $\sigma_t > 0$ a.s., $\mu, \sigma$ are $\mathcal{F}$-adapted processes and $X_t$ an unobservable exogenous stochastic factor satisfying

$$dX_t = b_t dt + a_t(\rho dW_t + \sqrt{1-\rho^2}dB_t), \quad X_0 = x_0 \in \mathbb{R}.$$

Let $\mathbf{F} = \{\mathcal{F}_t\},\ t \in [0,T]$ be the filtration given by $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{F}^H_t = \mathcal{F}^W_t \vee \mathcal{F}^B_t \vee \mathcal{F}^H_t$. Investors do not have a complete information on the market, they cannot observe neither the stochastic factor $X$ nor the Brownian motions $W$ and $B$ which drive the dynamics of the pair $(S, X)$ and as a consequence they cannot observe the $F$-hazard rate. At any time $t$, they may observe the risky asset price and know if default has occurred or not. The available information is given by

$$\widetilde{\mathcal{F}} := \mathcal{F}^S \vee \mathcal{F}^H \quad \subset \quad \mathcal{F} = \mathcal{G} \vee \mathcal{F}^H := \mathcal{F}^W \vee \mathcal{F}^B \vee \mathcal{F}^H.$$

A defaultable claim is a triplet $(\xi, Z, \tau)$, where promissed contigent claim $\xi$ is the promised payoff received by the owner at maturity $T$, $Z$ is the recovery process, which is paid at the default time if default has happened prior to or at time $T$, $\tau$

is the default time. Process $N = N_t, t \in [0,T]$ models the payment stream arising from the defaultable claim

$$N_t = Z_\tau I_{\tau \leqslant t} = \int_0^T Z_s dH_s,\ 0 \leqslant t < T, \qquad N_T = \xi I_{\tau > T},\ t = T. \tag{5.1}$$

By assumptions in [23] the hedging stops after default, hence is considered the stopped interval $[0, \tau \wedge T]$. If $M$ is an $(\mathbf{G}, P)$-martingale the stopped process $M^\tau = M_{t\wedge\tau}$ is a $(\mathbf{F}, P)$-martingale ($\{\mathcal{G}_t\}$ is increasing and subfiltration of $\{\mathcal{F}_t\}$, see Lemma 5.1.6 in [27]). The stopped processes $W^\tau$ and $B^\tau$ are $(\mathbf{F}, P)$-Brownian motions. The risky asset price $S^\tau$ is a $(\mathbf{F}, P)$-semimartingale with decomposition

$$S_t^\tau = s_0 + \int_0^{t\wedge\tau} S_u^\tau \mu_u du + \int_0^{t\wedge\tau} S_u^\tau \sigma(u, S_u^\tau) dW_u^\tau, \quad t \in [0,T],$$

$$M_t^{\mathcal{F}} = \int_0^{t\wedge\tau} S_u^\tau \sigma(u, S_u^\tau) dW_u^\tau.$$

Risk-minimization approach is introduced in [25].

The risky asset process $S$ is martingale, $\psi = (\theta, \eta)$ is an admissible strategy, $V(\psi) := \theta S + \eta$ its value process, and the cost process: $C_t(\psi) := V_t(\psi) - \int_0^t \theta_u dS_u$. An admissible strategy such that $V_T(\psi) = \xi$ is risk-minimizing if minimizes the risk process: $E[(C_T(\psi) - C_t(\psi))^2 | \mathcal{F}_t]$. Process $\theta^*$ is given by the Föllmer-Schweizer decomposition of $\xi$ (see [24]) $\xi = E[\xi] + \int_0^T \theta_u^* dS_u + A_T,\ P - a.s.$ where $A$ is a martingale strongly orthogonal to $S$.

Strategy $\psi^*$ is mean-self-financing and $C_t(\psi^*) = E[\xi] + A_t$.

In the semimartingale case such a strategy does not exist, hence Schweizer (in [26]) introduced the weaker concept of locally risk-minimizing strategy (under suitable assumptions it is equivalent to pseudo optimality).

This approach in the case of a defaultable claim and in partial information framework is considered in [23]. Here is assumed that hedging stops after default. This allows to work with hedging strategies only up to time $T \wedge \tau$.

The cost process $C(\varphi)$ of a $(\mathcal{F}, L^2)$-strategy (resp. $(\widetilde{\mathcal{F}}, L^2)$-strategy)$\varphi = (\theta, \eta)$ is given by

$$C_t(\varphi) = N_t + V_t(\varphi) - \int_0^t \theta_u dS_u^\tau, t \in [0, T \wedge \tau],$$

where $N$ is defined in (5.1).

Due to Definition 3.3 in [23], $\varphi$ is mean-self-financing if its cost process $C(\varphi)$ is a $(\mathcal{F}, P)$-martingale (resp. $(\widetilde{\mathcal{F}}, P)$-martingale).

Theorem 1.6 in [26], defines local risk minimization and formulates its equivalent characterisation. The extension of the local risk-minimization approach to payment streams requires to look for admissible strategies with the 0-achieving property, that is $V_{\tau\wedge T}(\varphi) = 0,\ P - a.s.$

Due to Definition 3.5 in [23] about the stopped Föllmer-Schweizer decomposition of random variable $\zeta \in L^2(\mathcal{F}_T, P)$ and Theorem 4.1 we can say that a random variable $\zeta$ admits a stopped Föllmer-Schweizer decomposition if it can be written as

$$\zeta = \zeta_0 + \int_0^T \theta_u^{\mathcal{F}} dS_u^{\tau} + A_{T\wedge\tau}^{\mathcal{F}}, \quad P - a.s.,$$

and if each of processes $A_t^{\mathcal{F}}$ and $M_t^{\mathcal{F}}, t \in [0, T\wedge\tau]$ is its own cause within $\mathcal{F}$, where $M_t^{\mathcal{F}}$ is the martingale part of $S^{\tau}$.
Adapting the results proved in [23] (see Proposition 3.6) to the concept of causality between stopped filtrations (Theorem 4.1) we get the following characterization.

**Proposition 5.1.** *Let $N$ be the payment stream associated to the defaultable claim $(\xi, Z, \tau)$. Then, $N$ admits an $(\mathcal{F}^S, \widetilde{\mathcal{F}})$-locally risk-minimizing strategy $\varphi^* = (\theta^*, \eta^*)$ if and only if there exists a process $\theta^{\mathcal{F}} \in \Theta^{\mathcal{F},\tau}$ a square integrable $(\mathcal{F}, P)$-martingale, which is its own cause within $\{\mathcal{F}_t\}$, null at zero such that the martingale part of $S^{\tau}$ is its own cause and*

$$N_{T\wedge\tau} = N_0 + \int_0^T \theta_u^{\mathcal{F}^S} dS_u^{\tau} + A_{t\wedge\tau}^{\widetilde{\mathcal{F}}} P - a.s.$$

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