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ON A SCALE OF CRITERIA ON n-DEPENDENCE

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Abstract. In this paper we prove that a planar set $\mathcal{X}$ of at most $mn-1$ points, where $m \le n$, is κ-dependent, if and only if there exists a number r, $1 \le r \le m-1$, and an essentially κ-dependent subset $\mathcal{Y} \subset \mathcal{X}$, $\#\mathcal{Y} \ge rs$, where $r+s-3=\kappa$, belonging to an algebraic curve of degree r, and not belonging to any curve of degree less than r. Moreover, if $\#\mathcal{Y} = rs$ then the set $\mathcal{Y}$ coincides with the set of intersection points of some two curves of degrees r and s, respectively. Let us mention that the first three criteria of the scale, for $m = 1, 2, 3$, are well-known results.

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1. Introduction, n-independence

Denote by Π_n the space of bivariate algebraic polynomials of total degree less than or equal to n. Its dimension is given by

$$N := \dim \Pi_n = \binom{n+2}{2}.$$

A plane algebraic curve is the zero set of some bivariate polynomial. To simplify notation, we shall use the same letter p, say, to denote the polynomial p and the curve given by the equation $p(x,y)=0$. More precisely, suppose p is a polynomial without multiple factors. Then the plane curve defined by the equation $p(x,y)=0$ shall also be denoted by p. So lines, conics, and cubics are equivalent to polynomials of degree 1, 2, and 3, respectively.

Suppose a set of k distinct points is given:

$$\mathcal{X}_k = \{(x_i, y_i) : i = 1, 2, \dots, k\} \subset \mathbb{C}^2.$$

The problem of finding a polynomial $p \in \Pi_n$ which satisfies the conditions

$$p(x_i, y_i) = c_i, \quad i = 1, \dots, k, \tag{1.1}$$

is called *interpolation problem*. We denote this problem by $(\Pi_n, \mathcal{X})$. The polynomial p is called *interpolating polynomial.*

Definition 1.1. The set of points $\mathcal{X}_k$ is called *n-poised*, if for any data $(c_1, \dots, c_k)$, there is a *unique* polynomial $p \in \Pi_n$ satisfying the conditions (1.1).

By a Linear Algebra argument a necessary condition for n-poisedness is

$$k = \#\mathcal{X}_k = \dim \Pi_n = N.$$

Definition 1.2. The interpolating problem $(\Pi_n, \mathcal{X}_k)$ is called n-solvable, if for any data $(c_1, \dots c_k)$, there exists a (not necessarily unique) polynomial $p \in \Pi_n$ satisfying the conditions (1.1).

A polynomial $p \in \Pi_n$ is called *n-fundamental polynomial* of a point $A \in \mathcal{X}$, if $p(A) = 1$ and $p\big|_{\mathcal{X}\setminus\{A\}} = 0$, where $p\big|_{\mathcal{X}}$ means the restriction of p to $\mathcal{X}$. We shall denote such a polynomial by $p^{\star}_{A,\mathcal{X}}$.
Sometimes we call n-fundamental also a polynomial from Π_n that just vanishes at all the points of $\mathcal{X}$ but A, since such a polynomial is a nonzero constant multiple of $p^{\star}_A$. A fundamental polynomial can be described as a plane curve containing all but one point of $\mathcal{X}$.

Next we consider an important concept of n-independence and n-dependence of point sets (see [1], [2], [4]).

Definition 1.3. A set of points $\mathcal{X}$ is called *n-independent*, if each its point has an n-fundamental polynomial. Otherwise, it is called *n-dependent.*

Since the fundamental polynomials are linearly independent, we get that $\#\mathcal{X} \leq N$ is a necessary condition for n-independence.

Proposition 1.1. *A set $\mathcal{X}$ is n-independent if and only if the interpolation problem $(\Pi_n, \mathcal{X})$ is n-solvable.*

Proof. Suppose $\mathcal{X} := \mathcal{X}_k$. In the case of n-independence we have the following Lagrange formula for a polynomial $p \in \Pi_n$ satisfying interpolating conditions (1.1):

$$p = \sum_{i=1}^{k} c_i p^{\star}_i.$$

On the other hand if the interpolation problem is n-solvable then for each point $(x_i, y_i),\ i = 1, \dots, k$, there exists an n-fundamental polynomial. Indeed, it is the solution of the interpolation problem (1.1), where $c_i = 1$, and $c_j = 0\ \forall j \neq i$. □

Definition 1.4. A set of points $\mathcal{X}$ is called *essentially n-dependent*, if none of its points has an n-fundamental polynomial.

If a point set $\mathcal{X}$ is n-dependent, then for some $A \in \mathcal{X}$, there is no n-fundamental polynomial, which means that for any polynomial $p \in \Pi_n$ we have that

$$p\big|_{\mathcal{X}\setminus\{A\}} = 0 \quad \Longrightarrow \quad p(A) = 0.$$

Thus a set $\mathcal{X}$ is essentially n-dependent means that any plane curve of degree n containing all but one point of $\mathcal{X}$, contains all of $\mathcal{X}$.

In the proof of the main result we will need the following

Proposition 1.2 ([5], Cor. 2.2)**.** *Suppose a set $\mathcal{X}$ is given. Denote by $\mathcal{Y}$ the subset of $\mathcal{X}$ that have n-fundamental polynomials with respect to $\mathcal{X}$. Then the set $\mathcal{X} \setminus \mathcal{Y}$ is essentially n-dependent.*

Corollary 1.1. *Any n-dependent point set has essentially n-dependent subset.*

Set $d(n,k) := \dim \Pi_n - \dim \Pi_{n-k}$. It is easily seen that $d(n,k) = (n+1) + n + \cdots + (n-k+2) = \frac{1}{2}k(2n-k+3)$, if $k \leq n$.

In the sequel we will need the following well-known proposition (see, e.g., [7], Proposition 3.1).

Proposition 1.3. *Let q be a curve of degree k without multiple components and $k \leq n$. Then the following assertions hold:*

(i) *Any set of more than $d(n,k)$ points located on the curve q is n-dependent;*
(ii) *Any set $\mathcal{X}$ of $d(n,k)$ points located on the curve q is n-independent if and only if*

$$p \in \Pi_n, \ p\big|_{\mathcal{X}} = 0 \Rightarrow p = fq, \ \text{where } f \in \Pi_{n-k}.$$

Corollary 1.2. *The following assertions hold:*

(i) *Any set of at least $n+2$ points located on a line is n-dependent;*
(ii) *Any set of at least $2n+2$ points located on a conic is n-dependent;*
(iii) *Any set of at least $3n+1$ points located on a cubic is n-dependent.*

2. Some known results

Let us start with the three known results which coincide with the first three items of the scale established in this paper, respectively.

Theorem 2.1 ([8])**.** *Any set $\mathcal{X}$ consisting of at most $n+1$ points is n-independent.*

Theorem 2.2 ([1], Prop. 1)**.** *A set $\mathcal{X}$ with no more than $2n+2$ points on the plane is n-dependent if and only if either $n+2$ of them are collinear or $\#\mathcal{X} = 2n+2$ and all the $2n+2$ points belong to a conic.*

Theorem 2.3 ([3], Thm. 5.1)**.** *A set $\mathcal{X}$ consisting of at most $3n$ points is n-dependent if and only if at least one of the following conditions hold:*

(i) *$n+2$ points are collinear;*
(ii) *$2n+2$ points belong to a (possibly reducible) conic;*

(iii) $\#\mathcal{X} = 3n$, *and there exist* $\sigma_3 \in \Pi_3$ *and* $\sigma_n \in \Pi_n$ *such that* $\mathcal{X} = \sigma_3 \cap \sigma_n$.

The following two results describe some properties of essentially dependent point sets laying in a curves of certain degrees.

Proposition 2.1 ([6], Prop. 3.3)**.** *Suppose that* $m \le n$. *If a set* $\mathcal{X}$ *of at most* mn *points is essentially* κ*-dependent then all the points of* $\mathcal{X}$ *lay in a curve of degree* m.

We say that a curve σ is not empty with respect to a set $\mathcal{X}$ if $X \cap \sigma \neq \varnothing$.

Theorem 2.4 ([6], Thm. 3.4)**.** *Assume that* σ_m *is a curve of degree* m*, which is either irreducible or is reducible such that all its irreducible components are not empty with respect to a set* $\mathcal{X} \subset \sigma_m$*, where* $\mathcal{X}$ *is essentially* κ*-dependent and* $m \le n+2$. *Then we have that* $\#\mathcal{X} \ge mn$.

The next result states a necessary and sufficient conditions for a set of mn points to coincide with the set of the intersection points of some two plane algebraic curves of degrees m and n, respectively.

Theorem 2.5 ([6], Thm. 3.1)**.** *A set* $\mathcal{X}$ *with* $\#\mathcal{X} = mn$, $m \le n$, *is the set of intersection points of some two plane curves of degrees* m *and* n*, respectively, if and only if the following two conditions are satisfied:*

(i) *The set* $\mathcal{X}$ *is essentially* $(m+n-3)$*-dependent;*

(ii) *No curve of degree less than* m *contains all of* $\mathcal{X}$.

3. Main result

By combining Proposition 2.1 and Theorem 2.4 we readily get the following

Proposition 3.1. *Suppose that* $\mathcal{X}$ *is an essentially* κ*-dependent point set with* $\#\mathcal{X} \le mn-1$*, where* $m \le n$*, and* $\kappa = m+n-3$. *Then there exists a number* r, $1 \le r \le m-1$*, and a curve* σ_r *of degree* r*, such that the following conditions hold:*

(i) $\#\mathcal{X} \ge rs$*, where* $r+s-3 = \kappa$*;*

(ii) σ_r *contains all of* $\mathcal{X}$

(iii) *There is no curve of degree less than* r *containing all of* $\mathcal{X}$.

Proof. We obtain from Proposition 2.1 that the set of points $\mathcal{X}$ lies in a curve σ of degree at most m. Without loss of generality we may assume that σ is either irreducible or is reducible such that all its irreducible components are not empty with respect to the set $\mathcal{X}$. Then, notice that the degree of the curve does not equal m, since in that case, in view of Theorem 2.4, we would have that $\#X \ge mn$. Finally, consider such a curve σ_r of the smallest possible degree $1 \le r \le m-1$.

Note that $r < \kappa - r + 3$ since $r < m \leq \kappa - m + 3$. Now, Theorem 2.4 implies that $\#\mathcal{X} \geq rs$, where $r + s - 3 = \kappa$. □

Now we are in a position to formulate the main result of the paper:

Theorem 3.1. *Suppose that $\mathcal{X}$ is a set of points such that $\#\mathcal{X} \leq mn - 1$, where $m \leq n$. Then $\mathcal{X}$ is κ-dependent, where $\kappa = m + n - 3$, if and only if there exist a number r, $1 \leq r \leq m-1$, and an essentially κ-dependent subset $\mathcal{Y} \subset \mathcal{X}$, $\#\mathcal{Y} \geq rs$, where $r + s - 3 = \kappa$, belonging to a curve of degree r, and not belonging to any curve of degree less than r.*
Moreover, if $\#\mathcal{Y} = rs$ then we have that $\mathcal{Y}$ coincides with the set of intersection points of some two plane curves of degrees r and s respectively.

Proof. The sufficiency part is obvious. If some set has a κ-dependent subset then the set itself is κ-dependent. Now let us prove the part of necessity.

We have that the set $\mathcal{X}$ is κ-dependent. By the Corollary 1.1 there exists some essentially κ-dependent subset $\mathcal{Y} \subset \mathcal{X}$. Now, applying Proposition 3.1 to the set of points $\mathcal{Y}$ we get that there exists a curve σ_r of degree r, $1 \leq r \leq m-1$, containing all of $\mathcal{Y}$, such that $\#\mathcal{Y} \geq rs$, where $r + s - 3 = \kappa$, and there is no curve of lower degree containing all of $\mathcal{Y}$.

Finally, let us prove the "moreover"part of the theorem. Here we have that $\mathcal{Y}$ is essentially κ-dependent, $\#\mathcal{Y} = rs$ and there is no curve of degree less than r passing through all the points of $\mathcal{Y}$. Notice also that $r < s$ since $r < m < \kappa - r + 3$. Hence we get from Theorem 2.5 that $\mathcal{Y}$ is the set of intersection points of some two plane curves of degrees r and s, respectively. □

Now, let us mention some necessary conditions for the set $\mathcal{X}$ to be able to apply Theorem 3.1. Suppose that we have a κ-dependent set $\mathcal{X}$. We need to find such numbers m and n, for which $\#\mathcal{X} \leq mn$, where $m + n - 3 = \kappa$ and $m \leq n$. Since the the expression mn achieves its maximum when $m = n = (\kappa + 3)/2$, then $\#\mathcal{X}$ must be not more than $\lfloor (\kappa + 3)^2/4 \rfloor$.

4. Some special cases of Theorem 3.1

In this section we verify that Theorem 3.1 is a generalization of Theorems 2.1, 2.2 and 2.3. For this purpose let us formulate Theorem 3.1 in the special cases with $m = 1, 2, 3, 4$.

Case $m = 1$. A set $\mathcal{X}$ of at most $\kappa+1$ points is never κ-dependent. This is equivalent to Theorem 2.1.

Case $m = 2$. A set $\mathcal{X}$ of at most $2\kappa + 1$ points is κ-dependent if and only if $\kappa + 2$ points of $\mathcal{X}$ are collinear.

Case $m = 3$. A set $\mathcal{X}$ of at most $3\kappa - 1$ points is κ-dependent, if and only if one of the following conditions hold:

(i) $\kappa + 2$ points of $\mathcal{X}$ belong to a line,

(ii) $2\kappa + 2$ points of $\mathcal{X}$ belong to a conic.

Clearly the statement in Case $m = 3$ is a generalization of Theorem 2.2.

Case $m = 4$. A set $\mathcal{X}$ of at most $4\kappa - 5$ points is κ-dependent, if and only if one of the following conditions hold:

(i) $\kappa + 2$ points of $\mathcal{X}$ belong to a line,

(ii) $2\kappa + 2$ points of $\mathcal{X}$ belong to a conic,

(iii) $\#\mathcal{X} = 3\kappa$ and $\mathcal{X}$ coincides with an intersection points of some two algebraic curves of degrees 3 and κ,

(iv) more than 3κ points of $\mathcal{X}$ belong to a cubic.

Finally, this statement generalizes Theorem 2.3.

Note that in above statements we used Corollary 1.2.

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