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On Compensation of Discrete Fourier Transform Error

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Abstract

A problem of reduction of the percentage (relative) error, appearing at a finite signal spectrum restoration by means of Discrete Fourier Transform is considered. A method based on using of a finite impulse filter with various impulse responses is offered. Depending on errors of the initial spectrum, a percentage error expression derivable after using the filtering is obtained. A criterion for comparison between specified errors at a large number of discrete points is offered. Conditions, when the filter of given response reduces the restoration error module, are obtained. Examples with uniform window and some finite functions of standard type are considered.

Keywords - FFT, discretization error, spectrum, spectrum restoration, finite impulse filter, error compensation

1 Introduction

Discrete (or Fast) Fourier Transform (DFT or FFT) [1] is one of mathematical methods mostly used at the solution of various applied problems. Particularly, its applications are well known in digital signal (image) processing, in determination of spatial spectrum in acoustics, geophysics, radio physics etc.

It can be noted that in many problems the discretization procedure is inevitable, and in some of them it is even essential, since data processing is performed by means of computers. Therefore the important problem is reduction or at least estimation of errors occurring at signal discretization and spectrum restoration.

Actuality of the problem of reduction of DFT error can be illustrated by example from the antenna theory and technology, when the restoration of the spatial distribution of radiation field (i.e. spatial spectrum) by discrete complex-valued distribution of electromagnetic field over the antenna aperture (so named NF-FF systems [2]) is required. In these systems the volume of measurements usually is not great (so the discretization error is significant), but the restored field, as well as some of its integrated parameters, are required to be determined exactly enough. In such systems the attempt to avoid the specified errors of processing usually transforms to very expensive hardware complications. Usually the absolute or mean-square discretization error is considered (see, for instance, [3]). But in these cases the expressions for error are complicated and its investigations meet certain difficulties.

A method of study of the percentage (or relative) discretization error (PED) of the Fourier Transform is offered in [4]. It is shown that for some certain types of real signals the discretization error decreases in inversely proportion to the squared number of discrete points. For some simple types of functions the exact expressions of discrete spectrum and corresponding simple approximated expressions of discretization error at a large number of points are calculated. In [5] a problem of determination and study of the Fourier Transform discretization error using different quadrature rules is considered. It is accentuated that the quadrature formula error depends not only on number of the integration nodes, but also on concrete type of integrand. Usually this error is estimated by available a priori information on belonging of the integrand to the certain class. In this paper a method of PED reduction of a finite signal spectrum restoration, based on using a finite impulse response (FIR) filter with various impulse responses is offered. Conditions, when the filter of given response reduces the restoration error module, are obtained. Examples with uniform window and some finite functions of standard type are considered.

2 Error Compensation Method

For simplicity we shall consider the univariate case. In [4] is studied PDE at discretization of the integral as follows

$$F(u) = \int_{-1}^{1} f(p)e^{-jup}dp, \tag{1}$$

by the formula

$$\tilde{F}(u) = \Delta p \sum_{i=0}^{N-1} f(p_i)e^{-jup_i}, \tag{2}$$

where $f(p)$ is a finite function, $N = 2^M$, $M \geq 1$ is a natural number, $\Delta p = 2/N$, $p_i = -1 + 2i/N$. The formula (2) has a final expression as follows

$$\tilde{F}(u) = (2/N)e^{j\pi k} \sum_{i=0}^{N-1} f(-1 + 2i/N)e^{-j2\pi ki/N}$$

and it is usually calculated by FFT algorithm.

Let $F_0(u)$ and $\tilde{F}_0(u)$ be the normed on the maximum value of (1) and (2) correspondingly. These functions are the continuous and the discrete spectrum functions respectively. Following [4], we shall define the percentage discretization error as follows

$$\tilde{\Delta}_N(u) = \frac{|F_0(u)| - |\tilde{F}_0(u)|}{|F_0(u)|} = 1 - \frac{|\tilde{F}_0(u)|}{|F_0(u)|}, \tag{3}$$

in the points of continuity of (3). It is obvious that $-\infty < \tilde{\Delta}_N(u) \leq 1$.

Let us consider the function $\tilde{H}_0(u)$ which represents the spectrum of some FIR of length N [1] and consider a discrete spectrum expression $\overline{F}(u) = \tilde{F}(u)\tilde{H}_0(u)$. We shall interpret $\overline{F}(u)$ as the spectrum of finite function $f(p)$, being adjusted by the spectrum $\tilde{H}_0(u)$. We assume that the maximum of $|\tilde{F}(u)\tilde{H}_0(u)|$ is accessed at the same point u_0, at which is accessed its maximum the function $|\tilde{F}_0(u)|$. Let $\overline{F}_0(u)$ be the spectrum $\overline{F}(u)$, normed to the maximum of module. Correspondingly we shall determine the PED as follows

$$\overline{\Delta}_N(u) = 1 - \frac{|\tilde{F}_0(u)|}{|F_0(u)|}. \tag{4}$$

One can show that (3) and (4) are connected via the expression

$$\overline{\Delta}_N(u) = 1 - [1 - \tilde{\Delta}_N(u)]|\tilde{H}_0(u)|. \tag{5}$$

We may note that when the function $\tilde{H}_0(u)$ is known, it is possible to express the errors (3) and (5) each by other.

Thus, the problem of error compensation is turned to finding of spectrum functions $\tilde{H}_0(u)$, which satisfy the inequality

$$|\overline{\Delta}_N(u)| \le |\tilde{\Delta}_N(u)| \tag{6}$$

The solution of inequality (6) is

$$1 \le |\tilde{H}_0(u)| \le \frac{1+\tilde{\Delta}_N(u)}{1-\tilde{\Delta}_N(u)}, \quad \text{when } \tilde{\Delta}_N(u) \ge 0 \tag{7}$$

$$\frac{1+\tilde{\Delta}_N(u)}{1-\tilde{\Delta}_N(u)} \le |\tilde{H}_0(u)| \le 1, \quad \text{when } \tilde{\Delta}_N(u) \le 0 \tag{8}$$

The formulas (7) and (8) explain the meaning of correction of the discrete spectrum $\tilde{F}_0(u)$. The positive value of the error (3) at some point shows that the discrete spectrum has the lower levels than the continuous one. In this case from formula (8) follows that the correcting function $\tilde{H}_0(u)$ enlarges the module of discrete spectrum, i.e. the discrete spectrum moves up to the continuous one. Similar results may be obtained for the case of negative value of (3).

3 Examples on Error Compensation

The case of large values of N is interesting. It is shown in [4] that at the small values of the ratio u/N and for some standard finite functions $f(p)$ PED looks as the following approximate expression

$$\tilde{\Delta}_N(u) \approx -c\left(\frac{u}{N}\right)^2, \quad c > 0. \tag{9}$$

Also there are functions $f(p)$ with $c < 0$.

At first, consider the case, when the discretization error is negative, i.e. $\tilde{\Delta}_N(u) \le 0$, and let the FIR spectrum be

$$\tilde{H}_0(u) = \frac{1 - e^{-ju\Delta p}}{u\Delta p} = sp\left(\frac{u}{N}\right)e^{-ju/N} \tag{10}$$

where $sp(x) = \sin x/x$. This function is often used in numerical integration with a Fourier kernel as a modification of ordinary rectangles method which looks as follows

$$\overline{F}_0(u) = \frac{1 - e^{-ju\Delta p}}{ju} \sum_{i=0}^{N-1} f(p_i)e^{-jup_i}.$$

At a small value of ratio u/N we have

$$sp\left(\frac{u}{N}\right) \approx 1 - \frac{1}{6}\left(\frac{u}{N}\right)^2 \tag{11}$$

so, from the formulas (8), (9) and (11) we obtain that the correction of the spectrum $\tilde{F}_0(u)$ by use of function (10) reduces PED for all functions $f(p)$, for which $c \geq 1/12$.

Let us consider some examples.

Example 1

Let $f(p) \equiv 1$ (rectangle window). Then

$$\tilde{\Delta}_N(u) = 1 - \left|sp\left(\frac{u}{N}\right)\right|^{-1} \approx 1 - \frac{1}{6}\left(\frac{u}{N}\right)^2 \quad (12)$$

Since the condition $c \geq 1/12$ is satisfied, we have $|\overline{\Delta}_N(u)| \leq |\tilde{\Delta}_N(u)|$. Moreover, replacing expression (12) in (5), we see that $\overline{\Delta}_N(u) = 0$, i.e. in this case the spectrum $F_0(u)$ is completely restored.

Example 2

Let $f(p) = \cos(\pi p/2)$ (cosine window). Then one can show by some direct calculations that

$$\tilde{\Delta}_N(u) = 1 - \frac{sp^2(\frac{\pi}{2N})}{sp(\frac{\pi-2u}{2N})sp(\frac{\pi+2u}{2N})}. \quad (13)$$

At a small u/N and $N \to \infty$ we have $\tilde{\Delta}_N(u) \approx -u^2/3N^2$, so $|\overline{\Delta}_N(u)| \leq |\tilde{\Delta}_N(u)|$. The error after correction is equal to $\overline{\Delta}_N(u) \approx u^2/6N^2$, i.e. the error became smaller, although it has exchanged the sign.

Example 3

Let $f(p) = \cos^2(\pi p/2)$ (squared cosine window). Then one can show that

$$\tilde{\Delta}_N(u) = 1 - \left|\frac{u(\pi^2 - u^2)}{N\pi^2} ctg\left(\frac{u}{N}\right) + \frac{u^2 sp(2u/N)}{\pi^2 sp(\frac{u-\pi}{N})sp(\frac{u+\pi}{N})}\right| \quad (14)$$

At fixed u and $N \to \infty$ it is turned to $\tilde{\Delta}_N(u) \approx -u^4/15N^4$. In this case we have $c = 0$, therefore using of the filter (11) increases the discretization error, i.e. $|\overline{\Delta}_N(u)| \geq |\tilde{\Delta}_N(u)|$. In the case when the discretization error (4) is positive, the similar results are obtained. We notice that it is possible to use many other spectrum windows $\tilde{H}_0(u)$, particularly the spectrum functions $\tilde{F}_0(u)$ considered in the examples above.

4 Conclusion

In this paper we have considered a problem of compensation of Discrete Fourier Transform discretization error at a finite function spectrum restoration. We have introduced the percentage discretization error which has simple expression for many standard finite functions at a large number of discrete points. Compensation method based on using a finite impulse response filter of known spectrum is applied. Conditions, when the filter of given response reduces the restoration error module, are obtained. Examples with uniform window and some finite functions of standard type are considered.

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Ֆուրյեի ընդհատ ձևափոխության սխալի նվազեցման մասին

Դ. Գ. Ասատրյան

Ամփոփում

Դիտարկվել է Ֆուրյեի ընդհատ ձևափոխության միջոցով ստացվող՝ վերջավոր ազդանշանի սպեկտրի վերականգնման հարաբերական սխալի նվազեցման խնդիր: Առաջարկվել է տարբեր իմպուլսային բնութագրերով զտիչների օգտագործման վրա հենված մեթոդ: Ստացվել է զտումից հետո առաջացած հարաբերական սխալի արտահայտությունը: Սխալները համեմատելու նպատակով առաջարկվել է չափանիշ՝ ընդհատ կետերի մեծ քանակի համար: Ստացվել են տրված բնութագրերով զտիչի կիրառման դեպքում սխալի նվազեցման պայմանները: Վերջավոր ֆունկցիաների մի քանի տիպերի համար բերվել են հավասարաչափ զտիչի կիրառման օրինակներ: