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CONJUGATE TRANSFORMS ON DYADIC GROUP

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Abstract. In this paper we study the properties of the Lebesgue constant of the conjugate transforms. For conjugate Fejér means we will find necessary and sufficient condition on t for which the estimation $E\left|\widetilde{\sigma}_n^{(t)}f\right| \leq cE\,|f|$ holds. We also prove that for dyadic irrational t, $L\log L$ is the maximal Orlicz space for which the estimation $E\left|\widetilde{\sigma}_n^{(t)}f\right| \leq c_1 + c_2 E\left(|f|\log^{+}|f|\right)$ is valid.

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1. Dyadic Hardy spaces and conjugate transforms

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N}$:=$\mathbb{P}\cup\{0\}$, the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval $\mathbb{I} := [0,1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p,n \in \mathbb{N}$, $0 \leq p < 2^n$.

Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0,1\}\,(k \in \mathbb{N})$. The group operation on G is the coordinate-wise addition, the measure (denote by μ) and the topology are the product measure and topology. The compact Abelian group G is called the dyadic group. A base for the neighborhoods of G can be given in the following way:

$$\begin{aligned} I_0(x) \quad &: \quad = G,\;\; I_n(x) := I_n(x_0,...,x_{n-1}) \\ &: \quad = \{y \in G :\; y = (x_0,...,x_{n-1},y_n,y_{n+1},...)\}\,,\; (x \in G, n \in \mathbb{N})\,. \end{aligned}$$

These sets are called the dyadic intervals. Let $0 = (0 : i \in \mathbb{N}) \in G$ denote the null element of G, $I_n := I_n(0)\ (n \in \mathbb{N})$. For every finite set E the number of elents in E we denote by $|E|$, i. e. $|E| := (E)^{\#}$.

For $k \in \mathbb{N}$ and $x \in G$ denote $r_k(x) := (-1)^{x_k}$, the k-th Rademacher function on dyadic group G.

Denote the dyadic expansion of $t \in \mathbb{I}$ by

$$t = \sum_{j=0}^{\infty} \frac{t_j}{2^{j+1}},\ t_j = 0, 1.$$

In the case of $t \in \mathbb{Q}$ choose the expansion which terminates in zeros. For $t \in \mathbb{I}$ we denote $\rho(t) := (t_0, t_1, ...) \in G$.

The σ-algebra generated by the dyadic intervals $\{I_n(x) : x \in G\}$ is denoted by A^n, more precisely, $A^n := \sigma\{I_n(x) : x \in G\}$. The expectation and the conditional expectation operators relative to A^n $(n \in \mathbb{N})$ are denoted by E and E_n, respectively.

The norm (or quasinorm) of the space L_p is defined by

$$\|f\|_p := (E|f|^p)^{1/p} \quad (0 < p < +\infty).$$

Denote by $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ martingale with respect to $(A^n, n \in \mathbb{N})$ (for details see, e. g. [18, 19]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n\in\mathbb{N}} \left|f^{(n)}\right|.$$

In case of $f \in L_1$, the maximal function can also be given by

$$f^* = \sup_{n\in\mathbb{N}} |E_n f|.$$

For $0 < p < \infty$ the Hardy martingale space H_p consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

For a martingale

$$f \sim \sum_{n=0}^{\infty} \left(f^{(n)} - f^{(n-1)}\right),\ f^{(-1)} = 0,$$

the conjugate transforms are defined by

$$\widetilde{f}^{(t)} \sim \sum_{n=0}^{\infty} r_n(\rho(t)) \left(f^{(n)} - f^{(n-1)}\right),$$

where $t \in \mathbb{I}$ is fixed.

Note that $\widetilde{f}^{(0)} = f$. As is well known, if f is an integrable function, then conjugate transforms $\widetilde{f}^{(t)}$ do exist almost everywhere, but they are not integrable in general.

The following equation holds ([18, 19])

$$\left\|\widetilde{f}^{(t)}\right\|_{H_p} = \|f\|_{H_p} \quad (0 < p < \infty, t \in \mathbb{I}).$$

Furthermore, Khintchin's inequality implies that

$$\|f\|_{H_p}^p \sim \int_{\mathbb{I}} \left\|\widetilde{f}^{(t)}\right\|_{H_p}^p dt \qquad (0 < p < \infty).$$

Let $Q(L) = Q(L)(\mathbb{I})$ be the Orlicz space [9] generated by the Young function Q, i.e. Q is convex continuous even function such that $Q(0) = 0$ and

$$\lim_{u\to+\infty} \frac{Q(u)}{u} = +\infty, \quad \lim_{u\to 0} \frac{Q(u)}{u} = 0.$$

This space is endowed with the norm

$$\|f\|_{Q(L)(\mathbb{I})} = \inf\{k > 0 : \int_{\mathbb{I}} Q(|f|/k) \leq 1\}.$$

In particular, if $Q(u) = u\log(1+u), u > 0$ then the corresponding space will be denoted by $L\log^+ L(\mathbb{I})$.

2. Walsh system and conjugate Fejйr means

Let $m \in \mathbb{N}$, then $m = \sum_{i=0}^{\infty} m_i 2^i$, where $m_i \in \{0,1\}$ $(i \in \mathbb{N})$, i.e. m is expressed in the number system of base 2. Denote $|m| := \max\{j \in \mathbb{N}: m_j \neq 0\}$, that is, $2^{|m|} \leq m < 2^{|m|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_m(x) := \prod_{k=0}^{\infty} (r_k(x))^{m_k} = r_{|m|}(x)(-1)^{\sum_{k=0}^{|m|-1} m_k x_k} \quad (x \in G, m \in \mathbb{P}).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that ([14], [8])

$$D_{2^n}(x) = \begin{cases} 2^n, \text{if } x \in I_n, \\ 0, \ \text{ if } x \in G\backslash I_n, \end{cases} \tag{2.1}$$

and

$$D_n(x) = w_n(x) \sum_{k=0}^{\infty} n_k \left(D_{2^{k+1}}(x) - D_{2^k}(x)\right). \tag{2.2}$$

Let $x \in I_j \backslash I_{j+1}$. Then from (2.1) and (2.2) we have

$$D_n(x) = w_n(x) \left(\sum_{k=0}^{j-1} n_k 2^k - n_j 2^j \right).$$

Hence,

$$|D_n(x)| = \alpha_j(n),$$

where

$$\alpha_j(n) = \left| \sum_{k=0}^{j-1} n_k 2^k - n_j 2^j \right|.$$

The partial sums of the Walsh-Fourier series are defined as follows:

$$S_M f := \sum_{i=0}^{M-1} \widehat{f}(i) w_i,$$

where the number $\widehat{f}(i) = E(fw_i)$ is said to be the ith Walsh-Fourier coefficient of the function f. It is easy to see that $E_n(f) = S_{2^n}(f)$.

For any given $n \in \mathbb{N}$ it is possible to write n uniquely as $n = \sum_{k=0}^{\infty} n_k 2^k$, where $n_k = 0$ or 1 for $k \in \mathbb{N}$. This expression will be called the binary expansion of n and the numbers n_k will be called the binary coefficients of n.

Define the variation of an $n \in \mathbb{N}$ with binary coefficients $(n_k : k \in \mathbb{N})$ by

$$V(n) := \sum_{k=1}^{\infty} |n_k - n_{k-1}| + n_0.$$

The Fejér means of Walsh-Fourier series is defined as follows

$$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f \qquad (n \in \mathbb{P}).$$

If $f \in L_1$ then it is easy to show that the sequence $(E_n(f) : n \in \mathbb{N})$ is a martingale. If f is a martingale, that is $f = (f^{(n)} : n \in \mathbb{N})$ then the Walsh-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i) = \lim_{k \to \infty} E\left(f^{(k)} w_i\right). \tag{2.3}$$

The Walsh-Fourier coefficients of $f \in L_1$ are the same as the ones of the martingale $(E_n(f) : n \in \mathbb{N})$ obtained from f.

Let

$$\beta_0(t) := r_0(\rho(t)), \beta_k(t) := r_n(\rho(t)) \text{ if } 2^{n-1} \le k < 2^n.$$

Then the nth partial sums of the conjugate transforms is given by

$$\widetilde{S}_n^{(t)} f := \sum_{k=0}^{n-1} \beta_k(t) \widehat{f}(k) w_k \quad (t \in \mathbb{I}, n \in \mathbb{P}).$$

Let $2^N \le n < 2^{N+1}$. Then we have

$$\begin{aligned}
\widetilde{S}_n^{(t)} f &= r_0(\rho(t)) \widehat{f}(0) w_0 + \sum_{l=1}^{N} r_l(\rho(t)) (E_l f - E_{l-1} f) \\
&\quad + r_{N+1}(\rho(t)) (S_n f - E_N f) \\
&= (-1)^{t_0} E f + \sum_{l=1}^{N} r_l(\rho(t)) (E_l f - E_{l-1} f) \\
&\quad + r_{N+1}(\rho(t)) (S_n f - E_N f) = f * \widetilde{D}_n^{(t)},
\end{aligned}$$

where

$$\widetilde{D}_n^{(t)}=\sum_{i=-1}^{N-1}(-1)^{t_{i+1}}\left(D_{2^{i+1}}-D_{2^i}\right)+(-1)^{t_{N+1}}\left(D_n-D_{2^N}\right), D_{2^{-1}}=0.$$

It is easy to see that $(-1)^{t_l}=1-2t_l$. Then for $\widetilde{D}_n^{(t)}$ we can write

$$\begin{aligned}\widetilde{D}_n^{(t)} &= \sum_{i=-1}^{N-1}\left(1-2t_{i+1}\right)\left(D_{2^{i+1}}-D_{2^i}\right)+\left(1-2t_{N+1}\right)\left(D_n-D_{2^N}\right)\\ &= D_n-2\sum_{i=-1}^{N-1}t_{i+1}\left(D_{2^{i+1}}-D_{2^i}\right)-2t_{N+1}\left(D_n-D_{2^N}\right).\end{aligned}$$

Set

$$m:=\sum_{i=0}^{N-1}t_{i+1}2^i<2^N.$$

Then from (2.2) we get

$$\widetilde{D}_n^{(t)}=D_n-2w_mD_m-2t_{N+1}\left(D_n-D_{2^N}\right)-2t_0.$$

The conjugate $(C,1)$-means of a martingale f are introduced by

$$\widetilde{\sigma}_n^{(t)}f:=\frac{1}{n}\sum_{k=0}^{n-1}\widetilde{S}_k^{(t)}f \qquad \left(t\in\mathbb{I},n\in\mathbb{P}\right).$$

The notiation $a\lesssim b$ in the proofs stands for $a<c\cdot b$, where c is an absolute constant.

3. Two sides estimation of Lebesgue constant of conjugate Walsh-Fourier series

Denote by L_n the lebesgue constants of the Walsh system:

$$L_n:=\int\limits_G\left|D_n\right|d\mu.$$

These constants were studied by many authors. For the trigonometric system it is important to note that results of Fejйr and Szegx, the letter gave in [16] an explicit formula for Lebesgue constants, namely,

$$L_n=\frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2k\left(2n+1\right)-1}\right).$$

Along with the trigonometric system the Walsh-Paley system is also severely studied for its importance in applications. The most properties of Lebesgue constants with respect to the Walsh-Paley system were obtained by Fine in [2]. In ([14], p. 34), the two-sided estimate

$$\frac{V\left(n\right)}{8}\leq L_n\leq V\left(n\right) \tag{3.1}$$

is proved. In [10], Lukomskii presented the estimate $L_n \geq V(n)/4$. Malykhin, Telyakovskii and Kholshchevnikova [11] improved the estimation (3.1) and proved the following

Theorem MTK. *For any positive integer n, the two-sided inequality*

$$\frac{V(n)+1}{3} \leq L_n < V(n)$$

is valid. Here the factors $1/3$ and 1 are sharp.

We would like to mention the work of Astashkin and Semenov [1] in which the sharp two-sided estimate for Lebesgue constants with respect to Walsh-Paley system are obtained.

The Walsh- Fejér kernels were studied by Toledo [17], in particular the following is proved

$$\sup\{\|K_n\|_1 : n \in \mathbb{P}\} = \frac{17}{15}.$$

Denote by $L_n^{(t)}$ the lebesgue constants of the conjugate transforms:

$$L_n^{(t)} := \int\limits_G \left| \widetilde{D}_n^{(t)} + 2t_0 \right| d\mu.$$

For $n, m \in \mathbb{N}$ and $2^N \leq n < 2^{N+1}$ we define

$$T(n,m) := \{i : n_i \neq n_{i-1}, m_i = m_{i-1}, i = 0, 1, ..., N-1\}.$$

In this section we study the properties of the Lebesgue constant of the conjugate transforms.

Theorem 3.1. *Let n is positive integer and*

$$m = \sum_{i=0}^{N-1} t_{i+1} 2^i, \quad 2^N \leq n < 2^{N+1}$$

and $t \in \mathbb{I}$. The two-sides inequality

$$\max\left(\frac{1}{2}|T(n,m)| + \frac{1}{3}V(n) - 1, \frac{1}{4}|T(n,m)| + \frac{2}{3}V(m)) - 1\right)$$
$$\leq \quad L_n^{(t)} \leq 2V(m) + |T(n,m)| + 2$$

is valid.

Proof. We have

$$(3.2) \qquad L_n^{(t)} = \sum_{i=0}^{N-1} \int\limits_{I_i \backslash I_{i+1}} \left| \widetilde{D}_n^{(t)} + 2t_0 \right| d\mu + \int\limits_{I_N \backslash I_{N+1}} \left| \widetilde{D}_n^{(t)} + 2t_0 \right| d\mu + + \int\limits_{I_{N+1}} \left| \widetilde{D}_n^{(t)} + 2t_0 \right| d\mu := J_1 + J_2 + J_3.$$

From (2.1), it is easy to see that

$$\int\limits_{I_i \setminus I_{i+1}} \left| \widetilde{D}_n^{(t)} + 2t_0 \right| d\mu$$

$$= \sum_{x_{i+1}=0}^{1} \cdots \sum_{x_N=0}^{1} \int\limits_{I_{N+1}(0,\ldots,0,x_i=1,x_{i+1},\ldots,x_N)} \left| \widetilde{D}_n^{(t)} + 2t_0 \right| d\mu$$

$$= \sum_{x_{i+1}=0}^{1} \cdots \sum_{x_N=0}^{1} \int\limits_{I_{N+1}(0,\ldots,0,x_i=1,x_{i+1},\ldots,x_{N-1},x_N)} \left| (1-2t_{N+1}) D_n - 2w_m D_m \right| d\mu$$

$$= \sum_{x_{i+1}=0}^{1} \cdots \sum_{x_{N-1}=0}^{1} \left(\int\limits_{I_{N+1}(0,\ldots,0,x_i=1,x_{i+1},\ldots,x_{N-1},0)} \left| (1-2t_{N+1}) D_n - 2w_m D_m \right| d\mu \right)$$

$$+ \sum_{x_{i+1}=0}^{1} \cdots \sum_{x_{N-1}=0}^{1} \left(\int\limits_{I_{N+1}(0,\ldots,0,x_i=1,x_{i+1},\ldots,x_{N-1},1)} \left| (1-2t_{N+1}) D_n - 2w_m D_m \right| d\mu \right)$$

$$= \sum_{x_{i+1}=0}^{1} \cdots \sum_{x_{N-1}=0}^{1} \left(\int\limits_{I_{N+1}(0,\ldots,0,x_i=1,x_{i+1},\ldots x_{N-1},0)} \left(\left| (1-2t_{N+1}) D_n - 2w_m D_m \right| + \right. \right.$$

$$\left. \left. \left| (1-2t_{N+1}) D_n + 2w_m D_m \right| \right) d\mu \right).$$

Since

$$\frac{|a-b| + |a+b|}{2} = \max\{|a|, |b|\}$$

we have

$$\begin{aligned} \int\limits_{I_i \setminus I_{i+1}} \left| \widetilde{D}_n^{(t)} + 2t_0 \right| d\mu &= \int\limits_{I_i \setminus I_{i+1}} \max\{\alpha_i(n), 2\alpha_i(m)\} d\mu \\ &= \frac{\max\{\alpha_i(n), 2\alpha_i(m)\}}{2^{i+1}}. \end{aligned}$$

Hence

$$J_1 = \sum_{i=0}^{N-1} \frac{\max\{\alpha_i(n), 2\alpha_i(m)\}}{2^{i+1}}. \tag{3.3}$$

For J_2 and J_3 we can write

$$\begin{aligned} J_2 &= \int\limits_{I_N \setminus I_{N+1}} \left| D_n - 2w_m D_m - 2t_{N+1}(D_n - D_{2^N}) \right| d\mu \\ &= \frac{\left| 2^N - n' - 2m - 2t_{N+1}\left(2^N - n' - 2^N\right) \right|}{2^{N+1}} \\ &= \frac{\left| 2^N - n' - 2m + 2t_{N+1} n' \right|}{2^{N+1}}, \end{aligned} \tag{3.4}$$

where $n=2^N+n', n^{'}<2^N$ and

$$J_3 = \int\limits_{I_{N+1}} \left|D_n - 2w_m D_m - 2t_{N+1}\left(D_n - D_{2^N}\right)\right| d\mu = \frac{\left|n-2m-2t_{N+1}\left(n-2^N\right)\right|}{2^{N+1}}. \tag{3.5}$$

Combining (3.2)-(3.5) we conclude that

$$\left\|\widetilde{D}_n^{(t)}+2t_0\right\|_1 = \sum_{i=0}^{N-1}\frac{\max\left\{\alpha_i\left(n\right),2\alpha_i\left(m\right)\right\}}{2^{i+1}}$$

$$+\frac{\left|2^{N+1}-n-2m+2t_{N+1}\left(n-2^N\right)\right|}{2^{N+1}}+\frac{\left|n-2m-2t_{N+1}\left(n-2^N\right)\right|}{2^{N+1}}.$$

Set $A(n)=\{i:|n_i-n_{i-1}|=1)$ and

$$S\left(n\right) := \sum_{i\in A(n)}\frac{\alpha_i\left(n\right)}{2^{i+1}}.$$

First, we prove that $S(n)\geq S(n(e))$, where

$$n(e)=n_N 2^N+...+n_{e+1}2^{e+1}+n_{e-1}2^e+....+n_0 2^1$$

and $n_e=n_{e-1}\neq n_{e+1}$. Set $A_e(n)=\{i:i\in A(n), i>e\}$. Then,

$$S(n)-S(n(e)) = \sum_{i\in A_e(n)}\left(\frac{\alpha_i\left(n\right)}{2^{i+1}}-\frac{\alpha_i\left(n(e)\right)}{2^{i+1}}\right) + \sum_{i\in A(n)\backslash A_e(n)}\left(\frac{\alpha_i\left(n\right)}{2^{i+1}}-\frac{\alpha_{i+1}\left(n(e)\right)}{2^{i+2}}\right).$$

Since

$$\frac{\alpha_i\left(n\right)}{2^{i+1}}-\frac{\alpha_{i+1}\left(n(e)\right)}{2^{i+2}}=0,\quad i\in A(n)\backslash A_e(n),$$

we get

$$S(n)-S(n(e))=\sum_{i\in A_e(n)}\left(\frac{\alpha_i\left(n\right)}{2^{i+1}}-\frac{\alpha_i\left(n(e)\right)}{2^{i+1}}\right).$$

For $i>e$ we can write

$$\alpha_i\left(n\right)-\alpha_i\left(n(e)\right)=\left|n_0 2^0+\cdots+n_{i-1}2^{i-1}-n_i 2^i\right| - \left|n\left(e\right)_0 2^0+\cdots+n\left(e\right)_{i-1}2^{i-1}-n\left(e\right)_i 2^i\right|$$

$$= \left|n_0 2^0+\cdots+n_{i-1}2^{i-1}-n_i 2^i\right| - \left|n_0 2^1+\cdots+n_{e-1}2^e+n_{e+1}2^{e+1}+\cdots+n_{i-1}2^{i-1}-n_i 2^i\right|.$$

Suppose that $n_i = 1$. Then we can write

$$\begin{aligned} &\alpha_i(n) - \alpha_i(n(e)) = 2^i - n_0 2^0 - \cdots - n_{i-1}2^{i-1} \\ &-(2^i - n_0 2^1 - \cdots - n_{e-1}2^e - n_{e+1}2^{e+1} - \cdots - n_{i-1}2^{i-1}) \\ = &\sum_{j=0}^{e-1} n_j 2^j - n_e 2^e. \end{aligned}$$

Let now consider the case when $n_i = 0$. Then we have

$$\alpha_i(n) - \alpha_i(n(e)) = -\left(\sum_{j=0}^{e-1} n_j 2^j - n_e 2^e\right).$$

So, we get following

$$\alpha_i(n) - \alpha_i(n(e)) = (2|n_i - n_e| - 1)\left|\sum_{j=0}^{e-1} n_j 2^j - 2^e n_e\right|.$$

And finally, since $n_{e+1} \neq n_e$, we get

$$\begin{aligned} S(n) - S(n(e)) = \left|\sum_{j=0}^{e-1} n_j 2^j - 2^e n_e\right| \sum_{i \in A_e(n)} \frac{(2|n_i - n_e| - 1)}{2^{i+1}} \\ > \frac{1}{2^{e+2}} - \sum_{i=e+2}^{\infty} \frac{1}{2^{i+1}} \geq 0. \end{aligned}$$

From the definition of function $V(n)$ it is easy to see that$V(n) = V(n(e))$. If we continue this process it is easy to see that we will get

$$n' = \sum_{i=0}^{|n'|-1} n_i' 2^i,$$

where, for $0 \leq i < |n'|$ $\quad n_i' + n_{i+1}' = 1$. First, we suppose that $n_0' = 0$. Set $n' = (010101...)$. Then we can write $V(n) = V(n')$ and $S(n) \geq S(n')$. Now, we calculate $S(n')$. We suppose that $V(n') = 2s$. It is easy to see that

$$\alpha_{2m}(n') = 2^1 + 2^3 + \cdots + 2^{2m-1} = \frac{2^{2m+1} - 2}{3}$$

and

$$\alpha_{2m-1}(n') = \left|2^1 + 2^3 + \cdots + 2^{2m-3} - 2^{2m-1}\right| = \frac{2^{2m} + 2}{3}.$$

Then we can write

$$S(n') = \sum_{m=1}^{s} \frac{\alpha_{2m}(n')}{2^{2m+1}} + \sum_{m=1}^{s} \frac{\alpha_{2m-1}(n')}{2^{2m}} \tag{3.6}$$
$$= \sum_{m=1}^{s} \frac{1}{2^{2m+1}} \frac{2^{2m+1}-2}{3} + \sum_{m=1}^{s} \frac{1}{2^{2m}} \frac{2^{2m}+2}{3}$$
$$= \frac{2}{3}s - \frac{2}{3}\sum_{m=1}^{s} \frac{1}{2^{2m+1}} + \frac{2}{3}\sum_{m=1}^{s} \frac{1}{2^{2m}}$$
$$= \frac{2}{3}s + \frac{1}{9}\left(1 - \frac{1}{2^{2s}}\right).$$

Now, we suppose that $n'_0 = 1$. Set $n' = (101010...)$. Then we can write

$$\alpha_{2m}(n') = \left|2^0 + 2^2 + \cdots + 2^{2m-2} - 2^{2m}\right| = \frac{2^{2m+1}+1}{3}$$

and

$$\alpha_{2m-1}(n') = 2^0 + 2^2 + \cdots + 2^{2m-2} = \frac{2^{2m}-1}{3}.$$

Hence, we have

$$S(n') = \sum_{m=0}^{s} \frac{\alpha_{2m}(n')}{2^{2m+1}} + \sum_{m=1}^{s} \frac{\alpha_{2m-1}(n')}{2^{2m}} \tag{3.7}$$
$$= \frac{2}{3}s + \frac{1}{2} - \frac{1}{18}\left(1 - \frac{1}{2^{2s}}\right) > \frac{2}{3}s.$$

Combining (3.6) and (3.7), we conclude

$$\frac{S(n)}{V(n)} \geq \frac{S(n')}{V(n')} > \frac{1}{3}. \tag{3.8}$$

Since $T(n,m) \cap A(m) = \varnothing$, we can write

$$\sum_{i=0}^{N-1} \frac{\max\{\alpha_i(n), 2\alpha_i(m)\}}{2^{i+1}} \tag{3.9}$$
$$\geqslant \sum_{i\in A(m)\setminus\{N\}} \frac{2\alpha_i(m)}{2^{i+1}} + \sum_{i\in T(n,m)} \frac{\alpha_i(n)}{2^{i+1}}$$
$$= \sum_{i\in A(m)} \frac{2\alpha_i(m)}{2^{i+1}} + \sum_{i\in T(n,m)} \frac{\alpha_i(n)}{2^{i+1}} - \frac{2\alpha_N(m)}{2^{N+1}}.$$

From (3.8) we have

$$\sum_{i\in A(m)} \frac{2\alpha_i(m)}{2^{i+1}} = 2S(m) > \frac{2}{3}V(m). \tag{3.10}$$

It is easy to see that if $n_i \neq n_{i-1}$, then $\alpha_i(n) \geq 2^{i-1}$. So, we have

$$\sum_{i\in T(n,m)} \frac{\alpha_i(n)}{2^{i+1}} \geq \frac{1}{4}\left|T(n,m)\right|. \tag{3.11}$$

Combining (3.9)-(3.11) we have

$$\sum_{i=0}^{N-1}\frac{\max\left\{\alpha_i\left(n\right),2\alpha_i\left(m\right)\right\}}{2^{i+1}}\geq\frac{2}{3}V(m)+\frac{1}{4}\left|T(n,m)\right|-1. \tag{3.12}$$

On the other hand, we can write

$$\begin{aligned}&\sum_{i=0}^{N-1}\frac{\max\left\{\alpha_i\left(n\right),2\alpha_i\left(m\right)\right\}}{2^{i+1}} \\ \geqslant&\sum_{i\in A(n)\backslash\{N,N+1\}}\frac{\alpha_i\left(n\right)}{2^{i+1}}+\sum_{i\in T(m,n)}\frac{2\alpha_i\left(m\right)}{2^{i+1}} \\ \geq&\frac{1}{3}V\left(n\right)+\frac{1}{2}\left|T\left(m,n\right)\right|-\frac{\alpha_N\left(N\right)}{2^{N+1}}-\frac{\alpha_{N+1}\left(N\right)}{2^{N+2}} \\ \geq&\frac{1}{3}V\left(n\right)+\frac{1}{2}\left|T\left(m,n\right)\right|-1.\end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13) we have

$$\begin{aligned}&\max\left(\frac{1}{2}\left|T(n,m)\right|+\frac{1}{3}V(n)-1,\frac{1}{4}\left|T(n,m)\right|+\frac{2}{3}V(m))-1\right) \\ \leqslant&\sum_{i=0}^{N-1}\frac{\max\left\{\alpha_i\left(n\right),2\alpha_i\left(m\right)\right\}}{2^{i+1}}\leq L_n^{(t)}.\end{aligned} \tag{3.14}$$

Now, we prove upper estimation. First, we prove

$$\begin{aligned}&\sum_{i=0}^{N-1}\frac{\max\left\{\alpha_i\left(n\right),2\alpha_i\left(m\right)\right\}}{2^{i+1}} \\ \leq&\ 2\sum_{i\in A(m)\cup T(n,m)}\frac{\max\left\{\alpha_i\left(n\right),2\alpha_i\left(m\right)\right\}}{2^{i+1}}.\end{aligned} \tag{3.15}$$

Suppose that $A(m)\cup T(n,m)=\{r_{1<}r_2......<r_s\}$. Then

$$\sum_{i=0}^{N-1}\frac{\max\left\{\alpha_i\left(n\right),2\alpha_i\left(m\right)\right\}}{2^{i+1}}=\sum_{i=1}^{s-1}\sum_{j=r_i}^{r_{i+1}-1}\frac{\max\left\{\alpha_j\left(n\right),2\alpha_j\left(m\right)\right\}}{2^{j+1}}.$$

Let $n_i=n_{i+1}$ for some i. Then it is easy to see that

$$\alpha_{i+1}\left(n\right)=\left|2^{i+1}n_{i+1}-2^in_i..-2^0n_0\right|=\left|2^in_i-2^{i-1}n_{i-1}..-2^0n_0\right|=\alpha_i\left(n\right).$$

Consequently,

$$\sum_{j=r_i}^{r_{i+1}-1}\frac{\max\left\{\alpha_j\left(n\right),2\alpha_j\left(m\right)\right\}}{2^{j+1}}\leq\frac{\max\left\{\alpha_{r_i}\left(n\right),2\alpha_{r_i}\left(m\right)\right\}}{2^{r_i}}.$$

Hence (3.15) is proved. Since $\alpha_i(n) \leq 2^i \quad (i \in \mathbb{N})$ and $\alpha_i(n) \leq 2^{i-1} \quad (i \notin A(n))$ from (3.15) we have

$$\sum_{i=0}^{N-1} \frac{\max\{\alpha_i(n), 2\alpha_i(m)\}}{2^{i+1}} \leq 2 \sum_{i \in A(m)} \frac{\max\{\alpha_i(n), 2\alpha_i(m)\}}{2^{i+1}} \tag{3.16}$$

$$+2 \sum_{i \in T(n,m)} \frac{\max\{\alpha_i(n), 2\alpha_i(m)\}}{2^{i+1}} \leq 2|A(m)| + |T(n,m)|.$$

It is easy to see that

$$\frac{\left|2^{N+1} - n - 2m + 2t_{N+1}\left(n - 2^N\right)\right|}{2^{N+1}} \tag{3.17}$$

$$+\frac{\left|n - 2m - 2t_{N+1}\left(n - 2^N\right)\right|}{2^{N+1}} \leq 2.$$

From (3.16) and (3.17)

$$L_n^{(t)} \leq 2V(m) + |T(n,m)| + 2. \tag{3.18}$$

Combining (3.14) and (3.18) we complete the proof of Theorem 3.1. □

4. Uniformly boundedness of conjugate Fejér means

The first result with respect to the a.e. convergence of the Walsh- Fejér means $\sigma_n f$ is due to Fine [3]. Later, Schipp [12] showed that the maximal operator $\sigma^* f := \sup_n |\sigma_n f|$ is of weak type (1, 1), from which the a. e. convergence follows by standard argument. Schipp's result implies by interpolation also the boundedness of $\sigma^* : L_p \to L_p$ $(1 < p \leq \infty)$. This fails to hold for $p = 1$ but Fujii [4] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1 (see also Simon [13]). Fujii's theorem was extended by Weisz [20]. Namely, he proved that the maximal operator $\sigma^* f$ and the conjugate maximal operator $\widetilde{\sigma}_*^{(t)}$ are bounded from the martingale Hardy space H_p to the space L_p for $p > 1/2$. Simon [15] gave a counterexample, which shows that this boundedness does not hold for $0 < p < 1/2$. In [7] (see also [6], [5]) the first author proved that the maximal operator $\widetilde{\sigma}_*^{(t)}$ is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

Weisz [20], [21] considered the norm convergence of conjugate Fejŭr means. In particular, the following is true

Theorem A (Weisz). *If* $t \in \mathbb{I}$ *then*

$$\left\|\widetilde{\sigma}_n^{(t)} f\right\|_{H_p} \leq c_p \|f\|_{H_p}, \quad (f \in H_p),$$

whenever $p > 1/2$.

Since $\|f\|_{H_1} \lesssim 1 + E\left(|f| \log^+ |f|\right)$ and $E|f| \leq \|f\|_{H_1}$, Theorem A implies that the following is true.

Theorem B. *Let $f \in L\log L$ and $t \in \mathbb{I}$. Then*

$$E\left|\widetilde{\sigma}_n^{(t)} f\right| \lesssim 1 + E\left(|f| \log^+ |f|\right). \tag{4.1}$$

On the other hand, for $t = 0$ we have following estimation.

Theorem C. *Let $f \in L_1$. Then*

$$E\left|\widetilde{\sigma}_n^{(t)} f\right| = E\left|\sigma_n f\right| \lesssim E\left|f\right|. \tag{4.2}$$

In this paper we find necessary and sufficient condition on t for which the estimation (4.2) holds for conjugate Fejér means. We also prove that for dyadic irrational t, $L\log L$ is maximal Orlicz space for which the estimation (4.1) is valid.

Theorem 4.1. *Let $t \in \mathbb{Q}$ and $f \in L_1$. Then*$E\left|\widetilde{\sigma}_n^{(t)} f\right| \lesssim E\left|f\right|$.

Theorem 4.2. *Let $t \notin \mathbb{Q}$ and $Q(L)$ be an Orlicz space for which*$Q(L) \nsubseteq L\log L$. *Then*

$$\sup_A \left\|\widetilde{\sigma}_{2^A}^{(t)}\right\|_{Q(L)\to L_1} = \infty.$$

Proof of Theorem 4.1. Let $2^A \le n < 2^{A+1}$. Then we can write

$$\widetilde{\sigma}_n^{(t)} f = \frac{1}{n} \sum_{m=1}^{A} \sum_{k=2^{m-1}}^{2^m-1} \widetilde{S}_k^{(t)} f + \frac{1}{n} \sum_{k=2^A}^{n-1} \widetilde{S}_k^{(t)} f. \tag{4.3}$$

Since for $2^{m-1} \le k < 2^m$ $\quad(E_{-1}f = 0)$

$$\begin{aligned} \widetilde{S}_k^{(t)} f &= r_0(\rho(t)) \widehat{f}(0) w_0 + \sum_{l=1}^{m-1} r_l(\rho(t)) (E_l f - E_{l-1} f) \\ &\quad + r_m(\rho(t)) (S_k f - E_{m-1} f) \\ &= \sum_{l=0}^{m-1} r_l(\rho(t)) (E_l f - E_{l-1} f) + r_m(\rho(t)) (S_k f - E_{m-1} f) \end{aligned} \tag{4.4}$$

from (4.3) we have

$$\begin{aligned} \widetilde{\sigma}_n^{(t)} f &= \frac{1}{n} \sum_{m=1}^{A} 2^{m-1} \sum_{l=0}^{m-1} r_l(\rho(t)) (E_l f - E_{l-1} f) \\ &\quad + \frac{1}{n} \sum_{m=1}^{A} r_m(\rho(t)) \sum_{k=2^{m-1}}^{2^m-1} (S_k f - E_{m-1} f) \\ &\quad + \frac{n-2^A}{n} \sum_{l=0}^{A} r_l(\rho(t)) (E_l f - E_{l-1} f) \\ &\quad + \frac{r_{A+1}(\rho(t))}{n} \sum_{k=2^A}^{n-1} (S_k f - E_{m-1} f) \\ &= \frac{1}{n} \sum_{m=1}^{A} 2^{m-1} \sum_{l=0}^{m-1} r_l(\rho(t)) (E_l f - E_{l-1} f) \end{aligned}$$

$$
\begin{aligned}
(4.5)\qquad & +\frac{1}{n}\sum_{m=1}^{A} r_m(\rho(t))\left(2^m\sigma_{2^m}f-2^{m-1}\sigma_{2^{m-1}}f\right)\\
& -\frac{1}{n}\sum_{m=1}^{A} r_m(\rho(t))\,2^{m-1}E_{m-1}f\\
& +\frac{n-2^A}{n}\sum_{l=0}^{A} r_l(\rho(t))\,(E_lf-E_{l-1}f)\\
& +\frac{r_{A+1}(\rho(t))}{n}\left(n\sigma_nf-2^A\sigma_{2^A}f\right)\\
& -\frac{r_{A+1}(\rho(t))}{n}\left(n-2^A\right)E_{m-1}f=:\sum_{j=1}^{6}J_jf.
\end{aligned}
$$

Since

$$E\,|E_mf|\le E\,|f| \tag{4.6}$$

and

$$E\,|\sigma_nf|\lesssim E\,|f| \tag{4.7}$$

we can write

$$E\,|J_jf|\lesssim E\,|f|\,,j=2,3,5,6. \tag{4.8}$$

For J_1f we can write

$$J_1f=\frac{1}{n}\sum_{m=1}^{A}2^{m-1}\widetilde{E}_m^{(t)}f,$$

where

$$\widetilde{E}_m^{(t)}f:=\sum_{l=0}^{m-1}r_l(\rho(t))\,(E_lf-E_{l-1}f)\,. \tag{4.9}$$

Since $E_lf=f*D_{2^l}$ we have

$$\widetilde{E}_m^{(t)}f=f*\sum_{l=0}^{m-1}r_l(\rho(t))\,(D_{2^l}-D_{2^{l-1}}):=f*\widetilde{D}_{2m}^{(t)}, \tag{4.10}$$

where

$$\widetilde{D}_{2^m}^{(t)}:=\sum_{l=0}^{m-1}r_l(\rho(t))\,(D_{2^l}-D_{2^{l-1}})\,.$$

It is easy to see that $r_l(t) = (-1)^{t_l} = 1 - 2t_l$. Then for $\widetilde{D}_{2^l}^{(t)}$ we can write

$$\begin{aligned}\widetilde{D}_{2^l}^{(t)} &= \sum_{l=0}^{m-1} (1-2t_l)(D_{2^l} - D_{2^{l-1}}) \\ &= D_{2^{m-1}} - 2\sum_{l=0}^{m-1} t_l (D_{2^l} - D_{2^{l-1}}) \\ &= (1-2t_{m-1}) D_{2^{m-1}} - 2\sum_{l=0}^{m-2} (t_l - t_{l+1}) D_{2^l}.\end{aligned}$$

Consequently,

$$f * \widetilde{D}_{2^m}^{(t)} = (1-2t_{m-1}) E_{m-1} f - 2\sum_{l=0}^{m-2} (t_l - t_{l+1}) E_l f. \tag{4.11}$$

$t \in \mathbb{Q}$ imply that

$$\sum_{l=0}^{\infty} |t_l - t_{l+1}| < \infty.$$

From (4.6) we get

$$E\left|\widetilde{E}_m^{(t)} f\right| \lesssim \left(|2t_{m-1} - 1| + \sum_{l=0}^{\infty} |t_l - t_{l+1}|\right) E|f| \lesssim E|f|.$$

Consequently,

$$E|J_1 f| = \frac{1}{n}\sum_{m=1}^{A} 2^{m-1} E\left|\widetilde{E}_m^{(t)} f\right| \lesssim E|f|. \tag{4.12}$$

Combining (4.5)-(4.12) we complete the proof of Theorem 4.1. □

Proof of Theorem 4.2. Since $t \notin \mathbb{Q}$ there exists a sequences $\{p_i : i \in \mathbb{P}\}$ and $\{q_i : i \in \mathbb{P}\}$ such that

$$0 \le q_1 \le p_1 < q_2 \le p_2 < \cdots < q_A \le p_A < \cdots$$

and

$$t_j = \begin{cases} 1, q_i \le j \le p_i \\ 0, p_i < j < q_{i+1} \end{cases}, i = 1, 2, ...$$

Set

$$\Delta_A := \bigcup_{t_j = 0 \text{ or } 1, j \in \{0,1,\ldots,p_A\} \setminus \{q_1, p_1, \ldots, q_A, p_A\}} I_{p_A+1}(t_0, ..., t_{q_1-1}, 0, t_{q_1+1}, ..., t_{p_1-1}, 0, t_{p_1+1}, ..., t_{q_A-1}, 0, t_{q_A+1}, ..., t_{p_A-1}, 0).$$

Define the function

$$f_A(x) := 2^{2A} \mathbb{I}_{\Delta_A}(x),$$

where $\mathbb{I}_E$ is characteristic function of the set E. It is easy to see that

$$\mu(\Delta_A) = \frac{2^{p_A+1-2A}}{2^{p_A+1}} = \frac{1}{2^{2A}}. \tag{4.13}$$

We can write (see (4.5) and (4.9))

$$\widetilde{\sigma}^{(t)}_{2^{2p_A+1}} f_A = \frac{1}{2^{2p_A+1}} \sum_{m=1}^{2p_A+1} 2^{m-1} \widetilde{E}^{(t)}_m f_A \tag{4.14}$$

$$+\frac{1}{2^{2p_A+1}} \sum_{m=1}^{2p_A+1} r_m(\rho(t)) \left(2^m \sigma_{2^m} f_A - 2^{m-1} \sigma_{2^{m-1}} f_A\right)$$

$$-\frac{1}{2^{2p_A+1}} \sum_{m=1}^{2p_A+1} r_m(\rho(t)) 2^{m-1} E_{m-1} f_A =: F_1 f_A + F_2 f_A + F_3 f_A.$$

From (4.6), (4.7) and (4.13) we have

$$E\left|F_j f_A\right| \lesssim E\left|f_A\right| \lesssim 1, j = 2, 3. \tag{4.15}$$

Set

$$\widetilde{\Delta}_i := I_{p_i+1}\left(x_0, ..., x_{q_1-1}, 0, x_{q_1+1}, ..., x_{p_1-1}, 0, x_{p_1+1}, ..., x_{q_i-1}, 0, x_{q_i+1}, ..., x_{p_i-1}, 1\right).$$

Suppose that $x \in \widetilde{\Delta}_i$ for some $i = 1, 2, ..., A$. Then

$$E_a f_A(x) = 2^a \int_{I_a(x)} f_A(s)\, d\mu(s) = 0, a > p_i.$$

Therefore, for $m > p_A$ we obtain (see 4.11)

$$\widetilde{E}^{(t)}_m f_A(x) = -2 \sum_{a=0}^{p_i} (t_a - t_{a+1}) E_a f_A(x) \tag{4.16}$$

$$= \sum_{k=1}^{i} \left[2E_{q_k-1} f_A(x) - 2E_{p_k} f_A(x)\right]$$

$$= \sum_{k=1}^{i} \left[2^{q_k+2A} \mu\left(I_{q_k-1}(x) \cap \Delta_A\right) - 2^{p_k+1+2A} \mu\left(I_{p_k}(x) \cap \Delta_A\right)\right].$$

it is easy to calculate that

$$\mu\left(I_{q_k-1}(x) \cap \Delta_A\right) = \frac{2^{p_A-(q_k-1)-2(A-k+1)}}{2^{p_A+1}} = 2^{-q_k-2(A-k)-2}$$

and

$$\mu\left(I_{p_k}(x) \cap \Delta_A\right) = \frac{2^{p_A-p_k-[2(A-k)+1]}}{2^{p_A+1}} = 2^{-p_k-2(A-k)-2}.$$

Hence, from (4.16) for $m \geq p_A + 1$ and $x \in \widetilde{\Delta}_i, i = 1, 2, ..., A$ we have

$$\widetilde{E}^{(t)}_m f_A \tag{4.17}$$

$$= \sum_{k=1}^{i} \left[2^{q_k+2A} \cdot 2^{-q_k-2(A-k)-2} - 2^{p_k+1+2A} \cdot 2^{-p_k-2(A-k)-2}\right]$$

$$= \sum_{k=1}^{i} \left[2^{2k-2} - 2^{2k-1}\right] = -\frac{2^{2i}-4}{3}.$$

Consequently, for $x \in \widetilde{\Delta}_i$ we get

$$\frac{1}{2^{2p_A+1}} \sum_{m=p_A+1}^{2p_A+1} 2^{m-1} \widetilde{E}_m^{(t)} f_A = -\frac{1}{2^{2p_A+1}} \sum_{m=p_A+2}^{2p_A+1} 2^{m-1} \frac{2^{2i}-4}{3}$$

$$= \quad -\frac{1}{2^{2p_A+1}} \left(2^{2p_A+1} - 2^{p_A+1}\right) \frac{2^{2i}-4}{3}.$$

Since

$$E\left|\widetilde{E}_m^{(t)} f_A\right| \quad \leq \quad E\left|E_m f_A\right| + 2\sum_{a=0}^{m-1} E\left|E_a f_A\right| \lesssim mE\left|f_A\right| \lesssim m$$

and

$$F_1 f_A = \frac{1}{2^{2p_A+1}} \sum_{m=1}^{p_A+1} 2^{m-1} \widetilde{E}_m^{(t)} f_A + \frac{1}{2^{2p_A+1}} \sum_{m=p_A+2}^{2p_A+1} 2^{m-1} \widetilde{E}_m^{(t)} f_A,$$

we have

$$\begin{aligned} E\left|F_1 f_A\right| \quad &\geq \quad E\left|\frac{1}{2^{2p_A+1}} \sum_{m=p_A+2}^{2p_A+1} 2^{m-1} \widetilde{E}_m^{(t)} f_A\right| \\ &\quad -\frac{1}{2^{2p_A+1}} \sum_{m=1}^{p_A+1} 2^{m-1} E\left|\widetilde{E}_m^{(t)} f_A\right| \\ &\geq \quad \sum_{i=1}^{A} \int\limits_{\widetilde{\Delta}_i} \left|\frac{1}{2^{2p_A+1}} \sum_{m=p_A+1}^{2p_A+1} 2^{m-1} \widetilde{E}_m^{(t)} f_A\right| d\mu \\ &\quad -\frac{1}{2^{2p_A+1}} \sum_{m=1}^{p_A+1} 2^{m-1} m \\ &\gtrsim \quad \sum_{i=1}^{A} 2^{2i} \mu\left(\widetilde{\Delta}_i\right) - \frac{p_A 2^{p_A}}{2^{2p_A+1}} \gtrsim A. \end{aligned} \tag{4.18}$$

Combining (4.14), (4.15) and (4.18) we have

$$E\left|\widetilde{\sigma}_{2^{2p_A+1}}^{(t)} f_A\right| \gtrsim A. \tag{4.19}$$

Let

$$Q\left(2^{2A}\right) \geq 2^{2A}, \qquad A \geq A_0. \tag{4.20}$$

By virtue of estimate (see ([9]))$\|f_A\|_{Q(L)} \leq 1 + E\left|Q\left(f_A\right)\right|$, from (4.13) and (4.20) we can write

$$\begin{aligned} E\left|\widetilde{\sigma}^{(t)}_{2^{2p_A+1}} f_A\right| &\leq \left\|\widetilde{\sigma}^{(t)}_{2^{2p_A+1}}\right\|_{Q(L)\to L_1} \|f_A\|_{Q(L)} \\ &\leq \left\|\widetilde{\sigma}^{(t)}_{2^{2p_A+1}}\right\|_{Q(L)\to L_1} \left(1 + E\left|Q\left(f_A\right)\right|\right) \\ &= \left\|\widetilde{\sigma}^{(t)}_{2^{2p_A+1}}\right\|_{Q(L)\to L_1} \left(1 + Q\left(2^{2A}\right)\mu\left(\Delta_A\right)\right) \\ &= \left\|\widetilde{\sigma}^{(t)}_{2^{2p_A+1}}\right\|_{Q(L)\to L_1} \left(1 + \frac{Q\left(2^{2A}\right)}{2^{2A}}\right) \\ &\lesssim \left\|\widetilde{\sigma}^{(t)}_{2^{2p_A+1}}\right\|_{Q(L)\to L_1} \frac{Q\left(2^{2A}\right)}{2^{2A}}, A \geq A_0. \end{aligned}$$

Consequently, by (4.19) we have

$$\left\|\widetilde{\sigma}^{(t)}_{2^{2p_A+1}}\right\|_{Q(L)\to L_1} \gtrsim \frac{A2^{2A}}{Q\left(2^{2A}\right)}, \quad A \geq A_0. \tag{4.21}$$

The fact that$Q\left(L\right) \subsetneqq L\log L$ is equalent to the condition

$$\overline{\lim_{u\to\infty}} \frac{u\log u}{Q\left(u\right)} = \infty.$$

Then there exists $\{u_k : k \in \mathbb{P}\}$ such that

$$\lim_{k\to\infty} \frac{u_k\log u_k}{Q\left(u_k\right)} = \infty, \quad u_{k+1} > u_k, k = 1, 2, ...$$

and a monotonically increasing sequence of positive integers $\{A_k : k \in \mathbb{P}\}$ such that

$$2^{2A_k} \leq u_k < 2^{2(A_k+1)}.$$

Then we have

$$\frac{2^{2A_k}A_k}{Q\left(2^{2A_k}\right)} \gtrsim \frac{u_k\log u_k}{Q\left(u_k\right)} \to \infty \quad \text{as} \quad k \to \infty.$$

Then, from (4.21) we conclude that

$$\sup_k \left\|\widetilde{\sigma}^{(t)}_{2^{2p_{A_k}+1}}\right\|_{Q(L)\to L_1} = \infty.$$

Theorem 4.2 is proved. □

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