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ON THE EXISTENCE OF POSITIVE WEAK SOLUTION FOR NONLINEAR SYSTEM WITH SINGULAR WEIGHTS

S. KHAFAGY, H. SERAG

Majmaah University, Majmaah, Saudi Arabia[1]
Al-Azhar University, Nasr City, Cairo, Egypt
E-mails: *s.khafagy@mu.edu.sa*; *serraghm@yahoo.com*

Abstract. In this article, we study the existence results of large positive weak solution for nonlinear system with singular weights (1.4), where Ω is a bounded domain of R^n with boundary $\partial\Omega$, $0 \in \Omega$, $1 < p, q < n$, $0 \leq r < \frac{n-p}{p}$, $0 \leq s < \frac{n-q}{q}$, and $\Lambda_p u = |u|^{p-2}u$, ϱ_p, ϱ_q, $\lambda, \mu, \gamma, \delta$ are positive constants and a, b are weight functions. We prove the existence of a large positive weak solutions for mappings. λ, μ large when $\lim\limits_{x\to+\infty} \frac{f^{\frac{1}{p-1}}(M(g(x))^{\frac{1}{q-1}})}{x} = 0$, for every $M > 0$. Here, there is no any sign-changing conditions on a or b. The proof of the main results is based on the sub-supersolutions method. Application and concluding remark are provided to demonstrate the effectiveness of our results.

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1. Introduction

Positive weak solutions for systems involving Laplacian, $p-$Laplacian, weighted $p-$Laplacian or singular $p-$Laplacian operators have been obtained by many authors through the sub-supersolutions method (see [1, 2, 5],[9]-[17],[20, 22]).

In the recent past, systems involving singular p-Laplacian nonlinear systems have been studied by many authors ([10, 16, 19, 26]).

On the other hand, some other authors have obtained the existence of weak solutions for $p-$Laplacian, weighted $p-$Laplacian or singular $p-$Laplacian nonlinear systems using the method of the theory of nonlinear monotone operators (see [18, 24]) and an approximation method (see [3, 25]).

In ([6]), existence and uniqueness of positive solutions have been studied by Dalmas for the following semilinear elliptic system

$$\left.\begin{array}{rl} -\Delta u = f(v) & \text{in } \Omega, \\ -\Delta v = g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{1.1}$$

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when $f(cg(x))$ is sublinear at 0 and ∞ for every $c > 0$. Relevant results are obtained in [8], in the case $f(0) < 0$ or $g(0) < 0$, where the authors extended the study of [6] to the case with no sign conditions on $f(0)$ or $g(0)$.

In [7], the authors considered the existence results of positive solutions for the nonlinear p-Laplacian system

$$\left.\begin{array}{ll} -\Delta_p u = \lambda f(v) & \text{in } \Omega, \\ -\Delta_p v = \lambda g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{1.2}$$

in the semiposotone case, i.e., $f(0)$ or $g(0)$ is negative. Under the condition

$$\lim_{x\to+\infty} \frac{f[M(g(x))^{\frac{1}{p-1}}]}{x^{p-1}} = 0, \qquad \text{for every} \quad M > 0, \tag{1.3}$$

the existence results of positive weak solutions was given for system (1.2) when λ is large enough.

In this paper, we discuss the existence of positive weak solution for λ, μ large for the singular nonlinear system

$$\left.\begin{array}{ll} -\mathrm{div}[|\mathrm{x}|^{-\mathrm{rp}}|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u}] - \varrho_{\mathrm{p}}\Lambda_{\mathrm{p}}\mathrm{u} = \lambda \mathrm{a}(\mathrm{x})|\mathrm{x}|^{-(\mathrm{r}+1)\mathrm{p}+\gamma}\mathrm{f}(\mathrm{v}) & \text{in } \Omega, \\ -\mathrm{div}[|\mathrm{x}|^{-\mathrm{sq}}|\nabla \mathrm{v}|^{\mathrm{q}-2}\nabla \mathrm{v}] - \varrho_{\mathrm{q}}\Lambda_{\mathrm{q}}\mathrm{v} = \mu \mathrm{b}(\mathrm{x})|\mathrm{x}|^{-(\mathrm{s}+1)\mathrm{q}+\delta}\mathrm{g}(\mathrm{u}) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{1.4}$$

where Ω is a bounded domain of R^n with boundary $\partial\Omega$, $0 \in \Omega$, $1 < p, q < n$, $0 \le r < \frac{n-p}{p}$, $0 \le s < \frac{n-q}{q}$, $\Lambda_p u = |u|^{p-2}u$, $\varrho_p, \varrho_q, \lambda, \mu, \gamma, \delta$ are positive constants, a, b are weight functions and f, g are given functions. We prove through the sub–supersolutions method, the existence of a large positive weak solutions for λ, μ large when $\lim\limits_{x\to+\infty} \frac{f^{\frac{1}{p-1}}(M(g(x))^{\frac{1}{q-1}})}{x} = 0$, for every $M > 0$

In this paper we are dealing with the singular $(p, q)-$Laplacian nonlinear system with no any sign-changing conditions on the functions a or b.

This paper is organized as follows: we introduce some basic definitions, technical results, notations and some assumptions on the functions a, b, f, g in section 2. Section 3 is devoted to the study of the existence and nonexistence of positive weak solutions for (1.4) by using the sub–supersolutions method. In section 4, we present an application and concluding remark showing that our results complement previously reported results.

2. Definitions and technical results

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. If $0 < r < \frac{n-p}{p}$ and $p \ge 1$, we define the weighted $L_p(\Omega, |x|^{-(r+1)p+\gamma})$ space with the norm (see

[27])

$$\|u\|_{L_p(\Omega,|x|^{-(r+1)p+\gamma})} = \left[\int\limits_{\Omega} |x|^{-(r+1)p+\gamma}|\nabla u|^p\right]^{\frac{1}{p}} < \infty. \tag{2.1}$$

If $1 < p < n$ and $0 < r < \frac{n-p}{p}$, we define $W_0^{1,p}(\Omega,|x|^{-rp})$ as being the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p}(\Omega,|x|^{-rp})$ with respect to the norm defined by

$$\|u\|_{W_0^{1,p}(\Omega,|x|^{-rp})} = \left[\int\limits_{\Omega} |x|^{-rp}|\nabla u|^p\right]^{\frac{1}{p}} < \infty. \tag{2.2}$$

The space $W_0^{1,p}(\Omega,|x|^{-rp})$ is a separable reflexive Banach space.

Definition 2.1. *We say that a pair of functions $(u,v) \in W_0^{1,p}(\Omega,|x|^{-rp})\times W_0^{1,q}(\Omega,|x|^{-sq})$ is a weak solution for system (1.4) if and only if:*

$$\left.\begin{array}{l} \int\limits_{\Omega} |x|^{-rp}|\nabla u|^{p-2}\nabla u\nabla\zeta dx - \varrho_p\int\limits_{\Omega} |u|^{p-2}u\zeta dx = \lambda\int\limits_{\Omega} a(x)|x|^{-(r+1)p+\gamma}f(v)\zeta dx,\\ \int\limits_{\Omega} |x|^{-sq}|\nabla v|^{q-2}\nabla v\nabla\eta dx - \varrho_q\int\limits_{\Omega} |v|^{q-2}v\eta dx = \mu\int\limits_{\Omega} b(x)|x|^{-(s+1)q+\delta}g(u)\eta dx, \end{array}\right\} \tag{2.3}$$

for all test functions $(\zeta,\eta) \in W_0^{1,p}(\Omega,|x|^{-rp}) \times W_0^{1,q}(\Omega,|x|^{-sq})$ with $\zeta,\eta \geq 0$.

Definition 2.2. *We say that a pair of functions $(\psi_1,\psi_2) \in W_0^{1,p}(\Omega,|x|^{-rp}) \times W_0^{1,q}(\Omega,|x|^{-sq})$ is a weak subsolution of (1.4) if and only if:*

$$\left.\begin{array}{l} \int\limits_{\Omega} |x|^{-rp}|\nabla\psi_1|^{p-2}\nabla\psi_1\nabla\zeta dx - \varrho_p\int\limits_{\Omega} |\psi_1|^{p-2}\psi_1\zeta dx \leq \lambda\int\limits_{\Omega} a(x)|x|^{-(r+1)p+\gamma}f(\psi_2)\zeta dx,\\ \int\limits_{\Omega} |x|^{-sq}|\nabla\psi_2|^{q-2}\nabla\psi_2\nabla\eta dx - \varrho_q\int\limits_{\Omega} |\psi_2|^{q-2}\psi_2\eta dx \leq \mu\int\limits_{\Omega} b(x)|x|^{-(s+1)q+\delta}g(\psi_1)\eta dx, \end{array}\right\} \tag{2.4}$$

for all test functions $(\zeta,\eta) \in W_0^{1,p}(\Omega,|x|^{-rp}) \times W_0^{1,q}(\Omega,|x|^{-sq})$ with $\zeta,\eta \geq 0$.

Definition 2.3. *We say that a pair of functions $(z_1,z_2) \in W_0^{1,p}(\Omega,|x|^{-rp}) \times W_0^{1,q}(\Omega,|x|^{-sq})$ is a weak supersolution of (1.4) if and only if:*

$$\left.\begin{array}{l} \int\limits_{\Omega} |x|^{-rp}|\nabla z_1|^{p-2}\nabla z_1\nabla\zeta dx - \varrho_p\int\limits_{\Omega} |z_1|^{p-2}z_1\zeta dx \geq \lambda\int\limits_{\Omega} a(x)|x|^{-(r+1)p+\gamma}f(z_2)\zeta dx,\\ \int\limits_{\Omega} |x|^{-sq}|\nabla z_2|^{q-2}\nabla z_2\nabla\eta dx - \varrho_q\int\limits_{\Omega} |z_2|^{q-2}z_2\eta dx \geq \mu\int\limits_{\Omega} b(x)|x|^{-(s+1)q+\delta}g(z_1)\eta dx, \end{array}\right\} \tag{2.5}$$

for all test functions $(\zeta,\eta) \in W_0^{1,p}(\Omega,|x|^{-rp}) \times W_0^{1,q}(\Omega,|x|^{-sq})$ with $\zeta,\eta \geq 0$.

Then the following result holds:

Lemma 2.1. *([4]) Suppose there exist subsolutions and supersolutions (ψ_1,ψ_2) and (z_1,z_2) respectively of system (1.4) such that $(\psi_1,\psi_2) \leq (z_1,z_2)$. Then system (1.4) has a solution (u,v) such that $(u,v) \in [(\psi_1,\psi_2),(z_1,z_2)]$.*

We suppose that a, b, f and g verify the following hypotheses:

($\mathbf{H}_1$) $\exists\ a_0, a_1, b_0, b_1 > 0$ such that $a_0 \leq a(x)|x|^{-(r+1)p+\gamma} \leq a_1$ and $b_0 \leq b(x)|x|^{-(s+1)q+\delta} \leq b_1$.

($\mathbf{H}_2$) $f, g : [0, \infty) \longrightarrow [0, \infty)$ are C^1 nondecreasing continuous functions such that $f(s), g(s) > 0$ for $s > 0$ and $\exists\ k_0 > 0$ such that $f(s), g(s) \geq -k_0$ for all $s \geq 0$.

($\mathbf{H}_3$) $\exists\ \xi, \kappa, \alpha, \beta, \Delta, \Gamma > 0$ such that $f(v) \leq \xi v^{q(\frac{p-1}{p})}$, $g(u) \leq \kappa u^{p(\frac{q-1}{q})}$ with $\frac{1}{p} + \frac{1}{q} = 1$.

($\mathbf{H}_4$) For all $M > 0$,

$$\lim_{x \to +\infty} \frac{f^{\frac{1}{p-1}}(M(g(x))^{\frac{1}{q-1}})}{x} = 0. \tag{2.6}$$

For simplicity, we make use of the following notations

$$\lambda_* = \frac{\varrho_p}{a_0} + \frac{\lambda \xi a_1}{p a_0} + \frac{\mu \kappa b_1}{q b_0}, \quad \mu_* = \frac{\varrho_q}{b_0} + \frac{\mu \kappa b_1}{q b_0} + \frac{\lambda \xi a_1}{p a_0}, \tag{2.7}$$

where the unknown quantities will be defined later.

Now, for the following singular eigenvalue problems ([27])

$$\left.\begin{array}{lr} -\mathrm{div}[|\mathrm{x}|^{-\mathrm{rp}}|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u}] = \lambda \mathrm{a}(\mathrm{x})|\mathrm{x}|^{-(\mathrm{r}+1)\mathrm{p}+\gamma}|\mathrm{u}|^{\mathrm{p}-2}\mathrm{u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{2.8}$$

and

$$\left.\begin{array}{lr} -\mathrm{div}[|\mathrm{x}|^{-\mathrm{sq}}|\nabla \mathrm{v}|^{\mathrm{q}-2}\nabla \mathrm{v}] = \lambda \mathrm{b}(\mathrm{x})|\mathrm{x}|^{-(\mathrm{s}+1)\mathrm{q}+\delta}|\mathrm{v}|^{\mathrm{q}-2}\mathrm{v} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{2.9}$$

we introduce the following technical results.

Theorem 2.1. *([27]) There exists the simple isolated first eigenvalue $\lambda_p > 0$ (respectively $\lambda_q > 0$) and precisely one corresponding eigenfunction $\phi_p \geq 0$ (respectively $\phi_q \geq 0$) a.e. in Ω of the eigenvalue problems (2.8) (respectively (2.9)). Moreover, they are characterized by*

$$\begin{array}{c} \lambda_p \int\limits_\Omega a(x)|x|^{-r(p+1)+\gamma}|\phi_p|^p \leq \int\limits_\Omega |x|^{-rp}|\nabla \phi_p|^p. \quad \text{and} \\ \lambda_q \int\limits_\Omega b(x)|x|^{-s(q+1)+\delta}|\phi_q|^q \leq \int\limits_\Omega |x|^{-sq}|\nabla \phi_q|^q. \end{array} \tag{2.10}$$

Theorem 2.2. *(Young's Inequality) Let $1 < p < \infty$ and let q be the conjugate index of p, that is $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $X, Y \geq 0$,*

$$XY \leq \frac{X^p}{p} + \frac{Y^q}{q}. \tag{2.11}$$

3. Existence and nonexistence results

Theorem 3.1. *Let ($\mathbf{H}_1$)– ($\mathbf{H}_4$) are hold. Then system (1.4) has a positive weak solution $(u, v) \in W_0^{1,p}(\Omega, |x|^{-rp}) \times W_0^{1,q}(\Omega, |x|^{-sq})$ for λ, μ large.*

Proof. Let $m,\, \sigma > 0$ be such that $|x|^{-rp}|\nabla\phi_p|^p \; - \lambda_p a(x)|x|^{-r(p+1)+\gamma}\phi_p^p \geq m$ and $|x|^{-sq}|\nabla\phi_q|^q \; - \lambda_q b(x)|x|^{-s(q+1)+\delta}\phi_q^q \geq m$ on $\overline{\Omega}_\sigma = \{x \in \Omega : d(x,\partial\Omega) \leq \sigma\}$. We shall verify that (ψ_1,ψ_2) is a subsolution of (1.4) for λ,μ large where $(\psi_1,\psi_2) = \left(\frac{p-1}{p}(\frac{\lambda a_0 k_0}{m})^{\frac{1}{p-1}}\phi_p^{\frac{p}{p-1}}, \frac{q-1}{q}(\frac{\mu b_0 k_0}{m})^{\frac{1}{q-1}}\phi_q^{\frac{q}{q-1}}\right)$. Let $\zeta \in W_0^{1,p}(P,\Omega)$ with $\zeta \geq 0$. A calculation shows that

$$\begin{aligned}
&\int_\Omega |x|^{-rp}|\nabla\psi_1|^{p-2}\nabla\psi_1\nabla\zeta dx - \varrho_p\int_\Omega |\psi_1|^{p-2}\psi_1\zeta dx\\
\leq\;& \frac{\lambda a_0 k_0}{m}\int_\Omega |x|^{-rp}\phi_p|\nabla\phi_p|^{p-2}\nabla\phi_p\nabla\zeta dx\\
=\;& \frac{\lambda a_0 k_0}{m}\left\{\int_\Omega |x|^{-rp}|\nabla\phi_p|^{p-2}\nabla\phi_p\nabla(\phi_p\zeta)dx - \int_\Omega |x|^{-rp}|\nabla\phi_p|^p\zeta dx\right\}\\
=\;& \frac{\lambda a_0 k_0}{m}\int_\Omega (\lambda_p a(x)|x|^{-r(p+1)+\gamma}\phi_p^p - |x|^{-rp}|\nabla\phi_p|^p)\zeta dx. \qquad (3.1)
\end{aligned}$$

Similarly, for $\eta \in W_0^{1,q}(Q,\Omega)$ with $\eta \geq 0$, we have

$$\begin{aligned}
&\int_\Omega |x|^{-sq}|\nabla\psi_2|^{q-2}\nabla\psi_2\nabla\eta dx - \varrho_q\int_\Omega |x|^{-s(q+1)+\delta}|\psi_2|^{q-2}\psi_2\eta dx\\
\leq\;& \frac{\mu b_0 k_0}{m}\int_\Omega (\lambda_q b(x)|x|^{-s(q+1)+\delta}\phi_q^q - |x|^{-sq}|\nabla\phi_q|^q)\eta dx. \qquad (3.2)
\end{aligned}$$

Now, on $\overline{\Omega}_\sigma$, we have $|x|^{-rp}|\nabla\phi_p|^p - \lambda_p a(x)|x|^{-r(p+1)+\gamma}\phi_p^p \geq m$. Hence, Using $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, we have

$$\frac{\lambda a_0 k_0}{m}(\lambda_p a(x)|x|^{-r(p+1)+\gamma}\phi_p^p - |x|^{-rp}|\nabla\phi_p|^p) \leq -\lambda a_0 k_0 \leq \lambda a(x)|x|^{-r(p+1)+\gamma}f(\psi_2).$$

A similar argument shows that

$$\frac{\mu b_0 k_0}{m}(\lambda_q b(x)|x|^{-s(q+1)+\delta}\phi_q^q - |x|^{-sq}|\nabla\phi_q|^q) \leq -\mu b_0 k_0 \leq \mu b(x)|x|^{-(s+1)q+\delta}g(\psi_1).$$

Next, on $\Omega - \overline{\Omega}_\sigma$, we have $\phi_p \geq \rho_p$, $\phi_q \geq \rho_q$ for some $\rho_p, \rho_q > 0$. Also $g(\psi_1)$ and $f(\psi_2)$ are depending on λ,μ respectively and nondecreasing continuous functions and therefore for λ,μ large we have, using (2.10), $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$,

$$\begin{aligned}
\frac{\lambda a_0 k_0}{m}(\lambda_p a(x)|x|^{-r(p+1)+\gamma}\phi_p^p - |x|^{-rp}|\nabla\phi_p|^p) \;&\leq\; \frac{\lambda a_0 k_0}{m}\lambda_p \leq \lambda a(x)|x|^{-r(p+1)+\gamma}f(\psi_2),\\
\frac{\mu b_0 k_0}{m}(\lambda_q b(x)|x|^{-s(q+1)+\delta}\phi_q^q - |x|^{-sq}|\nabla\phi_q|^q) \;&\leq\; \frac{\mu b_0 k_0}{m}\lambda_q \leq \mu b(x)|x|^{-(s+1)q+\delta}g(\psi_1).
\end{aligned}$$

Hence, (3.1) becomes

$$\int_\Omega |x|^{-rp}|\nabla\psi_1|^{p-2}\nabla\psi_1\nabla\zeta dx - \varrho_p\int_\Omega |\psi_1|^{p-2}\psi_1\zeta dx \leq \lambda\int_\Omega a(x)|x|^{-r(p+1)+\gamma}f(\psi_2)\zeta dx.$$

Similarly, for $\eta \in W_0^{1,q}(Q,\Omega)$ with $\eta \geq 0$, (3.2) becomes

$$\int_\Omega |x|^{-sq}|\nabla\psi_2|^{q-2}\nabla\psi_2\nabla\eta dx - \varrho_q\int_\Omega |\psi_2|^{q-2}\psi_2\eta dx \leq \lambda\int_\Omega b(x)|x|^{-(s+1)q+\delta}g(\psi_1)\eta dx.$$

That is, according to (2.4), (ψ_1,ψ_2) is a subsolution of (1.4) for λ,μ large.

Supersolution:

Next, let (e_p,e_q) be the unique solution of (see [21])

$$\left.\begin{array}{rl} -\mathrm{div}[|x|^{-rp}|\nabla e_p|^{p-2}\nabla e_p] = 1 & \text{in } \Omega, \\ -\mathrm{div}[|x|^{-sq}|\nabla e_q|^{q-2}\nabla e_q] = 1 & \text{in } \Omega, \\ e_p = e_q = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{3.3}$$

where as it is known $e_p,e_q > 0$ on Ω and $\frac{\partial e_p}{\partial n},\frac{\partial e_q}{\partial n} < 0$ on $\partial\Omega$.

Let

$$\begin{aligned} z_1 &= \frac{C}{\nu_p}\left(\frac{\lambda}{1-\varrho_p\nu_p^{p-1}}\right)^{\frac{1}{p-1}}e_p, \\ z_2 &= \left(\frac{\mu b_1}{1-\varrho_q\nu_q^{q-1}}\right)^{\frac{1}{q-1}}\ [g(C(\frac{\lambda}{1-\varrho_p\nu_p^{p-1}})^{\frac{1}{p-1}})]^{\frac{1}{q-1}}e_q, \end{aligned} \tag{3.4}$$

where $C > 0$ is a large enough to be chosen later, $\nu_p = \|e_p\|_\infty$ and $\nu_q = \|e_q\|_\infty$. Now, let us verify that (z_1,z_2) is a supersolution of (1.4) for λ,μ large. To this end, let $\zeta \in W_0^{1,p}(\Omega,|x|^{-rp})$ with $\zeta \geq 0$. Then we have, using (3.3) and (3.4)

$$\begin{aligned} &\int_\Omega |x|^{-rp}|\nabla z_1|^{p-2}\nabla z_1\nabla\zeta dx - \varrho_p\int_\Omega |z_1|^{p-2}z_1\zeta dx \\ = \ &\frac{\lambda}{1-\varrho_p\nu_p^{p-1}}\left(\frac{C}{\nu_p}\right)^{p-1}\int_\Omega |x|^{-rp}|\nabla e_p|^{p-2}\nabla e_p\nabla\zeta dx \\ &-\frac{\lambda}{1-\varrho_p\nu_p^{p-1}}\left(\frac{C}{\nu_p}\right)^{p-1}\varrho_p\int_\Omega |e_p|^{p-2}e_p\zeta dx \\ \geq \ &\lambda\left(\frac{C}{\nu_p}\right)^{p-1}\int_\Omega \zeta dx. \end{aligned} \tag{3.5}$$

By $(\mathbf{H}_4)$, we can choose C large enough so that

$$\left(\frac{C}{\nu_p}\right)^{p-1} \geq a_1 f([(\frac{\mu b_1}{1-\varrho_q\nu_q^{q-1}})^{\frac{1}{q-1}}][g(C(\frac{\lambda}{1-\varrho_p\nu_p^{p-1}})^{\frac{1}{p-1}})]^{\frac{1}{q-1}}\nu_q).$$

Hence (3.5) becomes,

$$\begin{aligned} &\int_\Omega |x|^{-rp}|\nabla z_1|^{p-2}\nabla z_1\cdot\nabla\zeta dx - \varrho_p\int_\Omega |z_1|^{p-2}z_1\zeta dx \\ \geq \ &\lambda a_1\int_\Omega f([(\frac{\mu b_1}{1-\varrho_q\nu_q^{q-1}})^{\frac{1}{q-1}}][g(C(\frac{\lambda}{1-\varrho_p\nu_p^{p-1}})^{\frac{1}{p-1}})]^{\frac{1}{q-1}}\nu_q)\zeta dx \\ \geq \ &\lambda\int_\Omega a(x)|x|^{-(r+1)p+\gamma}f(z_2)\zeta dx. \end{aligned}$$

Next, for $\eta \in W_0^{1,q}(\Omega, |x|^{-sq})$ with $\eta \geq 0$, one can have

$$\begin{aligned}
& \int_\Omega |x|^{-sq}|\nabla z_2|^{q-2}\nabla z_2 \nabla \eta dx - \varrho_q \int_\Omega |z_2|^{q-2} z_2 \eta dx \\
= & \frac{\mu b_1}{1-\varrho_q \nu_q^{q-1}} g(C(\frac{\lambda}{1-\varrho_p \nu_p^{p-1}})^{\frac{1}{p-1}}) \int_\Omega |x|^{-sq}|\nabla e_q|^{q-2}\nabla e_q \nabla \eta dx \\
& - \frac{\mu b_1}{1-\varrho_q \nu_q^{q-1}} g(C(\frac{\lambda}{1-\varrho_p \nu_p^{p-1}})^{\frac{1}{p-1}}) \varrho_q \int_\Omega |e_q|^{q-2} e_q \eta dx \\
\geq & \frac{\mu b_1}{1-\varrho_q \nu_q^{q-1}} g(\frac{C}{\nu_p}(\frac{\lambda}{1-\varrho_p \nu_p^{p-1}})^{\frac{1}{p-1}} e_p)[\int_\Omega \eta dx - \int_\Omega \varrho_q \nu_q^{q-1} \eta dx] \\
= & \frac{\mu b_1}{1-\varrho_q \nu_q^{q-1}} g(\frac{C}{\nu_p}(\frac{\lambda}{1-\varrho_p \nu_p^{p-1}})^{\frac{1}{p-1}} e_p)[1-\varrho_q \nu_q^{q-1}] \int_\Omega \eta dx \\
\geq & \mu b_1 \int_\Omega g(\frac{C}{\nu_p}(\frac{\lambda}{1-\varrho_p \nu_p^{p-1}})^{\frac{1}{p-1}} e_p) \eta dx \geq \mu \int_\Omega b(x)|x|^{-(s+1)q+\delta} g(z_1) \eta dx.
\end{aligned}$$

That is, according to (2.5), (z_1, z_2) is a supersolution of (1.4) with $z_i \geq \psi_i$ for C large, $i = 1, 2$. Thus, according to (Lemma 1), there exists a positive weak solution (u, v) of (1.4) with $\psi_1 \leq u \leq z_1,\ \psi_2 \leq v \leq z_2$. This completes the proof.

Theorem 3.2. *If f and g satisfy hypothesis $(\mathbf{H}_3)$, then system (1.4) has not nontrivial positive weak solution if*

$$0 < \lambda_* < \lambda_p \text{ and } 0 < \mu_* < \mu_q \ , \tag{3.6}$$

where λ_*, μ_* are given by (2.7).

Proof. Multiplying the first equation of system (1.4) by u, and using Young inequality given by (2.11), we get

$$\begin{aligned}
\int_\Omega |x|^{-rp}|\nabla u|^p dx - \varrho_p \int_\Omega |u|^p dx & = \lambda \int_\Omega a(x)|x|^{-r(p+1)+\gamma} f(v) dx \leq \lambda \xi a_1 \int_\Omega u v^{q(\frac{p-1}{p})} dx \\
& \leq \frac{\lambda \xi a_1}{p} \int_\Omega [u^p + (p-1)v^q] dx.
\end{aligned}$$

It follows that

$$\int_\Omega |x|^{-rp}|\nabla u|^p dx \leq [\varrho_p + \frac{\lambda \xi a_1}{p}] \int_\Omega u^p dx + \frac{\lambda \xi a_1}{q} \int_\Omega v^q dx.$$

Using (2.10), we have

$$\begin{aligned}\lambda_p a_0 \int_\Omega |u|^p dx &\leq \lambda_p \int_\Omega a(x)|x|^{-r(p+1)+\gamma}|u|^p dx \leq \int_\Omega |x|^{-rp}|\nabla u|^p dx \\ &\leq [\varrho_p + \frac{\lambda\xi a_1}{p}] \int_\Omega u^p dx + \frac{\lambda\xi a_1}{q}\int_\Omega v^q dx,\end{aligned}$$

and hence,

$$[\lambda_p - \frac{\varrho_p}{a_0} - \frac{\lambda\xi a_1}{pa_0}]\,\|u\|_p^p - \frac{\lambda\xi a_1}{qa_0}\,\|v\|_q^q \leq 0. \tag{3.7}$$

On the other hand, multiplying the second equation of system (1.4) by v, and using (2.11), we have

$$\begin{aligned}\int_\Omega |x|^{-sq}|\nabla v|^q dx - \varrho_q \int_\Omega |v|^q dx &= \mu \int_\Omega b(x)|x|^{-s(q+1)+\delta} g(u) v dx \leq \mu\kappa b_1 \int_\Omega u^{p(\frac{q-1}{q})} v dx \\ &\leq \frac{\mu\kappa b_1}{q}\int_\Omega [(q-1)u^p + v^q] dx,\end{aligned}$$

which implies

$$\int_\Omega |x|^{-sq}|\nabla v|^q dx \leq \frac{\mu\kappa b_1}{p}\int_\Omega u^p dx + [\varrho_q + \frac{\mu\kappa b_1}{q}]\int_\Omega v^q dx.$$

Using (2.10), we have

$$\begin{aligned}\lambda_q b_0 \int_\Omega |v|^q dx &\leq \lambda_q \int_\Omega b(x)|x|^{-s(q+1)+\delta}|v|^q dx \leq \int_\Omega |x|^{-sq}|\nabla v|^q dx \\ &\leq \frac{\mu\kappa b_1}{p}\int_\Omega u^p dx + [\varrho_q + \frac{\mu\kappa b_1}{q}]\int_\Omega v^q dx,\end{aligned}$$

and hence,

$$-\frac{\mu\kappa b_1}{pb_0}\,\|u\|_p^p + [\lambda_q - \frac{\varrho_q}{b_0} - \frac{\mu\kappa b_1}{qb_0}]\,\|v\|_q^q \leq 0. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$[\lambda_p - (\frac{\varrho_p}{a_0} + \frac{\lambda\xi a_1}{pa_0} + \frac{\mu\kappa b_1}{qb_0})]\,\|u\|_p^p + [\lambda_q - (\frac{\varrho_q}{b_0} + \frac{\mu\kappa b_1}{qb_0} + \frac{\lambda\xi a_1}{pa_0})]\,\|v\|_q^q \leq 0, \tag{3.9}$$

which is a contradiction if (3.6) holds. This completes the proof.

4. Application and concluding remark

Consider the following nonlinear system

$$\left.\begin{array}{lr} -\mathrm{div}[|x|^{-rp}|\nabla u|^{p-2}\nabla u] = \lambda a(x)|x|^{-(r+1)p+\gamma} v^{\beta} & \text{in } \Omega, \\ -\mathrm{div}[|x|^{-sq}|\nabla v|^{q-2}\nabla v] = \mu b(x)|x|^{-(s+1)q+\delta} u^{\alpha} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{4.1}$$

under the assumptions

(a_1) Let $a(x), b(x)$ be weight functions such that $a_0 \le a(x)|x|^{-(r+1)p+\gamma} \le a_1$, $b_0 \le b(x)|x|^{-(s+1)q+\delta} \le b_1$.

(a_2) $0 < \alpha < p-1$ and $0 < \beta < q-1$.

Corollary 4.1. *If (a_1) and (a_2) are hold, then system (4.1) has a large positive weak solution for λ, μ large.*

Proof. If $f(v) = v^{\beta}, g(u) = u^{\alpha}$, then (H_4) implies $\alpha\beta < (p-1)(q-1)$. Then, according to theorem 3.1 with $\varrho_p = \varrho_q = 0$, system (4.1) has a large positive weak solution for λ, μ large.

Remark 4.1. *Existence results obtained in this paper still hold if we replace the condition (2.6), given in (H_4), by the condition $\lim\limits_{x\to+\infty} \frac{f[M(g(x))^{\frac{1}{q-1}}]}{x^{p-1}} = 0$, for every $M > 0$.*

Remark 4.2. *The results of this paper generalize and complement some results reported in the literature:*

1) If $|x|^{-rp} = |x|^{-sq} = 1, p = q = 2, \varrho_p = \varrho_q = 1, \lambda = \mu = 1$ and $a(x)|x|^{-(r+1)p+\gamma} = b(x)|x|^{-(s+1)q+\delta} = 1$, then we have some results for the existence theorem in [1].

2) If $|x|^{-rp} = |x|^{-sq} = 1, p = q, \varrho_p = \varrho_q = 0, \lambda = \mu$ and $a(x)|x|^{-(r+1)p+\gamma} = b(x)|x|^{-(s+1)q+\delta} = 1$, then we have some results for the existence theorem in [7].

3) If $|x|^{-rp} = |x|^{-sq} = 1, p = q = 2, \varrho_p = \varrho_q = 0, |x|^{-(r+1)p+\gamma} = |x|^{-(s+1)q+\delta} = 1$ and $\lambda = \mu$, then we have some results for the existence theorem in [22].

4) If $|x|^{-rp} = |x|^{-sq} = 1, \varrho_p = \varrho_q = 0, |x|^{-(r+1)p+\gamma} = |x|^{-(s+1)q+\delta} = 1$ and $\lambda = \mu$, then we have some results for the existence and nonexistence theorems in [23].

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