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QUASI SURE STRASSEN'S LAW OF THE ITERATED LOGARITHM FOR INCREMENTS OF FBM IN HÖLDER NORM

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Abstract. In this paper, we present functional Strassen's law of the iterated logarithm for Csörgő-Révész (C-R) increments of a fractional Brownian motion in Hölder norm with respect to (r, p)-capacity. The method of the proof for our main results is based on the large deviation for the fractional Brownian motion.

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1. Introduction and main results

The functional limit theorems for fractional Brownian motion (FBM) have been investigated in various directions. For example, Wang [5] studied functional limit theorems for increments of a fractional Brownian motion under the Sup-norm. At the same time, Lin, Wang and Hwang [3] obtained functional limit theorems for d-dimensional FBM under Hölder norm.

The capacity is a set function on B with the property that it sometimes takes positive values even for $\mu-$null sets, while a set of capacity zero has always $\mu-$measure zero. Therefore, an interesting problem is to find out what property holds not only almost sure but also quasi sure. In this paper, we shall discuss this topic. Liu [4] established quasi sure Strassen-type law of the iterated logarithm for Csörgő-Révész (C-R) increments of Brownian Motion (BM) in Hölder norm with respect to $(r, p)-$capacity. In this paper, we present Strassen's law of the iterated logarithm for C-R increments of

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FBM in Hölder norm with respect to $(r,p)-$capacity. We generalize Strassen-type law of the iterated logarithm of FBM in terms of $(r,p)-$capacities under Hölder norm, the corresponding results in [3], [4] and [7] are extended.

Let (B,H,μ) be an abstract Wiener space and $D^{r,p}$ denotes the Sobolev space, i.e.,

$$D^{r,p}=(1-\mathcal{L})^{-\frac{r}{2}}\boldsymbol{L}^p,\ \|F\|_{r,p}=\|(1-\mathcal{L})^{-\frac{r}{2}}F\|_p,\ F\in L^p,\ r\geq 0,\ 1\leq p<\infty,$$

where $\boldsymbol{L}^p$ denotes L^p-space of real-valued functions on (B,μ) and $\mathcal{L}$ is the Ornstein-Uhlenbeck operator on (B,H,μ). For $r\geq 0,\ p>1$, $(r,p)-$capacity is defined by

$$C_{r,p}(O)=\inf\{\|F\|_{r,p}^p;F\in D_{r,p},F\geq 1,\mu-a.s.\ \ on\ \ O\}\text{ for open set }O\subset B,$$

and for any set $A\subset B$, we define $C_{r,p}(A)$ by

$$C_{r,p}(A)=\inf\{C_{r,p}(O);A\subset O\subset\ B,O\text{ is open}\}.$$

Let $\{X(t);t\geq 0\}$ be a standard γ-fractional Brownian motion with $0<\gamma<1$ and $X(0)=0$. The $\{X(t);t\geq 0\}$ has a covariance function

$$R(s,t)=E(X(s)X(t))=\frac{1}{2}(s^{2\gamma}+t^{2\gamma}-|s-t|^{2\gamma}),\quad s,t\geq 0,$$

and a representation

$$X(t)=\int_{R^1}\frac{1}{k_\gamma}\left\{|x-t|^{(2\gamma-1)/2}-|x|^{(2\gamma-1)/2}\right\}dB(x),$$

where $k_\gamma^2=\displaystyle\int_{R^1}\left\{|x-1|^{(2\gamma-1)/2}-|x|^{(2\gamma-1)/2}\right\}^2dx$, $\{B(t);-\infty<t<+\infty\}$ is a Brownian motion and $\dfrac{1}{k_\gamma}\{|x-t|^{(2\gamma-1)/2}-|x|^{(2\gamma-1)/2}\}$ is interpreted to be $I_{(0,t]}$ when $\gamma=\frac{1}{2}$. $\{X(t);t\geq 0\}$ has stationary increments with $E(X(s+t)-X(s))^2=t^{2\gamma},\ s,t\geq 0$ and when $\gamma=\frac{1}{2}$, it is a standard Brownian motion. Let $C_0[0,1]$ be the space of continuous functions from $[0,1]$ to $\mathbb{R}$ with value zero at the origin, endowed with usual norm $\|f\|=\sup_{0\leq t\leq 1}|f(t)|$ and we denote by

$$\mathcal{H}=\{f\in C_0[0,1]:f\text{ is an absolutely continuous function},$$
$$\|f\|_{\mathcal{H}}^2=\int_0^1(\dot{f}(s))^2ds<\infty\}.$$

Then $\mathcal{H}$ is a Hilbert space with respect to the scalar product

$$\langle f,g\rangle=\int_0^1\dot{f}(x)\dot{g}(x)dx,\qquad\text{for}\quad f,g\in\mathcal{H}.$$

Let μ be the Wiener measure on $C_0[0,1]$, then $(C_0, \mathcal{H}, \mu)$ is an abstract Wienner space. Let us consider two Banach spaces as follows

$$\mathcal{C}^\alpha = \left\{ f \in C_0[0,1] : \|f(\cdot)\|_\alpha = \sup_{s,t\in[0,1], s\neq t} \frac{|f(t)-f(s)|}{|t-s|^\alpha} < \infty \right\},$$

$$\mathcal{C}^{\alpha,0} = \left\{ f \in \mathcal{C}^\alpha : \lim_{\delta\to 0} \sup_{s,t\in[0,1], 0<|t-s|<\delta} \frac{|f(t)-f(s)|}{|t-s|^\alpha} = 0 \right\},$$

where $0 < \alpha < \frac{1}{2}$. In this paper, we assume $0 < \alpha < \gamma < \frac{1}{2}$. Then $(C^{\alpha,0}, \mathcal{H}, \mu)$ is also an abstract Wiener space, see [1, Theorem 2.4] for details. Define a mapping $I : \mathcal{C}^{\alpha,0} \to [0,\infty]$ by

$$I(f) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{f}(t)|^2 dt, & f \in \mathcal{H}, \\ +\infty, & \text{otherwise.} \end{cases}$$

The limit set associated with functional laws of the iterated logarithm for $\{X(t); t \geq 0\}$ is K_γ and it is a subset of functions in $\mathcal{C}^{\alpha,0}$ with the form

$$f(t) = \int_{R^1} \frac{1}{k_\gamma}\{|x-t|^{(2\gamma-1)/2} - |x|^{(2\gamma-1)/2}\} g(x)dx, \quad 0 \leq t \leq 1.$$

Here the function $g(x)$ ranges over the unit ball of $L^2(R^1)$ and $\int_{R^1} g^2(s)ds \leq 1$. The subset K of $\mathcal{C}^{\alpha,0}$ is defined by

$$K = \{f \in \mathcal{H} : f \in K_\gamma, 2I(f) \leq 1\}.$$

Let a_u be a non-decreasing function from $(0,+\infty)$ to $(0,+\infty)$ such that

(H$_1$) $a_u \leq u$, for any $u \in (0,+\infty)$;

(H$_2$) The function $\dfrac{u}{a_u}$ is non-decreasing;

(H$_3$) $\lim_{u\to+\infty} \dfrac{a_u}{u} = \rho$ with the constant $\rho < 1$;

(H$_4$) $\lim_{u\to+\infty} \dfrac{\log \frac{u}{a_u}}{\log\log u} = +\infty$.

We set $\ell_u = \log \frac{u \log u}{a_u}$, $\beta_u = (2a_u^{2\gamma}\ell_u)^{-\frac{1}{2}}$ and let $\Delta(t,u)$ denote the path $s \to X(ut + a_u s) - X(ut)$, $s \in [0,1]$.

In the following, we state our main results.

Theorem 1.1. *Assume that* (H$_1$) *and* (H$_2$) *hold, then we have*

$$\lim_{u\to\infty} \sup_{t\in[0,1-\frac{a_u}{u}]} \inf_{f\in K} \|\beta_u \Delta(t,u)(\cdot) - f(\cdot)\|_\alpha = 0, \quad C_{r,p} - q.s. \tag{1.1}$$

Theorem 1.2. *Assume that* (H_1), (H_2) *and* (H_3) *hold, then for any* $f \in K$, *we have*

$$\liminf_{u\to\infty} \inf_{t\in[0,1-\frac{a_u}{u}]} \|\beta_u \Delta(t,u)(\cdot) - f(\cdot)\|_\alpha = 0, \quad C_{r,p} - q.s. \tag{1.2}$$

Moreover, assume that (H_4) *also holds, then for any* $f \in K$, *we have*

$$\lim_{u\to\infty} \inf_{t\in[0,1-\frac{a_u}{u}]} \|\beta_u \Delta(t,u)(\cdot) - f(\cdot)\|_\alpha = 0, \quad C_{r,p} - q.s. \tag{1.3}$$

Remark 1.1. *In* [3], *authors proved the equations* (1.1)-(1.3) *in the sense of probability. Theorem* 1.1 *and Theorem* 1.2 *generalized them from probability to* (r,p)*-capacity. At the same time, in* [4], *Liu established quasi sure Strassen-type law of the iterated logarithm for C-R increments of BM in Hölder norm with respect to* $(r,p)-$*capacity. Theorem* 1.1 *and Theorem* 1.2 *generalized them from BM to FBM. From the proof below, we can see that the method we use is different from that in* [3] *and* [4], *and it is more complicated.*

This paper is organized as follows. In Section 2, we introduce some basic lemmas which will be used in this paper. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2.

2. Some lemmas

Our proofs are based on the following lemmas. We followed Theorem 3.3 in [7], we have following lemma.

Lemma 2.1. *Assume* $\{X(t); t \geq 0\}$ *is FBM with* $X(0) = 0$. *Then for any closed set* $A \subset \mathcal{C}^{\alpha,0}$, *we have*

$$\limsup_{\varepsilon\to 0} \varepsilon \left(\log C_{r,p} \left(\bigcup_{0\le t\le T-h} \left\{ \sqrt{\frac{\varepsilon}{h^{2\gamma}}} \left(X(t+h\cdot) - X(t)\right) \in A \right\} \right) + \log \frac{h}{T} \right) \leq$$
$$\leq - \inf_{f\in A} I(f),$$

where $0 < h < 1$, $0 \leq \alpha < \gamma < 1$.

We followed the method in [6], we have following lemma.

Lemma 2.2. *Let* $1 \leq k < Z$, $q_1, q_2 \in (1,\infty)$ *be given so that* $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2}$. *For any* $f \in K$, *put*

$$F_\varepsilon^{(i)} = \left\| \varepsilon \frac{X(t_i + h_i\cdot) - X(t_i)}{\sqrt{h_i^{2\gamma}}} - f \right\|_\alpha, i = 1, 2, ..., n,$$

where $0 \le t_i < \infty, h_i > 0$. *Then there exits a constant* $C = C(r, p, q_1, f)$, *such that for any* $\delta \in (0,1]$ *and* $\varepsilon \in (0,1]$, *we have*

$$C_{r,p}\left(\bigcap_{i=1}^{n}\left\{Z; a_i < F_\varepsilon^{(i)} < b_i\right\}\right)^{1/p} \le$$

$$\le C\delta^{-2r^2-r}n^r\mu\left(\bigcap_{i=1}^{n}\left\{Z; a_i-\delta < F_\varepsilon^{(i)} < b_i+\delta\right\}\right)^{1/q_2}.$$

The following lemmas can be found in the corresponding literature, so we list them directly without proof.

Lemma 2.3. ([3]) *Let* $0 < \gamma < 1$ *and fix* $0 < \alpha < q < \gamma$. *Let* $d_k = k^{k+(1-\iota)}$, $s_k = k^{-k}$ *for* $k \ge 1$ *and* $0 < \iota < 1$. *Let*

$$Y_k(s_k, t) = \int_{|x| \notin I_k} \frac{1}{k_\gamma}\left\{|x - s_k t|^{(2\gamma-1)/2} - |x|^{(2\gamma-1)/2}\right\} dB(x), \qquad 0 \le t \le 1,$$

where $B(x)$ *is a standard Brownian motion,* $I_k = (s_k d_{k-1}, s_k d_k]$. *Let* $0 < \beta < \iota$. *Then, for* $\delta = \min\{2\beta(\gamma - q), \iota - \beta, (1-\iota)(2-2\gamma), (2\gamma - 2q)\iota\}$, *there is a constant* $C > 0$ *depending only on* γ *such that uniformly in* t, h, k, *we have*

$$\sigma_k^2(t,h) = E\{[Y_k(s_k, t+h) - Y_k(s_k,t)]^2\} \le Ch^{2q}s_k^{2\gamma}k^{-\delta}.$$

Lemma 2.4. ([3]) *Let* $\{X(t); t \ge 0\}$ *be FBM with* $X(0) = 0$ *and* $\sigma^2(t-s) = E(X(t) - X(s))^2$, *For any* $\epsilon > 0$, *there exists a positive constant* $C_1 = C(\epsilon)$, $\forall x \ge x_0 > 0$ *such that*

$$\mu\left(\sup_{0\le s<t\le 1}|X(t+s) - X(t)| \ge x\sigma(t-s)\right) \le C\exp\left(-\frac{x^2}{2+\epsilon}\right).$$

Lemma 2.5. ([2]) *Consider a separable Banach space* E *with dual* E^* *and a centered Gaussian measure* μ *on* E. *Let* V *be a convex, symmetric, measurable subset of* E. *For all* $f \in H$,

$$\mu(f+V) \ge \mu(V)\exp\left\{-\frac{1}{2}\|f\|_\mu^2\right\}.$$

3. Proof of Theorem 1.1

Put $u_n = \theta^n$ with $1 < \theta < 2$. For any u, there exists an integer n such that $u_n \le u \le u_{n+1}$. Let $\Psi_{t,u}(s) = \beta_u[X(t+a_u s) - X(t)]$, $s \in [0,1]$, $t \in [0, u - a_u]$, then $\Psi_{t,u}(s) = \frac{\beta_u}{\beta_{u_{n+1}}}\Psi_{t,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}s)$. We have

$$\begin{aligned}
&\sup_{t\in[0,1-\frac{a_u}{u}]}\inf_{f\in K}\|\beta_u\Delta(t,u)(\cdot)-f(\cdot)\|_\alpha \\
&= \sup_{t\in[0,1-\frac{a_u}{u}]}\inf_{f\in K}\|\beta_u[X(ut+a_us)-X(ut)]-f(s)\|_\alpha \\
&= \sup_{x\in[0,u-a_u]}\inf_{f\in K}\|\beta_u[X(x+a_us)-X(x)]-f(s)\|_\alpha \\
&= \sup_{x\in[0,u-a_u]}\inf_{f\in K}\|\Psi_{x,u}(s)-f(s)\|_\alpha \\
&= \sup_{x\in[0,u-a_u]}\inf_{f\in K}\|\frac{\beta_u}{\beta_{u_{n+1}}}\Psi_{x,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}s)-f(s)\|_\alpha \qquad (3.1)\\
&\le \sup_{x\in[0,u_{n+1}-a_{u_n}]}\inf_{f\in K}\left\|\Psi_{x,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}\cdot)-f(\frac{a_u}{a_{u_{n+1}}}\cdot)\right\|_\alpha \\
&+\left(\frac{\beta_u}{\beta_{u_{n+1}}}-1\right)\sup_{x\in[0,u_{n+1}-a_{u_n}]}\left\|\Psi_{x,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}\cdot)\right\|_\alpha \\
&+\inf_{f\in K}\left\|f(\frac{a_u}{a_{u_{n+1}}}\cdot)-f(\cdot)\right\|_\alpha := I_1+I_2+I_3.
\end{aligned}$$

For any $\varepsilon > 0$, let $A = \{f \in \mathcal{C}^{\alpha,0}; \|f-K\|_\alpha \ge \varepsilon\}$. If $f \in A$, $f \notin K$, thus there exits $\delta > 0$, $2\inf_{f\in K} I(f) > 1+\delta := \eta > 1$. By Lemma *2.1*, *(H1)* and *(H2)*, we have

$$\begin{aligned}
&C_{r,p}(I_1\ge\varepsilon) = \\
&= C_{r,p}\left(\sup_{x\in[0,u_{n+1}-a_{u_n}]}\inf_{f\in K}\left\|\beta_{u_{n+1}}[X(x+a_{n+1}s)-X(x)]-f(s)\right\|_\alpha\ge\varepsilon\right) \\
&= C_{r,p}\left(\sup_{x\in[0,u_{n+1}-a_{u_n}]}\left\|\sqrt{\frac{1}{2\ell_{n+1}}}\frac{X(x+a_{n+1}s)-X(x)}{\sqrt{a^{2\gamma}_{u_{n+1}}}}-K\right\|_\alpha\ge\varepsilon\right) \\
&\le C_{r,p}\left(\bigcup_{0\le x\le u_{n+1}-a_{u_n}}\left\{\sqrt{\frac{1}{2\ell_{n+1}}}\frac{X(x+a_{n+1}s)-X(x)}{\sqrt{a^{2\gamma}_{u_{n+1}}}}\in A\right\}\right) \\
&\le \frac{u_{n+1}}{a_{u_n}}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^\eta \le \theta\left(\frac{a_{u_n}}{u_n}\right)^{\eta-1}\left(\frac{1}{\log u_{n+1}}\right)^\eta \le \frac{\theta}{[(n+1)\log\theta]^\eta}.
\end{aligned}$$

It is clear that

$$\sum_{n=1}^\infty \frac{\theta}{[(n+1)\log\theta]^\eta} < \infty.$$

By the Borel-Cantelli lemma, we get

$$\lim_{n\to\infty} I_1 = 0,\ C_{r,p} - q.s. \qquad (3.2)$$

For any $f \in K \subset \mathcal{C}^{\alpha,0}$, $\|f\|_\alpha \leq 1$ and large n, by (3.2), we have

$$\sup_{x\in[0,u_{n+1}-a_{u_n}]} \left\| \Psi_{x,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}\cdot) \right\|_\alpha \leq 2,\ C_{r,p}-q.s.$$

Obviously,

$$\frac{\beta_u}{\beta_{u_{n+1}}} - 1 \leq \theta^\gamma - 1.$$

By (3.4) in [3], we have

$$I_3 = \sup_{f\in K} \left\| f(\frac{a_u}{a_{u_{n+1}}}\cdot) - f(\cdot) \right\|_\alpha \leq 2(\theta-1)^{\gamma-\alpha}.$$

Letting $\theta \to 1$, we get

$$\lim_{n\to\infty} (I_2 + I_3) = 0, \qquad C_{r,p}-q.s. \tag{3.3}$$

Combining (3.1) with (3.2) and (3.3), we obtain (1.1).

4. Proof of Theorem 1.2

First of all, let's prove the following two lemmas.

Lemma 4.1. *If conditions* (H$_1$)-(H$_3$) *hold, for any* $f \in K$, *we have*

$$\liminf_{u\to\infty} \|\beta_u \Delta(1-\frac{a_u}{u}, u) - f(s)\|_\alpha = 0, \qquad C_{r,p}-q.s.$$

Proof. Case (I). $\limsup\limits_{u\to\infty} \frac{\log u a_u^{-1}}{\log\log u} < \infty$. We take u_m, such that $u_1 = u_0,\ u_{m+1} - a_{u_{m+1}} = u_m,\ m \geq 1$. For $k = 1, 2, \cdots$, we define

$$Z_k(t) = \int_{|x|\in(d_{k-1},d_k]} \frac{1}{K_\gamma} \left\{ |x-t|^{(2\gamma-1)/2} - |x|^{(2\gamma-1)/2} \right\} dB(x)$$

and

$$X_k(t) = X(t) - Z_k(t),$$

for $0 \leq t \leq 1,\ d_k = k^{k+(1-\iota)},\ s_k = k^{-k},\ 0 < \gamma < 1$. Then $\{Z_k(\cdot)\},\ k = 1, 2, \cdots$, are independent and

$$\{s_k^\gamma X_k(\cdot)\} \overset{\mathcal{D}}{=} \{Y_k(s_k, \cdot)\}.$$

where $Y_k(s_k,\cdot)$ is as in Lemma 2.2. For any $\varepsilon > 0$, we have

$$
\begin{aligned}
&C_{r,p}\left\{\|\beta_{u_m}\Delta(1-\frac{a_{u_m}}{u_m},u_m)(\cdot)-f(\cdot)\|_\alpha \geq 4\varepsilon\right\}\\
&=C_{r,p}\left\{\|\beta_{u_m}[X(u_{m-1}+a_{u_m}s)-X(u_{m-1})]-f(s)\|_\alpha \geq 4\varepsilon\right\}\\
&\leq C_{r,p}\left\{\|\beta_{u_m}[Z_m(u_{m-1}+a_{u_m}s)-Z_m(u_{m-1})]-f(s)\|_\alpha \geq 2\varepsilon\right\}\\
&+C_{r,p}\left\{\|\beta_{u_m}[X_m(u_{m-1}+a_{u_m}s)-X_m(u_{m-1})]\|_\alpha \geq 2\varepsilon\right\}\\
&:=I_{41}+I_{42}.
\end{aligned}
\tag{4.1}
$$

By Lemmas 2.2-2.4, we have

$$
\begin{aligned}
&I_{42}=C_{r,p}\left\{\|\beta_{u_m}[X_m(u_{m-1}+a_{u_m}s)-X_m(u_{m-1})]\|_\alpha \geq 2\varepsilon\right\}\\
&=C_{r,p}\left\{\left\|\frac{1}{\sqrt{2s_m^{2\gamma}\ell_{u_m}}}\frac{Y_m(s_m,u_{m-1}+a_{u_m}\cdot)-Y_m(s_m,u_{m-1})}{\sqrt{a_{u_m}^{2\gamma}}}\right\|_\alpha \geq 2\varepsilon\right\}\\
&\leq C\varepsilon^{(-2k^2-k)p}\mu\left(\left\|\frac{1}{\sqrt{2s_m^{2\gamma}\ell_{u_m}}}\frac{Y_m(s_m,u_{m-1}+a_{u_m}\cdot)-Y_m(s_m,u_{m-1})}{\sqrt{a_{u_m}^{2\gamma}}}\right\|_\alpha \geq \varepsilon\right)^{\frac{p}{q_2}}\\
&\leq C_1\mu\left(\left\|\frac{1}{\sqrt{2s_m^{2\gamma}\ell_{u_m}}}[Y_m(s_m,\frac{u_{m-1}}{a_{u_m}}+\cdot)-Y_m(s_m,\frac{u_{m-1}}{a_{u_m}})]\right\|_\alpha \geq \varepsilon\right)^{\frac{p}{q_2}}\\
&\leq C_1\mu\left(\sup_{0\leq x<y\leq 1}\left|\frac{Y_m(s_m,y)-Y_m(s_m,x)}{(y-x)^q\sqrt{Cs_m^{2\gamma}m^{-\delta}}}\right| \geq \varepsilon\sqrt{\frac{n^\delta}{C}2\ell_{u_m}}\right)^{\frac{p}{q_2}}\\
&\leq C_1\exp\left(-\frac{p}{q_2}\frac{2\varepsilon^2m^\delta}{(2+\epsilon)C}\log\frac{u_m\log u_m}{a_{u_m}}\right)\leq C_1\left(\frac{a_{u_m}}{u_m\log u_m}\right)^{C_2m^\delta},
\end{aligned}
$$

where

$$C_1=C\varepsilon^{(-2k^2-k)p}>0,\quad C_2=\frac{p}{q_2}\frac{2\varepsilon^2}{(2+\epsilon)C}>0.$$

For large m, we have $C_2m^\delta > 1$, thus $\sum_{m=1}^\infty I_{42}<\infty$. By the Borel-Cantelli lemma, we get

$$\lim_{m\to\infty}\{\|\beta_{u_m}[X_m(u_{m-1}+a_{u_m}s)-X_m(u_{m-1})]\|_\alpha\}=0, C_{r,p}-q.s. \tag{4.2}$$

Since $\{Z_k(\cdot)\},\ k=1,2,\cdots$ are independent, by Lemma 2.2, we have

(4.3)

$$C_{r,p}\left\{\bigcap_{m=l}^{n}\|\beta_{u_m}[Z_m(u_{m-1}+a_{u_m}\cdot)-Z_m(u_{m-1})]-f(\cdot)\|_\alpha\geq 2\varepsilon\right\}^{\frac{q_2}{p}}$$
$$\leq Cn^{rq_2}\varepsilon^{(-2r^2-r)q_2}\prod_{k=l}^{n}\mu(\|\beta_{u_m}[Z_m(u_{m-1}+a_{u_m}\cdot)-Z_m(u_{m-1})]-f(\cdot)\|_\alpha\geq\varepsilon)$$
$$:=C_3n^{rq_2}\prod_{m=l}^{n}J_m\left(C_3=C\varepsilon^{(-2r^2-r)q_2}>0\right).$$

Moreover,

$$\begin{aligned}J_m&=\mu(\|\beta_{u_m}[Z_m(u_{m-1}+a_{u_m}\cdot)-Z_m(u_{m-1})]-f(\cdot)\|_\alpha\geq\varepsilon)\\&\leq\mu\biggl(\|\beta_{u_m}[X(u_{m-1}+a_{u_m}\cdot)-X(u_{m-1})]-f(\cdot)\|_\alpha\geq\frac{1}{2}\varepsilon\biggr)\\&\quad+\mu\biggl(\|\beta_{u_m}[X_m(u_{m-1}+a_{u_m}\cdot)-X_m(u_{m-1})]\|_\alpha\geq\frac{1}{2}\varepsilon\biggr):=J_m^1+J_m^2.\end{aligned}\tag{4.4}$$

By the same method of the estimate of I_{42}, exist $C_{21}>0,\ C_{22}>0$, such that

$$J_m^2\leq C_{21}\left(\frac{a_{u_m}}{u_m\log u_m}\right)^{C_{22}m^\delta}.\tag{4.5}$$

Let $f^{(\epsilon)}=(1-\frac{\epsilon}{2})f$, for $f\in K$ and $0<\epsilon<1$. Then $f^{(\epsilon)}\in K$ and $\|f-f^{(\epsilon)}\|_\alpha<\frac{\epsilon}{2}$. For large n, by Lemma 2.5 and Lemma 2.3, we have

$$\mu\biggl(\|\beta_{u_m}[X(u_{m-1}+a_{u_m}\cdot)-X(u_{m-1})]-f(\cdot)\|_\alpha<\frac{1}{2}\varepsilon\biggr)$$
$$\geq\mu\biggl(\biggl\|\frac{X(u_{m-1}+a_{u_m}\cdot)-X(u_{m-1})}{a_{u_m}^\gamma}-f^{(\epsilon)}(\cdot)\sqrt{2\ell_{u_m}}\biggr\|_\alpha<\frac{\varepsilon}{4}\sqrt{2\ell_{u_m}}\biggr)$$
$$\geq\exp\left(-\|f^{(\epsilon)}\|_\gamma^2\ell_{u_m}\right)\mu\biggl(\biggl\|\frac{X(u_{m-1}+a_{u_m}\cdot)-X(u_{m-1})}{a_{u_m}^\gamma}\biggr\|_\alpha<\frac{\varepsilon}{4}\sqrt{2\ell_{u_m}}\biggr)$$
$$\geq\exp\left(-(1-\frac{\epsilon}{2})^2\|f\|_\gamma^2\log\frac{u_m\log u_m}{a_{u_m}}\right)\left(1-C\exp\left(-\frac{\varepsilon^2}{8(2+\epsilon)}\log\frac{u_m\log u_m}{a_{u_m}}\right)\right)$$
$$\geq\exp\left(-(1-\frac{\epsilon}{2})^2\log\frac{u_m\log u_m}{a_{u_m}}\right)\left(1-C\exp\left(-\frac{\varepsilon^2}{8(2+\epsilon)}\log\frac{u_m\log u_m}{a_{u_m}}\right)\right)$$
$$=\left(\frac{a_{u_m}}{u_m\log u_m}\right)^{C_{11}}\left(1-C\left(\frac{a_{u_m}}{u_m\log u_m}\right)^{C_{12}}\right),$$

where

$$C>0,\quad 0<C_{11}=(1-\frac{\epsilon}{2})^2<1,\quad C_{12}=\frac{\varepsilon^2}{8(2+\epsilon)}>0.$$

Therefore, for large m, $1 - C\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{12}} > \frac{1}{2}$, we have

$$
\begin{aligned}
J_m^1 &\le \exp\left\{-\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}}\left(1 - C\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{12}}\right)\right\} \\
&\le \exp\left\{-\frac{1}{2}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}}\right\}.
\end{aligned} \tag{4.6}
$$

By (4.3)-(4.6), for large n, we have

$$
\begin{aligned}
&C_{r,p}\left\{\bigcap_{m=l}^{n} \|\beta_{u_m}[Z_m(u_{m-1} + a_{u_m}\cdot) - Z_m(u_{m-1})] - f(\cdot)\|_\alpha \ge 2\varepsilon\right\}^{\frac{q_2}{p}} \\
&= C_1 n^{rq_2} \prod_{m=l}^{n}\left(C_{21}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{22} m^\delta} + \exp\left\{-\frac{1}{2}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}}\right\}\right). \\
&\le C_1 n^{rq_2} \prod_{m=l}^{n} \exp\left\{-\frac{1}{4}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}}\right\} \\
&= C_1 n^{rq_2} \exp\left\{-\frac{1}{4}\sum_{m=l}^{n}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}}\right\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&C_{r,p}\left\{\bigcap_{m=l}^{n} \|\beta_{u_m}[Z_m(u_{m-1} + a_{u_m}\cdot) - Z_m(u_{m-1})] - f(\cdot)\|_\alpha \ge 2\varepsilon\right\} \\
&\le C_1 n^{rp} \exp\left\{-\frac{p}{q_2}\frac{1}{4}\sum_{m=l}^{n}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}}\right\}.
\end{aligned} \tag{4.7}
$$

By *(H_3)* and definition of u_m, we have

$$
\sum_{m=l}^{n}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}} \ge A(\log u_n)^{\eta_0},
$$

where $A = A(l) > 0, \eta_0 = 1 - C_{11} > 0$. Since $\limsup_{u\to\infty} \frac{\log u a_u^{-1}}{\log\log u} < \infty$, we take $\theta > \frac{2}{\eta_0}$ and $u_0 = e^{(\log n_0)^\theta}$, for large n, we can proof

$$
\log u_n \ge (\log n)^\theta,\ n \ge n_0.
$$

Thus, for $n_0 < l < n$, we get

$$
\sum_{m=l}^{n}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}} \ge A(\log u_n)^{\eta_0} \ge A(\log n)^{\eta_0\theta} \ge (\log n)^2. \tag{4.8}
$$

By (4.7) and (4.8), we obtain

$$C_{r,p}\left\{\bigcap_{m=l}^{n}\|\beta_{u_m}[Z_m(u_{m-1}+a_{u_m}\cdot)-Z_m(u_{m-1})]-f(\cdot)\|_\alpha\geq 2\varepsilon\right\}$$
$$\leq C_1 n^{rp}\exp\left\{-\frac{p}{4q_2}(\log n)^2\right\}\to 0, \qquad n\to\infty.$$

So,

$$C_{r,p}\left\{\bigcup_{l=1}^{\infty}\bigcap_{m=l}^{\infty}\|\beta_{u_m}[Z_m(u_{m-1}+a_{u_m}\cdot)-Z_m(u_{m-1})]-f(\cdot)\|_\alpha\geq 2\varepsilon\right\}=0.$$

Thus,

$$\liminf_{u\to\infty}\{\|\beta_{u_m}[Z_m(u_{m-1}+a_{u_m}s)-Z_m(u_{m-1})]-f(s)\|_\alpha\}=0, \tag{4.9}$$

$C_{r,p}$ - q.s. By (4.1), (4.2) and (4.9), Lemma 4.1 holds.

Case (II) $\limsup_{u\to\infty}\frac{\log ua_u^{-1}}{\log\log u}=\infty$, see Lemma 4.2. Hence, the proof is completed.

Lemma 4.2. *If conditions* (H$_1$)-(H$_4$) *hold, for any* $f\in K$, *we have*

$$\lim_{u\to\infty}\inf_{t\in[0,1-\frac{a_u}{u}]}\|\beta_u\Delta(t,u)-f(s)\|_\alpha=0, \qquad C_{r,p}-q.s.$$

Proof. Since $\lim_{u\to\infty}\frac{\log ua_u^{-1}}{\log\log u}=\infty$, we choose u_n, such that $\frac{u_n}{a_{u_n}}=n^d,\ d>1$. Let $t_i=ia_{u_{n+1}},\ i=0,1,...,k_n=\left[\frac{u_n}{a_{u_{n+1}}}\right]-1,\ \ g(n)=\frac{\log\frac{u_n}{a_{u_n}}}{\log\log u_n}=\frac{\log n^d}{\log\log u_n}$. We have $u_n=\exp(n^{\frac{d}{g(n)}})$, $g(n)$ is non-decreasing and $g(n)\to\infty, n\to\infty$. Moreover, for any $b>0$, $\frac{n^b}{\log u_n}\to\infty$ and

$$1\leq\frac{u_{n+1}}{u_n}=\exp\left\{(n+1)^{\frac{d}{g(n+1)}}-n^{\frac{d}{g(n)}}\right\}\leq\exp\left(n^{\frac{d}{g(n)}-1}\right)\to 1,\ n\to\infty.$$

Therefore, we have

$$\begin{aligned}
&\inf_{t\in[0,1-\frac{a_u}{u}]}\|\beta_u\Delta(t,u)(\cdot)-f(\cdot)\|_\alpha\\
&=\inf_{x\in[0,u-a_u]}\|\beta_u[X(x+a_us)-X(x)]-f(s)\|_\alpha\\
&=\inf_{x\in[0,u-a_u]}\|\frac{\beta_u}{\beta_{u_{n+1}}}\Psi_{x,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}s)-f(s)\|_\alpha\\
&\leq\inf_{x\in[0,u_n-a_{u_{n+1}}]}\left\|\Psi_{x,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}\cdot)-f(\frac{a_u}{a_{u_{n+1}}}\cdot)\right\|_\alpha\\
&+\left(\frac{\beta_u}{\beta_{u_{n+1}}}-1\right)\sup_{x\in[0,u_n-a_{u_{n+1}}]}\left\|\Psi_{x,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}\cdot)\right\|_\alpha\\
&+\left\|f(\frac{a_u}{a_{u_{n+1}}}\cdot)-f(\cdot)\right\|_\alpha:=I_5+I_2'+I_3'.
\end{aligned}\tag{4.10}$$

Similarly to the proof of (3.3), we have

$$\lim_{n\to\infty}(I_2' + I_3') = 0, \qquad C_{r,p} - q.s. \tag{4.11}$$

On the other hand,

(4.12)

$$\begin{aligned}
&C_{r,p}\{I_5 \geq 4\varepsilon\} = C_{r,p}\{\inf_{x\in[0,u_n-a_{u_{n+1}}]} \left\|\Psi_{x,u_{n+1}}(\cdot) - f(\cdot)\right\|_\alpha \geq 4\varepsilon\} \\
&= C_{r,p}\{\inf_{x\in[0,u_n-a_{u_{n+1}}]} \left\|\beta_{u_{n+1}}[X(x + a_{u_{n+1}}\cdot) - X(x)] - f(\cdot)\right\|_\alpha \geq 4\varepsilon\} \\
&\leq C_{r,p}\{\min_{x_i\in[0,u_n-a_{u_{n+1}}]} \left\|\beta_{u_{n+1}}[Z_i(x_i + a_{u_{n+1}}\cdot) - Z_i(x_i)] - f(\cdot)\right\|_\alpha \geq 2\varepsilon\} \\
&+ \leq C_{r,p}\{\max_{x_i\in[0,u_n-a_{u_{n+1}}]} \left\|\beta_{u_{n+1}}[X_i(x_i + a_{u_{n+1}}\cdot) - X_i(x_i)] - f(\cdot)\right\|_\alpha \geq 2\varepsilon\} \\
&:= I_{51} + I_{52}.
\end{aligned}$$

By Lemmas 2.2-2.4, we have

$$\begin{aligned}
I_{52} &\leq \sum_{i=0}^{[\frac{u_n}{a_{u_{n+1}}}]-1} C_{r,p}\left\{\|\beta_{u_n+1}[X_i(x_i + a_{u_{n+1}}s) - X_i(x_i)]\|_\alpha \geq 2\varepsilon\right\} \\
&= \sum_{i=0}^{[\frac{u_n}{a_{u_{n+1}}}]-1} C_{r,p}\left\{\left\|\frac{1}{\sqrt{2s_i^{2\gamma}\ell_{u_{n+1}}}}\frac{Y_i(s_i, x_i + a_{u_{n+1}}\cdot) - Y_i(s_i, x_i)}{\sqrt{a_{u_{n+1}}^{2\gamma}}}\right\|_\alpha \geq 2\varepsilon\right\} \\
&\leq C\varepsilon^{(-2k^2-k)p} \sum_{i=0}^{[\frac{u_n}{a_{u_{n+1}}}]-1} \mu\left(\left\|\frac{1}{\sqrt{2s_i^{2\gamma}\ell_{u_{n+1}}}}\frac{Y_i(s_i, x_i + a_{u_{n+1}}\cdot) - Y_i(s_i, x_i)}{\sqrt{a_{u_{n+1}}^{2\gamma}}}\right\|_\alpha \geq \varepsilon\right)^{\frac{p}{q_2}} \\
&\leq C_1 \sum_{i=0}^{[\frac{u_n}{a_{u_{n+1}}}]-1} \mu\left(\left\|\frac{1}{\sqrt{2s_i^{2\gamma}\ell_{u_{n+1}}}}[Y_i(s_i, i + \cdot) - Y_i(s_i, i)]\right\|_\alpha \geq \varepsilon\right)^{\frac{p}{q_2}} \\
&\leq C_1 \sum_{i=0}^{[\frac{u_n}{a_{u_{n+1}}}]-1} \mu\left(\sup_{0\leq x<y\leq 1}\left|\frac{Y_i(s_i, y) - Y_i(s_i, x)}{(y-x)^q\sqrt{Cs_i^{2\gamma}i^{-\delta}}}\right| \geq \varepsilon\sqrt{\frac{i^\delta}{C}2\ell_{u_{n+1}}}\right)^{\frac{p}{q_2}} \\
&\leq C_1 \sum_{i=0}^{[\frac{u_n}{a_{u_n}}]-1} \exp\left(-\frac{p}{q_2}\frac{2\varepsilon^2 i^\delta}{(2+\epsilon)C}\log\frac{u_{n+1}\log u_{n+1}}{a_{u_{n+1}}}\right) \\
&\leq C_1 \sum_{i=n_0}^{[\frac{u_n}{a_{u_{n+1}}}]-1} \left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_2 i^\delta} \leq C_1\frac{u_{n+1}}{a_{u_{n+1}}}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_2 {n_0}^\delta} \\
&\leq C_1\,(n+1)^{-d(C_2 n_0^\delta - 1) - C_2 {n_0}^\delta}.
\end{aligned}$$

where

$$C_1 = C\varepsilon^{(-2k^2-k)p} > 0, \quad C_2 = \frac{p}{q_2}\frac{\varepsilon^2}{(2+\epsilon)C} > 0.$$

Taking n_0 to be large enough such that $d(C_2 n_0^\delta - 1) + C_2 {n_0}^\delta > 1$, thus

$$\sum_{n=1}^{\infty} I_{52} < \infty.$$

By the Borel-Cantelli Lemma, we have

$$\limsup_{n\to\infty} \max_{x_i\in[0,u_n-a_{u_{n+1}}]} \left\|\beta_{u_{n+1}}[X_i(x_i + a_{u_{n+1}}\cdot) - X_i(x_i)]\right\|_\alpha = 0, \tag{4.13}$$

$C_{r,p}$ - q.s. Since $\{Z_k(\cdot)\}, k = 1, 2, \cdots$ are independent, by Lemma 2.2, we have

(4.14)

$$\begin{aligned}
I_{51} &= C_{r,p}\{\min_{x_i\in[0,u_n-a_{u_{n+1}}]} \left\|\beta_{u_{n+1}}[Z_i(x_i + a_{u_{n+1}}\cdot) - Z_i(x_i)] - f(\cdot)\right\|_\alpha \geq 2\varepsilon\} \\
&= C_{r,p}\{\bigcap_{i=0}^{k_n} \left\|\beta_{u_{n+1}}[Z_i(x_i + a_{u_{n+1}}\cdot) - Z_i(x_i)] - f(\cdot)\right\|_\alpha \geq 2\varepsilon\} \\
&\leq C(k_n+1)^{rp}\varepsilon^{(-2r^2-r)p} \prod_{i=0}^{k_n} \mu\big(\|\beta_{u_{n+1}}[Z_i(x_i + a_{u_{n+1}}\cdot) - Z_i(x_i)] - f(\cdot)\|_\alpha \geq \varepsilon\big)^{\frac{p}{q_2}} \\
&:= C_1(k_n+1)^{rp} \prod_{i=0}^{k_n} L_i^{\frac{p}{q_2}}. \left(C_1 = C\varepsilon^{(-2r^2-r)p} > 0\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
L_i &= \mu\big(\|\beta_{u_{n+1}}[Z_i(x_i + a_{u_{n+1}}\cdot) - Z_i(x_i)] - f(\cdot)\|_\alpha \geq \varepsilon\big) \\
&\leq \mu\left(\|\beta_{u_{n+1}}[X(x_i + a_{u_{n+1}}\cdot) - X(x_i)] - f(\cdot)\|_\alpha \geq \frac{1}{2}\varepsilon\right) \\
&+ \mu\left(\|\beta_{u_{n+1}}[X_i(x_i + a_{u_{n+1}}\cdot) - X_i(x_i)] - f(\cdot)\|_\alpha \geq \frac{1}{2}\varepsilon\right) \\
&:= L_i^1 + L_i^2.
\end{aligned} \tag{4.15}$$

By the same method of the estimate of I_{52}, there exist $C_{21} > 0,\ C_{22} > 0$ such that

$$L_i^2 \leq C_{21}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{22}i^\delta}. \tag{4.16}$$

By the same method of the estimate of J_m^1, there exist $C>0,\ C_{11}>0,\ C_{12}>0$ such that

$$
\begin{aligned}
L_i^1 &\le \exp\left\{-\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{11}}\left(1-C\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{12}}\right)\right\}\\
&\le \exp\left\{-\frac{1}{2}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{11}}\right\}.
\end{aligned} \tag{4.17}
$$

Therefore, by (4.14)-(4.17), we have

$$
\begin{aligned}
&I_{51}=C_1(k_n+1)^{rp}\prod_{i=0}^{k_n}L_i^{\frac{p}{q_2}}\le C_1(k_n+1)^{rp}\times\\
&\times\prod_{i=0}^{k_n}\left[\exp\left\{-\frac{1}{2}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{11}}\right\}+C_{21}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{22}i^{\delta}}\right]^{\frac{p}{q_2}}\\
&\le C_1(k_n+1)^{rp}\prod_{i=0}^{k_n}\left[\exp\left\{-\frac{1}{4}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{11}}\right\}\right]^{\frac{p}{q_2}}\\
&\le C_1n^{drp}\exp\left\{-\frac{p}{4q_2}\frac{u_n}{a_{u_{n+1}}}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{11}}\right\}.
\end{aligned}
$$

Take a appropriate d, such that

$$
\sum_{n=1}^{\infty}I_{51}\le\sum_{n=1}^{\infty}C_1n^{drp}\exp\left\{-\frac{p}{4q_2}\frac{u_n}{a_{u_{n+1}}}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{11}}\right\}<\infty.
$$

By Borel-Cantelli Lemma, we have

$$
\limsup_{n\to\infty}\min_{x_i\in[0,u_n-a_{u_{n+1}}]}\left\|\beta_{u_{n+1}}[Z_i(x_i+a_{u_{n+1}}\cdot)-Z_i(x_i)]-f(\cdot)\right\|_{\alpha}=0, \tag{4.18}
$$

$C_{r,p}$-q.s. By (4.10)-(4.13) and (4.18), the proof of Lemma *4.2* is completed.

Below, we prove Theorem 1.2. Obviously, by Lemmas 4.1-4.2, (1.2) holds. Moreover, by Lemma 4.2, (1.3) also holds.

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