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On Interval Colorings of Complete $k-$ partite Graphs K_n^k

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Abstract

Problems of existence, construction and estimation of parameters of interval colorings of complete k-partite graphs K_n^k are investigated.

Let $G = (V, E)$ be an undirected graph without loops and multiple edges [1], $V(G)$ and $E(G)$ be the sets of vertices and edges of G, respectively. The degree of a vertex $x \in V(G)$ is denoted by $d_G(x)$, the maximum degree of a vertex of G-by $\Delta(G)$, and the chromatic index [2] of G-by $\chi'(G)$. A graph is regular, if all its vertices have the same degree. If α is a proper edge coloring of the graph G [3], then the color of an edge $e \in E(G)$ in the coloring α is denoted by $\alpha(e)$, and the set of colors of the edges that are incident to a vertex $x \in V(G)$, is denoted by $S(x, \alpha)$. For a non-empty subset D of $\mathbf{Z}_+$, let $l(D)$ and $L(D)$ be the minimal and maximal element of D, respectively. A non-empty subset D of $\mathbf{Z}_+$ is interval, if $l(D) \leq t \leq L(D)$, $t \in \mathbf{Z}_+$ implies that $t \in D$. Interval D is referred to be (q, h)-interval if $l(D) = q$, $|D| = h$ and is denoted by $Int(q, h)$. For intervals D_1 and D_2 with $|D_1| = |D_2| = h$, and a $p \in \mathbf{Z}_+$, the notation $D_1 \oplus p = D_2$ means: $l(D_1) + p = l(D_2)$, $L(D_1) + p = L(D_2)$.

A proper coloring α of edges of G with colors $1, 2, \ldots, t$ is an interval t-coloring of G [4], if for each color $i, 1 \leq i \leq t$, there exists at least one edge $e_i \in E(G)$ with $\alpha(e_i) = i$, and the edges incident with each vertex $x \in V(G)$ are colored by $d_G(x)$ consecutive colors.

A graph G is interval colorable, if there is $t \geq 1$, for which G has an interval t-coloring.

The set of all interval colorable graphs is denoted by $\mathcal{N}$ [5].

For $G \in \mathcal{N}$ the least and the greatest values of t, for which G has an interval t-coloring, is denoted by $w(G)$ and $W(G)$, respectively.

In [5] it is proved:

Theorem 1. Let G be a regular graph.

1) $G \in \mathcal{N}$ iff $\chi'(G) = \Delta(G)$.

2) If $G \in \mathcal{N}$ and $\Delta(G) \leq t \leq W(G)$, then G has an interval t-coloring.

Theorem 2 [6]. Let $n = p \cdot 2^q$, where p is odd, and $q \in \mathbf{Z}_+$. Then $W(K_{2n}) \geq 4n - 2 - p - q$.

In this paper interval colorings of complete k-partite graphs K_n^k [7] are investigated, where

$$V\left(K_n^k\right) = \left\{x_j^{(i)} \mid 1 \leq i \leq k, 1 \leq j \leq n\right\},$$
$$E\left(K_n^k\right) = \left\{\left(x_p^{(i)}, x_q^{(j)}\right) \mid 1 \leq i < j \leq k, 1 \leq p \leq n, 1 \leq q \leq n\right\}.$$

It is not hard to see that $\Delta(K_n^k) = (k-1)\cdot n$.
From the results of [8] we imply that

$$\chi'(K_n^k) = \begin{cases} (k-1)\cdot n, & \text{if } n\cdot k \text{ is even}, \\ (k-1)\cdot n+1, & \text{if } n\cdot k \text{ is odd}. \end{cases}$$

Theorem 1 implies:

Corollary 1.
1) $K_n^k \in \mathcal{N}$, if $n\cdot k$ is even;
2) $K_n^k \notin \mathcal{N}$, if $n\cdot k$ is odd.

Corollary 2. If $n\cdot k$ is even, then $w(K_n^k) = (k-1)\cdot n$.

Theorem 3. If k is even, then $W\left(K_n^k\right) \geq \left(\frac{3}{2}k-1\right)\cdot n - 1$.

Proof. Suppose $V\left(K_n^k\right) = \left\{x_j^{(i)} \mid 1\leq i\leq k, 1\leq j\leq n\right\}$,
$E\left(K_n^k\right) = \left\{\left(x_p^{(i)}, x_q^{(j)}\right) \mid 1\leq i<j\leq k, 1\leq p\leq n, 1\leq q\leq n\right\}$.
For the graph K_n^k define an edge coloring λ as follows:

for $i=1,\dots,\left\lfloor\frac{k}{4}\right\rfloor$, $j=2,\dots,\frac{k}{2}$, $i<j$, $i+j\leq\frac{k}{2}+1$, $p=1,\dots,n$, $q=1,\dots,n$ set:

$$\lambda\left(\left(x_p^{(i)}, x_q^{(j)}\right)\right) = (i+j-3)\cdot n+p+q-1;$$

for $i=2,\dots,\frac{k}{2}-1$, $j=\left\lfloor\frac{k}{4}\right\rfloor+2,\dots,\frac{k}{2}$, $i<j$, $i+j\geq\frac{k}{2}+2$, $p=1,\dots,n$, $q=1,\dots,n$ set:

$$\lambda\left(\left(x_p^{(i)}, x_q^{(j)}\right)\right) = \left(i+j+\frac{k}{2}-4\right)\cdot n+p+q-1;$$

for $i=3,\dots,\frac{k}{2}$, $j=\frac{k}{2}+1,\dots,k-2$, $j-i\leq\frac{k}{2}-2$, $p=1,\dots,n$, $q=1,\dots,n$ set:

$$\lambda\left(\left(x_p^{(i)}, x_q^{(j)}\right)\right) = \left(\frac{k}{2}+j-i-1\right)\cdot n+p+q-1;$$

for $i=1,\dots,\frac{k}{2}$, $j=\frac{k}{2}+1,\dots,k$, $j-i\geq\frac{k}{2}$, $p=1,\dots,n$, $q=1,\dots,n$ set:

$$\lambda\left(\left(x_p^{(i)}, x_q^{(j)}\right)\right) = (j-i-1)\cdot n+p+q-1;$$

for $i=2,\dots,1+\left\lfloor\frac{k-2}{4}\right\rfloor$, $j=\frac{k}{2}+1,\dots,\frac{k}{2}+\left\lfloor\frac{k-2}{4}\right\rfloor$, $j-i=\frac{k}{2}-1$, $p=1,\dots,n$, $q=1,\dots,n$ set:

$$\lambda\left(\left(x_p^{(i)}, x_q^{(j)}\right)\right) = (2i-3)\cdot n+p+q-1;$$

for $i=\left\lfloor\frac{k-2}{4}\right\rfloor+2,\dots,\frac{k}{2}$, $j=\frac{k}{2}+1+\left\lfloor\frac{k-2}{4}\right\rfloor,\dots,k-1$, $j-i=\frac{k}{2}-1$, $p=1,\dots,n$, $q=1,\dots,n$ set:

$$\lambda\left(\left(x_p^{(i)},x_q^{(j)}\right)\right)=(i+j-3)\cdot n+p+q-1;$$

for $i=\frac{k}{2}+1,\dots,\frac{k}{2}+\left\lfloor\frac{k}{4}\right\rfloor-1$, $j=\frac{k}{2}+2,\dots,k-2$, $i<j$, $i+j\leq\frac{3}{2}k-1$, $p=1,\dots,n$, $q=1,\dots,n$ set:

$$\lambda\left(\left(x_p^{(i)},x_q^{(j)}\right)\right)=(i+j-k-1)\cdot n+p+q-1;$$

for $i=\frac{k}{2}+1,\dots,k-1$, $j=\frac{k}{2}+\left\lfloor\frac{k}{4}\right\rfloor+1,\dots,k$, $i<j$, $i+j\geq\frac{3}{2}k$, $p=1,\dots,n$, $q=1,\dots,n$ set:

$$\lambda\left(\left(x_p^{(i)},x_q^{(j)}\right)\right)=\left(i+j-\frac{k}{2}-2\right)\cdot n+p+q-1.$$

Let us show that λ is an interval $\left(\left(\frac{3}{2}k-1\right)\cdot n-1\right)-$coloring of the graph K_n^k.

First of all let us show that for $i=1,2,\dots,\left(\frac{3}{2}k-1\right)\cdot n-1$ there is an edge $e_i\in E\left(K_n^k\right)$ such that $\lambda(e_i)=i$.

Consider the vertices $x_1^{(1)},x_2^{(1)},\dots,x_n^{(1)},x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}$. It is not hard to see that for $j=1,2,\dots,n$

$$S\left(x_j^{(1)},\lambda\right)=\bigcup_{l=1}^{k-1}\left(Int\left(j,n\right)\oplus n\cdot(l-1)\right)\ \text{and}\ S\left(x_j^{(k)},\lambda\right)=\bigcup_{l=\frac{k}{2}}^{\frac{3}{2}k-2}\left(Int\left(j,n\right)\oplus n\cdot(l-1)\right).$$

Let $\overline{C}$ and $\overline{\overline{C}}-$be the subsets of colors of the edges, that are incident to the vertices $x_1^{(1)},x_2^{(1)},\dots,x_n^{(1)}$ and $x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}$ in a coloring λ, respectively, that is:

$$\overline{C}=\bigcup_{j=1}^{n}S\left(x_j^{(1)},\lambda\right)\ \text{and}\ \overline{\overline{C}}=\bigcup_{j=1}^{n}S\left(x_j^{(k)},\lambda\right).$$

It is not hard to see that $\overline{C}\cup\overline{\overline{C}}=\left\{1,2,\dots,\left(\frac{3}{2}k-1\right)\cdot n-1\right\}$, and, therefore, for $i=1,2,\dots,\left(\frac{3}{2}k-1\right)\cdot n-1$ there is an edge $e_i\in E\left(K_n^k\right)$ such that $\lambda(e_i)=i$.

Now, let us show that the edges that are incident to a vertex $v\in V\left(K_n^k\right)$ are colored by $(k-1)\cdot n$ consecutive colors.

Let $x_j^{(i)}\in V\left(K_n^k\right)$, where $1\leq i\leq k,1\leq j\leq n$.

Case 1. $1\leq i\leq 2,1\leq j\leq n$.

It is not hard to see that

$$S\left(x_j^{(1)},\lambda\right)=S\left(x_j^{(2)},\lambda\right)=\bigcup_{l=1}^{k-1}\left(Int\left(j,n\right)\oplus n\cdot(l-1)\right)=Int\left(j,(k-1)\cdot n\right).$$

Case 2. $3\leq i\leq\frac{k}{2},1\leq j\leq n$.

It is not hard to see that

$$S\left(x_j^{(i)},\lambda\right)=\bigcup_{l=i-1}^{k-3+i}\left(Int\left(j,n\right)\oplus n\cdot(l-1)\right)=Int\left(j+n\cdot(i-2),(k-1)\cdot n\right).$$

Case 3. $\frac{k}{2}+1\leq i\leq k-2, 1\leq j\leq n.$
It is not hard to see that

$$S\left(x_j^{(i)},\lambda\right)=\bigcup_{l=i-\frac{k}{2}+1}^{\frac{k}{2}-1+i}\left(Int\left(j,n\right)\oplus n\cdot(l-1)\right)=Int\left(j+n\cdot\left(i-\tfrac{k}{2}\right),(k-1)\cdot n\right).$$

Case 4. $k-1\leq i\leq k, 1\leq j\leq n.$
It is not hard to see that

$$S\left(x_j^{(k-1)},\lambda\right)=S\left(x_j^{(k)},\lambda\right)=\bigcup_{l=\frac{k}{2}}^{\frac{3}{2}k-2}\left(Int\left(j,n\right)\oplus n\cdot(l-1)\right)=$$
$$Int\left(j+n\cdot\left(\tfrac{k}{2}-1\right),(k-1)\cdot n\right).$$

Theorem 3 is proved.

Corollary 3. If k is even and $(k-1)\cdot n\leq t\leq\left(\frac{3}{2}k-1\right)\cdot n-1$, then K_n^k has an interval $t-$coloring.

Theorem 4. Suppose $k=p\cdot 2^q$, where p is odd, and $q\in\mathbf{N}$. Then $W\left(K_n^k\right)\geq(2k-p-q)\cdot n-1$.

Proof. Let $V\left(K_n^k\right)=\left\{x_j^{(i)}|\ 1\leq i\leq k,1\leq j\leq n\right\}$,
$E\left(K_n^k\right)=\left\{\left(x_r^{(i)},x_s^{(j)}\right)|\ 1\leq i<j\leq k,1\leq r\leq n,1\leq s\leq n\right\}$.

Consider the graph K_k, where $V\left(K_k\right)=\{u_1,u_2,\ldots,u_k\}$, $E\left(K_k\right)=\{(u_i,u_j)\mid 1\leq i<j\leq k\}$. Theorem 2 implies that if $k=p\cdot 2^q$, where p is odd, and $q\in\mathbf{N}$, then $W\left(K_k\right)\geq 2k-1-p-q$.Suppose φ is an interval $(2k-1-p-q)-$coloring of K_k.

Define a coloring ψ of the edges of K_n^k as follows:
For $i=1,2,\ldots,k$ and $j=1,2,\ldots,k$, $i\neq j$ set:

$$\psi\left(\left(x_r^{(i)},x_s^{(j)}\right),K_n^k\right)=\left(\varphi\left((u_i,u_j),K_k\right)-1\right)\cdot n+r+s-1,\text{where } r=1,2,...,n,$$
$$s=1,2,...,n.$$

Let us show that ψ is an interval $((2k-p-q)\cdot n-1)-$coloring of the graph K_n^k.

The definition of ψ and the equalities $L\left(S\left(u_i,\varphi\right)\right)-l\left(S\left(u_i,\varphi\right)\right)=k-2$, $i=1,2,\ldots,k$ imply that:

1) $S\left(x_j^{(i)},\psi\right)=\bigcup_{m=l(S(u_i,\varphi))}^{L(S(u_i,\varphi))}\left(Int\left(j,n\right)\oplus n\cdot(m-1)\right)=$
$=Int\left(j+n\cdot\left(l\left(S\left(u_i,\varphi\right)\right)-1\right),(k-1)\cdot n\right)$
for $i=1,2,\ldots,k$ and $j=1,2,\ldots,n$;

2) $\bigcup_{i=1}^{k}\bigcup_{j=1}^{n}S\left(x_j^{(i)},\psi\right)=Int\left(1,(2k-p-q)\cdot n-1\right).$

This show that ψ is an interval $((2k-p-q)\cdot n-1)$ −coloring of the graph K_n^k.

Theorem 4 is proved.

Corollary 4. Suppose that $k=p\cdot 2^q$, where p is odd and $q\in\mathbf{N}$. If $(k-1)\cdot n\leq t\leq (2k-p-q)\cdot n-1$, then K_n^k has an interval t-coloring.

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K_n^k լրիվ k կողմանի գրաֆների միջակայքային ներկումների մասին

Ռ. Քամալյան, Պ. Պետրոսյան

Ամփոփում

Աշխատանքում հետազոտվում են K_n^k լրիվ k կողմանի գրաֆների միջակայքային ներկումների գոյության, կառուցման և թվային պարամետրերի գնահատման հարցեր: