Mathematical Problems of Computer Science 18, 1997, 1–13.

On a Representation of Hyperarithmetical Predicates

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Abstract

In Aslyan [1] the representability of arithmetical predicates by means of general form recursive equations is proved. Here we show that arbitrary hyperarithmetical predicate $P(x_1, ..., x_k)$ is representable by the system of recursive equations. On the other hand it is shown that the class of hyperarithmetical predicates is the largest class of predicates representable by means of recursive equations. This presents a new characterisation of hyperarithmetical sets "from outside", where, characteristic functions of hyperarithmetical sets are described as a unique solutions of G.F.R.E..

1 Introduction

In this article we are concerned in with a problem of determination of a class of number-theoretic (N-T) predicates represented by general form recursive equations (G.F.R.E.) introduced in Marandjian [11]. We define a language *Rec* of recursive terms and a set $RecIdx \subset \omega$ of indexes of recursive functionals, represented by means of terms of language *Rec* in a fixed interpretation, to make our consideration complete. Each recursive term of language *Rec* will denote a partial recursive functional as well as the corresponding index will reflect the generation of it by a series of applications of substitution, primitive recursion and minimization. If $Q \in Rec$ then by $\lceil Q \rceil \in RecIdx$ we will denote the index of corresponding functional represented by term Q.

Definition 1.1 *for arbitrary* $i, n, q, k, 0 < j \leq n$

1. $S \in Rec$; $\langle 1, 1 \rangle \in RecIdx$.
2. $C_q^m \in Rec$; $\langle 2, n, q \rangle \in RecIdx$.
3. $P_j^n \in Rec$; $\langle 3, n, j \rangle \in RecIdx$.
4. $f_i^n \in Rec$; $\langle 4, n, i \rangle \in RecIdx$.
5. *If* $g_j, f \in Rec$ *for* $0 < j \leq n$ *with indexes* $\lceil g_i \rceil, \lceil f \rceil \in RecIdx$ *correspondingly then* $Sub(f, g_1, ..., g_n) \in Rec$ *and* $\langle 5, m, \langle \lceil f \rceil, \lceil g_0 \rceil, ..., \lceil g_n \rceil \rangle \rangle \in RecIdx$.

6. *If $f, g \in Rec$ with indexes $\lceil g \rceil, \lceil f \rceil \in RecIdx$ correspondingly then*
   $Pr[f, g] \in Rec$,
   and $\langle 6, n+1, \lceil f \rceil, \lceil g \rceil \rangle \in RecIdx$.

7. *If $f \in Rec$ with index $\lceil f \rceil$ then*
   $\mu[f] \in Rec$ *and* $\langle 7, n-1, \lceil f \rceil \rangle \in RecIdx$.

It is easy to see that the set $RecIdx$ is primitive recursive. When the meaning is clear we use the term $RecIdx$ as a name for corresponding set and as a designation of its characteristic function. For any term F we write $F[f_0, ..., f_k](x_1, ..., x_m)$ to indicate that a term F contains occurrences of function symbols $f_0, ..., f_k$ and the value of F depend on integer variables $(x_1, ..., x_m)$.

Definition 1.2 *Let a finite set $F_0, ..., F_n, G_0, ..., G_n$ of recursive terms be given, then the system of equations of the form (1) is called a system of general form recursive equations (G.F.R.E.).*

$$\begin{aligned} F_0[f_0, ..., f_k](x_1, ..., x_m) &\simeq G_0[f_0, ..., f_k](x_1, ..., x_m) \\ &\;\;... \\ F_n[f_0, ..., f_k](x_1, ..., x_m) &\simeq G_n[f_0, ..., f_k](x_1, ..., x_m) \end{aligned} \tag{1}$$

Definition 1.3 *We will say that the solution $\mathsf{f}_0, ..., \mathsf{f}_k$ of the system of G.F.R.E. (1) with respect to the function symbol $f_j, 0 \le j \le k$ represents a N-T predicate $P(x_1, ..., x_q)$ if the system of recursive equations (1) has a unique solution in the class of N-T functions, and the function $\mathsf{f}_j(x_1, ..., x_q)$ from the list $\mathsf{f}_0, ..., \mathsf{f}_k$ is the characteristic function of predicate $P(x_1, ..., x_q)$.*

Definition 1.4 *Let the recursive term $F^*\left[f_{i_1}^{n_1}, ..., f_{i_k}^{n_k}\right](x_{j_1}, ..., x_{j_m})$ be given, then the number $\langle m, \langle n_1, ..., n_k \rangle \rangle$ we will call the signature of the functional F^*. Here by $\Phi_e^{\langle m, \langle n_1, ..., n_k \rangle \rangle}\left[\tilde{f}\right](\tilde{x})$ we denote a universal function for functionals of the signature $\langle m, \langle n_1, ..., n_k \rangle \rangle$ and by $T^{\langle m+2, \langle n_1, ..., n_k \rangle \rangle}\left[\tilde{f}\right](e, \tilde{x}, y)$ we denote Kleene normal form predicate. By W_e^m we denote recursively enumerable set usually defined as $W_e^m \overset{def}{\equiv} Arg\left\{\Phi_e^{\langle m, 0 \rangle}[](x_1, ..., x_m)\right\}$. By the symbols c, L, R we denote the terms for pairing function on ω and inverses to c respectively, with the properties below*

$$\begin{aligned} L(c(x, y)) &= x \\ R(c(x, y)) &= y \\ c(L(x), R(x)) &= x. \end{aligned}$$

Before performing our task we will construct the set $HypIdx \subset \omega$ of indexes of hyperarithmetical sets and assign corresponding hyperarithmetical set J_i^n from the set $J_i^n \in Hyp$ of hyperarithmetical sets to each element i of the set $HypIdx \subset \omega$. Roughly speaking an index $i \in HypIdx$ will encode an information on consecutive steps in the process of construction of the set J_i^n. The definition of hyperarithmetical sets given below is taken from Shoenfild [14] and differ from the original definition of Kleene [7].

Let O be the Church-Kleene [2] system of recursive ordinal notations. For each $e \in O$ Kleene defines H_e to be the subset of ω obtained by iterating the ordinary jump operator along e, beginning with the empty set. Then a subset of ω is said to be hyperarithmetical if it is recursive in H_e for some $e \in O$. Later in Kleene [8]it is shown that hyperarithmetical subsets of ω are exactly the Δ_1^1 sets (the sets which are both Σ_1^1 and Π_1^1 definable). Theorem **2.6** below states the representability of hyperarithmetical sets by means of G.F.R.E. providing a new characterization of them. Excellent sources of information on classical hyperarithmetical theory are Hinman [4],[5], Harrison [3].

Definition 1.5 *For arbitrary e, m.*

1. $\langle 0, e\rangle \in HypIdx$.

2. *Let* $e \in HypIdx$ *then* $\langle 1, e\rangle \in HypIdx$.

3. *Let* W_e^1 *be non empty and let* $\forall y \in W_e^1 \Rightarrow y \in HypIdx$ *then* $\langle 2, e\rangle \in HypIdx$.

Now for arbitrary m let define the mapping J_i^m from the set $HypIdx$ to the set Hyp of hyperarithmetical sets .

For arbitrary e, m.

1. *Let* $\langle 0, e\rangle \in HypIdx$ *then* $J^m_{\langle 0,e\rangle} = W_e^m \in Hyp$.

2. *Let* $\langle 1, e\rangle \in HypIdx$ *then* $J^m_{\langle 1,e\rangle} = \overline{J_e^m} \in Hyp$.

3. *Let* $\langle 2, e\rangle \in HypIdx$ *then* $J^m_{\langle 2,e\rangle} = \bigcup_{y \in W_e^1} J_y^m \in Hyp$.

2 Representability of Hyperarithmetical Sets

In this section we will prove that for arbitrary hyperarithmetical predicate $P(x_1, ..., x_m) \in Hyp$ there exists recursive equation

$$F\left[f_0^m, f_0^{m+1}\right](x_0, x_1, ..., x_m, y) \simeq 0,$$

so, that the solution of which with respect to the function symbol $f_0^m, 0 \le j \le k$ represents the N-T predicate $P(x_1, ..., x_m)$.

Theorem 2.6 *For arbitrary m there exists a primitive recursive function $p^{(m)} : N \to N$ such that for any $z \in HypIdx$ we have $p^{(m)}(z) \in RecIdx$ and the solution $\mathfrak{f}_0^m, \mathfrak{f}_0^{m+1}$ of the system of G.F.R.E. (2) with respect to the function symbol f_0^m*

$$\Phi_{p^{(m)}(z)}^{\langle m+1,\langle m,m+1\rangle\rangle}\left[f_0^m, f_0^{m+1}\right](\overline{x}, y) = 0 \tag{2}$$

represents the hyperarithmetical set J_z^m.

Proof. Before carrying out the proof let us assume that there exist recursive functions $q^{(m)}, k^{(m)} : N \to N; r^{(m)}, g : N \times N \to N$; with the properties 1-4 below. The existence of such functions are provided by Lemmas **6.14 -6.16** in appendix.

1. For any index e of the set W_e and partial recursive function $\varphi_z(x)$
$W_{g(e,z)}(x) = \left\{x \mid \exists p \Phi_z^{\langle 1,0 \rangle}(p) \downarrow = x \& p \in W_e\right\}$.
2. Let an index e of the set W_e be given. Then $q^{(m)}(e)$ is an index of a functional of the signature $\langle m+1, \langle m, m+1 \rangle\rangle$ such that the solution of the system of G.F.R.E. (3) with respect to the function symbol f_0^m represents the set W_e^m.
$$\Phi_{q^{(m)}(e)}^{\langle m+1,\langle m,m+1\rangle\rangle}\left[f_0^m, f_0^{m+1}\right](\overline{x}, y) = 0. \tag{3}$$
3. Let an index $e \in RecIdx$ of a functional of the signature $\langle m+1, \langle m, m+1 \rangle\rangle$ be given and let the solution of the system of G.F.R.E.
$\Phi_e^{\langle m+1,\langle m,m+1\rangle\rangle}\left[f_0^m, f_0^{m+1}\right](\overline{x}, y) = 0$ with respect to the function symbol f_0^m represents a N-T predicate $P(x_1, ..., x_m)$. Then $k^{(m)}(e) \in RecIdx$ is an index of a functional of the signature $\langle m+1, \langle m, m+1 \rangle\rangle$ such that the solution of the system of G.F.R.E. (4) with respect to the function symbol f_0^m represents the predicate $\overline{P}(x_1, ..., x_m)$.
$$\Phi_{k^{(m)}(e)}^{\langle m+1,\langle m,m+1\rangle\rangle}\left[f_0^m, f_0^{m+1}\right](\overline{x}, y) = 0 \tag{4}$$
4. Let an index e of the set W_e^1 be given such that for any $z \in W_e^1$ we have that $z \in RecIdx$ and z is an index of a functional of the signature $\langle m+1, \langle m, m+1 \rangle\rangle$ so that the solution of the system of G.F.R.E.
$\Phi_z^{\langle m+1,\langle m,m+1\rangle\rangle}\left[f_0^m, f_0^{m+1}\right](\overline{x}, y) = 0$ with respect to the function symbol f_0^m represents some predicate $P^{(Z)}(x_1, ..., x_m)$. Then $r^{(m)}(e) \in RecIdx$ is an index of a functional of the signature $\langle m+1, \langle m, m+1 \rangle\rangle$ such that the solution of the system of G.F.R.E. (5) with respect to the function symbol f_0^m represents the predicate $\bigcup_{Z \in W_e^1} P^{(Z)}(x_1, ..., x_m)$.
$$\Phi_{r^{(m)}(e)}^{\langle m+1,\langle m,m+1\rangle\rangle}\left[f_0^m, f_0^{m+1}\right](\overline{x}, y) = 0 \tag{5}$$

Now define the function $\Psi^{(m)}(z, y)$ in the following manner

$$\Psi^{(m)}(z, y) \overset{def}{\equiv} \begin{cases} q^{(m)}\left((y)_1\right) & \text{if } \ lx(y) = 2 \& (y)_0 = 0 \\ k^{(m)}\left(\Phi_z^{\langle 1,0 \rangle}\left((y)_1\right)\right) & \text{if } \ lx(y) = 2 \& (y)_0 = 1 \\ r^{(m)}\left(g\left((y)_1, z\right)\right) & \text{if } \ lx(y) = 2 \& (y)_0 = 2 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Kleene's second recursion theorem states that for any partial recursive $\Phi(x, y)$ there exists such x_0 that $\Phi(x_0, y) = \Phi_{x_0}^{\langle 1,0 \rangle}(y)$. In accord with this we define $p^{(m)}(y) = \Psi(z_0, y) = \Phi_{z_0}^{\langle 1,0 \rangle}(y)$. Then we have

$$p^{(m)}(y) = \begin{cases} q^{(m)}\left((y)_1\right) & \text{if } \ lx(y) = 2 \& (y)_0 = 0 \\ k^{(m)}\left(p^{(m)}\left((y)_1\right)\right) & \text{if } \ lx(y) = 2 \& (y)_0 = 1 \\ r^{(m)}\left(g\left((y)_1, z_0\right)\right) & \text{if } \ lx(y) = 2 \& (y)_0 = 2 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Now we will prove that the function $p^{(m)}(y)$ has the properties stated in the formulation of the theorem.

Basis : Let $z \in HypIdx$ be of the form $\langle 0, e\rangle$. Then according to the Definition **1.5** e is an index of the set $J^m_{\langle 0,e\rangle} = W^m_e$. It follows from (7) that for such z $p^{(m)}(z) = q^{(m)}(e)$. According to point 2 above this provides that $\Phi^{(m+1,\langle m,m+1\rangle)}_{q^{(m)}(e)}\left[\bar{f}\right](\overline{x}, y) = 0$ represents the set W^m_e with respect to the function symbol f^m_0.

Induction step:

Case 1: Let $z \in HypIdx$ be of the form $\langle 1, e\rangle$, then according to the Definition **1.5** , z is an index of the set $J^m_z = \overline{J^m_e}$. It follows from (7) that for such a z, $p^{(m)}(z) = k^{(m)}\left(p^{(m)}(e)\right)$. According to our inductive hyphothesis $p^{(m)}(e)$ is an index of a functional of the signature $\langle m+1, \langle m, m+1\rangle\rangle$ such that the solution of the system of G.F.R.E. $\Phi^{(m+1,\langle m,m+1\rangle)}_{p^{(m)}(e)}\left[\bar{f}\right](\overline{x}, y) = 0$ with respect to the function symbol f^m_0 represents the set J^m_e. Now from the properties of the function $k^{(m)}$ stated in the point 3 above $k^{(m)}\left(p^{(m)}(e)\right)$ is an index of a functional of the signature $\langle m+1, \langle m, m+1\rangle\rangle$ such that the solution of the system of G.F.R.E. $\Phi^{(m+1,\langle m,m+1\rangle)}_{k^{(m)}\left(p^{(m)}(e)\right)}\left[\bar{f}\right](\overline{x}, y) = 0$ with respect to the function symbol f^m_0 represents the predicate $J^m_z = \overline{J^m_e}$.

Case 2: Let $z \in HypIdx$ be of the form $\langle 2, e\rangle$, then according to the Definition **1.5** z is an index of the set $J^m_z = \bigcup_{y \in W^1_e} J^m_y$. From (7) it follows that for such a z, $p^{(m)}(z) = r^{(m)}(g(e, z_0))$. According to the properties of the function g stated in the point 1 above, $g(e, z_0)$ is an index of the set $W_{g(e,z_0)} = Val\left\{p^{(m)}(W_e)\right\}$. It follows from our inductive hyphothesis that $\forall k \in W_{g(e,z_0)} \Rightarrow k \in RecIdx$ and k is an index of a functional of the signature $\langle m+1, \langle m, m+1\rangle\rangle$ such that the solution of the system of G.F.R.E. $\Phi^{(m+1,\langle m,m+1\rangle)}_k\left[f^m_0, f^{m+1}_0\right](\overline{x}, y) = 0$ with respect to the function symbol f^m_0 represents the predicate J^m_y. Now from the properties of the function $r^{(m)}$ stated in the point 4 above, $r^{(m)}(g(e, z_0))$ is an index of a functional of the signature $\langle m+1, \langle m, m+1\rangle\rangle$ such that the solution of the system of G.F.R.E. $\Phi^{(m+1,\langle m,m+1\rangle)}_{r^{(m)}(g(e,z_0))}\left[\bar{f}\right](\overline{x}, y) = 0$ with respect to the function symbol f^m_0 represents the set J^m_z. □

3 The Class of Representable Sets

It is shown in the previous section that an arbitrary hyperarithmetical predicate $P \subset \omega^n$ could be characterized or represented as a set which characteristic function is a unique solution of a system of G.F.R.E. (1). Here we show that arbitrary predicate P representable by this mean belongs to the class Hyp.

Theorem 3.7 *Let a system of G.F.R.E. (1) be given and the set of functions* $\mathsf{f}_0, ..., \mathsf{f}_k$ *be the unique solution of the system (1). Let* $Val(\mathsf{f}_i) = \{0, 1\}$, *then the predicate* $P(\overline{x}) \overset{def}{\equiv} \{\overline{x} \mid \mathsf{f}_i(\overline{x}) = 0\}$ *belongs to the set* Hyp.

Proof. An easy verification shows that the predicate

$$Sol(\mathsf{f}_0, ..., \mathsf{f}_k) \rightleftharpoons \left\{ \begin{array}{c} \mathsf{f}_0, ..., \mathsf{f}_k \mid \mathsf{f}_0, ..., \mathsf{f}_k \text{ is a solution of the system} \\ \text{of general form recursive equations (1)} \end{array} \right\}$$

has a Π_2^0 representation for fixed system (1). Now we have that
$P(\overline{x}) \Leftrightarrow \exists \mathsf{f}_0, ..., \exists \mathsf{f}_k \left(Sol\left(\mathsf{f}_0, ..., \mathsf{f}_k\right) \& \mathsf{f}_i\left(\overline{x}\right) = 0\right)$, hence the predicate $P(\overline{x})$ has Σ_1^1 representation. On the other hand it is follows from the proof of the Theorem **2.6** that the predicate $\overline{P}(\overline{x})$ is also representable by some system (obtained from the given one by replacing all the occurrences of term $f_i(U_1, ..., U_n)$ with $\overline{sg}(f_i(U_1, ..., U_n))$)of general form recursive equation and also belongs to the class Σ_1^1. Hence $P(\overline{x}) \in Hyp$. □

4 The Number-Theoretic Functions Represented by Means of General Form Recursive Equations

In previous sections the notion of representation of N-T predicate by the system of G.F.R.E. is given and the class of N-T predicates representable by G.F.R.E. is determined. In particular it is shown that arbitrary representable predicate is hyperarithmetical and vise versa. From this follows that the class of N-T predicates represented by means of G.F.R.E. coincides with the class of hyperarithmetical sets.

In this section two definitions of representation of N-T function by means of G.F.R.E. are given and the equivalence of this definitions is shown. From the Definition **4.9** given below it is evident that the class of hyperarithmetical functions is those of representable functions.

Definition 4.8 *We will say that the solution of the system of G.F.R.E (8)*

$$\begin{array}{rcl} F_0\left[f_0, ..., f_k\right](x_1, ..., x_m) & \simeq & G_0\left[f_0, ..., f_k\right](x_1, ..., x_m) \\ & ... & \\ F_n\left[f_0, ..., f_k\right](x_1, ..., x_m) & \simeq & G_n\left[f_0, ..., f_k\right](x_1, ..., x_m) \end{array} \tag{8}$$

with respect to the function symbol f_i $0 \leq i \leq k$ represents a number theoretic function $\mathsf{f}^*(\overline{x})$, if the system of (G.F.R.E) (8) has a unique solution $\mathsf{f}_0, ..., \mathsf{f}_k$ in the class of N-T functions and the equality (9) holds for arbitrary $\overline{x} \in \omega^m$

$$\mathsf{f}_i(\overline{x}) = \mathsf{f}^*(\overline{x}) \tag{9}$$

Definition 4.9 *A N-T function $\mathsf{f}^*(\overline{x})$ is called representable by means of G.F.R.E (8) if the predicate $\Gamma_{\mathsf{f}^*}(\overline{x}, y) \stackrel{def}{\equiv} \begin{cases} 0 & if\ \ \mathsf{f}^*(\overline{x}) = y \\ 1 & if\ \ otherwise \end{cases}$ is representable with respect to the function symbol f_i $0 \leq i \leq k$ from the system of G.F.R.E (8).*

Proposition 4.10 *A N-T function is representable by means of Definition* **4.8** if and only if it is representable by means of Definition **4.9** .

Proof. $\Longrightarrow$Let a N-T function $\mathsf{f}^*(\overline{x})$ be representable by means of Definition **4.9** and let (8) be a system of G.F.R.E. which has a unique solution $\mathsf{f}_0, ..., \mathsf{f}_k$ such that $\mathsf{f}_i(\overline{x}) = \mathsf{f}^*(\overline{x})$. Then consider the system (10)

$$\begin{array}{lcl} F_0\left[f_0, ..., f_k\right](x_1, ..., x_m) & \simeq & G_0\left[f_0, ..., f_k\right](x_1, ..., x_m) \\ & ... & \\ F_n\left[f_0, ..., f_k\right](x_1, ..., x_m) & \simeq & G_n\left[f_0, ..., f_k\right](x_1, ..., x_m) \\ F_{n+1}\left[f_0, ..., f_k, f_{k+1}\right](x_1, ..., x_m, y) & & 0 \end{array} \tag{10}$$

where
$F_{n+1}[f_0, ..., f_k, f_{k+1}](x_1, ..., x_m, y) \stackrel{def}{\equiv}$
$\stackrel{def}{\equiv} f_{k+1}(\overline{x}, f_k(\overline{x})) + \sum_{i=1}^{y} (\overline{sg}(f_{k+1}(\overline{x}, i)) - 1)$. Now it is easy to see that the system of G.F.R.E (10) has a unique solution $\mathsf{f}_0, ..., \mathsf{f}_k, \mathsf{f}_{k+1}$ where $\mathsf{f}_{k+1}(\overline{x}) = \Gamma_{\mathsf{f}_k}(\overline{x})$.

$\Longleftarrow$ Let the system of G.F.R.E (8) be given and let a system of functions $\mathsf{f}_0, ..., \mathsf{f}_k$ be the unique solution of the system of G.F.R.E (8), such that $\mathsf{f}_k = \Gamma_{\mathsf{f}^*}(\overline{x})$. Then consider the system of G.F.R.E obtained from (8) by joining the equation $F^*_{n+1}[f_0, ..., f_k, f_{k+1}](x_1, ..., x_m, y) \simeq 0$ to the system (8) where $F^*_{n+1}[f_0, ..., f_k, f_{k+1}](x_1, ..., x_m) \stackrel{def}{\equiv} f_k(\overline{x}, f_{k+1}(\overline{x}))$. It is easy to verify that the obtained system has a unique solution $\mathsf{f}_0, ..., \mathsf{f}_k, \mathsf{f}_{k+1}$ where $\mathsf{f}_{k+1}(\overline{x}) = \mathsf{f}^*(\overline{x})$. □.

5 A Characterization of J. Myhill's Finitely Representable Functions Via General Form Recursive Equations

In Kuznecov and Trahtenbrot [10] a class of N-T functions 'effectively closed in Baire Space' have been introduced. Later, from entirely different starting-point, Myhill [12] introduced the same class of N-T functions (so called finitely representable functions) as N-T functions 'which can be effectively given'. In this section we will prove a characterization of the class of finitely representable functions, utilizing the notion of G.F.R.E.. To describe the class of finitely representable functions Myhill considers a formalism (the set of finite formulas) defined below.

For each non-negative integer n it contains a numeral $\overline{n}$ denoting n. It also contains one place functional constant f denoting a N-T function. Terms are built up from numerals $\overline{n}$ and one place constant function symbol f in usual manner. Then each identity $t_1 = t_2$, where t_1, t_2 are terms, is an atomic formula. Formulas of our formalism are formed from atomic formulas by connectives of propositional calculus.

1. Each atomic formula is a finite formula.
2. If A_1, A_2 are finite formulas then $A_1 \& A_2, A_1 \vee A_2, A_1 \to A_2, \neg A_1$ are finite formulas.

Each finite formula refers to only finitely many values of the function denoted by f and hence asserts finitely testable properties of the function denoted by f. It is obvious what does it mean a N-T function f satisfies a finite formula A. Now let some Gödel numbering of finite formulas be fixed. A function f is a model of a set Ξ of finite formulas if f satisfies each formula in Ξ. A function f is a standard model of a set Ξ of finite formulas if for each term t in Ξ the value of t is a number. A N-T function is called finitely representable if there exists an axiomitizable system Ξ of finite formulas such that f is the unique standard model of Ξ.

Definition 5.11 *Let a recursive equation (11) be given and let for arbitrary N-T function* f *and* $\overline{x} \in \omega$ $F[\mathsf{f}](\overline{x}) \downarrow$. *Then the recursive equation (11) we will call an effectively testable property (E.T.P.) on N-T functions. If the function* f *is a solution of recursive equation (11) then we will say that an effectively testable property (11) is satisfied by the function* f.

$$F[f](x) \simeq 0 \tag{11}$$

Theorem 5.12 *A N-T function* $\mathsf{f}(x)$ *is finitely representable if and only if there exists an effectively testable property (11) on N-T functions such that the function* $\mathsf{f}(x)$ *is a unique function satisfying (11).*

Proof. $\Longrightarrow$Let $\mathsf{f}(x)$ be finitely representable function and let W_z be recursively enumerable set such that for any y $y \in W_z \Rightarrow y$ is a Gödel number of a some finite formula and let $\mathsf{f}(x)$ be a unique standard model of axiomatizable system $\Xi \stackrel{def}{\equiv} \{F_y : y \in W_z\}$. It is easy to construct an everywhere defined functional on N-T functions

$$Test[f](x) \stackrel{def}{\equiv} \begin{cases} 0 & \text{if} \quad x \text{ is a Gödel number of finite formula} \\ & \quad \text{and } F_x \text{ satisfied by the N-T function } f. \\ 1 & \quad \text{otherwise} \end{cases}$$

Consider the equation

$$Test[f](g(x)) \simeq 0 \tag{12}$$

where $Val\{g\} = W_z$. It is evident that an equation (12) is an E.T.P. on N-T functions satisfied only by N-T function $\mathsf{f}(x)$.

$\Longleftarrow$Let the function $\mathsf{f}(x)$ be a unique solution of recursive equation (11). In Myhill [12] it is proved that a N-T function f is finitely representable if and only if there exists a partial recursive functional $\Omega[f]$ such that $\Omega[\mathsf{f}]$ is undefined, while for each N-T function $\mathsf{f}' \neq \mathsf{f}$, $\Omega[\mathsf{f}']$ is a number n such that $(\exists x \leq n)(\mathsf{f}'(x) \neq \mathsf{f}(x))$. For such Ω_{f} consider the partial recursive functional

$$\Omega_{\mathsf{f}}[f] \stackrel{def}{\equiv} \mu z T^{\langle 1\langle 1\rangle\rangle}(e, \mu y \overline{sg}(F[f](y)), z)$$

where $e \in RecIdx$ is an index of functional $F[f](y)$. It is easy to see that $\Omega_{\mathsf{f}}[\mathsf{f}]$ undefined ,while for each N-T function $\mathsf{f}' \neq \mathsf{f}$ $\Omega_{\mathsf{f}}[\mathsf{f}']$ is a number n such that

$$(\exists x \leq n)(\mathsf{f}'(x) \neq \mathsf{f}(x)).$$

This completes the proof of the theorem. □

6 Appendix

Lemma 6.13 *There exists a recursive function $g(e, z)$ such that if an indexes $e, z \in RecIdx$ then $W_{g(e,z)}(x) = \left\{x \mid \exists p \Phi_z^{\langle 1,0\rangle}(p) \downarrow= x \& p \in W_e\right\}$.*

Proof. Let f, g have the following property

$$\begin{array}{c} W_x = Val\,\Phi_{f(x)} \& \left[W_x \neq \emptyset \Rightarrow \Phi_{f(x)} \text{ is general recursive}\right] \\ W_{q(x)} = Val\,\Phi_x \end{array}. \tag{13}$$

For the existence of the functions f, g with the property (13) see Rogers [13]. Then one may define the function $g(e, z) \stackrel{def}{\equiv} q(\langle 5, \langle 1\rangle, \langle z, f(e)\rangle\rangle)$. □

Lemma 6.14 *There exists a recursive function $q^{(m)}(x)$ such that if an index $e \in RecIdx$ of the set W_e^m is given then $q^{(m)}(e)$ is an index of a functional of signature $\langle m+1, \langle m, m+1\rangle\rangle$, such that the solution of the system of G.F.R.E. (3) with respect to the function symbol f_0^m represents the set W_e^m.*

Proof. Let us define $F\left[f_0^m, f_1^{m+1}\right](\overline{x}, y, e) \overset{def}{\equiv}$
$\overset{def}{\equiv} \Phi_{w_0}^{\langle m+2,\langle m,m+1\rangle\rangle}\left[f_0^m, f_1^{m+1}\right](\overline{x}, y, e)$
$\overset{def}{\equiv} F_1\left[f_0^m, f_1^{m+1}\right](\overline{x}, y, e) + F_2\left[f_0^m, f_1^{m+1}\right](\overline{x}, y, e)$
where $F_1\left[f_0^m, f_1^{m+1}\right](\overline{x}, y, e) \overset{def}{\equiv} \left|f_1^{m+1}(\overline{x}, y) - T^{\langle m,0\rangle}(e, \overline{x}, y)\right|$ and
$F_2\left[f_0^m, f_1^{m+1}\right](\overline{x}, y, e) \overset{def}{\equiv}$
$$\overset{def}{\equiv} \begin{cases} \overline{sg}\left(\mu t\left[f_1^{m+1}(\overline{x}, t) = 0\right] + 1\right) & \text{if } f_0^m(\overline{x}) = 0 \\ 0 & \text{if } f_0^m(\overline{x}) = 1 \& f_1^{m+1}(\overline{x}, y) = 1 \\ 1 & \text{otherwise} \end{cases}.$$
From the definition of F it is easy to see that (3) holds if and only if

$$F_1\left[\mathrm{f}_0^m, \mathrm{f}_1^{m+1}\right](\overline{x}, y, e) = 0 \tag{14}$$

$$F_2\left[\mathrm{f}_0^m, \mathrm{f}_1^{m+1}\right](\overline{x}, y, e) = 0. \tag{15}$$

Then (14) is satisfied only in the case if $\mathrm{f}_1^{m+1}(\overline{x}, y) \equiv T^{\langle m,0\rangle}(e, \overline{x}, y)$. Now we will prove that the recursive equation $F\left[\mathrm{f}_0^m, \mathrm{f}_1^{m+1}\right](\overline{x}, y, e) = 0$ has a unique solution $\mathrm{f}_0^m, \mathrm{f}_1^{m+1}$ where f_0^m is the characteristic function of the set W_e^m.

Let $x_1, ..., x_m, y \in N$ be arbitrary non-negative integers.

Case 1: f_0^m is the characteristic function of the set W_e^m.

1.1: Let $\langle x_1, ..., x_m\rangle \in W_e^m$ then $\exists y T^{\langle m,0\rangle}(e, \overline{x}, y) \Leftrightarrow \exists y\left[\mathrm{f}_1^{m+1}(x_1, ..., x_m, y) = 0\right]$ and $\mathrm{f}_0^m(x_1, ..., x_m) = 0$. From the construction of F_2 it is evident that $F\left[\mathrm{f}_0^m, \mathrm{f}_1^{m+1}\right](\overline{x}, y, e) = 0$.

1.2: Let $\langle x_1, ..., x_m\rangle \notin W_e^m$ then $\forall y \overline{T^{\langle m,0\rangle}}(e, \overline{x}, y) \Leftrightarrow \Leftrightarrow \forall y\left[\mathrm{f}_1^{m+1}(x_1, ..., x_m, y) = 1\right]$ and $\mathrm{f}_0^m(x_1, ..., x_m) = 1$, hence for arbitrary y we have $F\left[\mathrm{f}_0^m, \mathrm{f}_1^{m+1}\right](\overline{x}, y, e) = 0$.

Case 2: f_0^m is not the characteristic function of the set W_e^m.

2.1: Let $\exists x_1^*, ..., x_m^* (\langle x_1, ..., x_m\rangle \in W_e^m \& \mathrm{f}_0^m(x_1, ..., x_m) = 1)$ then we have $F_2\left[\mathrm{f}_0^m, \mathrm{f}_1^{m+1}\right](x_1, ..., x_k; y) \uparrow$ by construction of F_2.

2.2: Let $\exists x_1^*, ..., x_k^* (\langle x_1, ..., x_m\rangle \notin W_e^m \& \mathrm{f}_0^m(x_1, ..., x_m) = 0)$ then $\forall y\left[\mathrm{f}_1^{m+1}(x_1, ..., x_m, y) = 1\right]$ hence $F_{n+1}\left[\mathrm{f}_0^m, \mathrm{f}_1^{m+1}\right](x_1, ..., x_m; y) \uparrow$.

Now let us define $q^{(m)}(e) \overset{def}{\equiv} S_1^{m+1}(w_0, e)$. □

Lemma 6.15 *There exists a recursive function $k^{(m)}(x_0)$ such that if an index $e \in RecIdx$ of functional of signature $\langle m+1, \langle m, m+1\rangle\rangle$ is given and if the solution of the system of G.F.R.E. $\Phi_e^{\langle m+1,\langle m,m+1\rangle\rangle}\left[f_0^m, f_0^{m+1}\right](\overline{x}, y) = 0$ with respect to the function symbol f_0^m represents N-T predicate $P(x_1, ..., x_m)$ then $k^{(m)}(e) \in RecIdx$ is an index of a functional of signature $\langle m+1, \langle m, m+1\rangle\rangle$, such that the solution of the system of G.F.R.E. (4) with respect to the function symbol f_0^m represents $\overline{P}(x_1, ..., x_m)$.*

Proof. Let $h = \ulcorner \overline{sg}(x_0) \urcorner$ and let $R(e,t) \Leftrightarrow RecIdx(e) \,\&\, (e)_0 = t$. Now we define the function $k^{(m)}(e)$ as following.

$$k^{(m)}(e) \overset{def}{\equiv}$$

$$\overset{def}{\equiv} \begin{cases} e & \text{if } R(e,1) \vee R(e,2) \vee \\ & \quad \vee R(e,3) \\ \langle 5, (e)_1, \langle h, e\rangle\rangle & \text{if } R(e,4) \,\&\, (e)_2 = 0 \\ e & \text{if } R(e,4) \,\&\, (e)_2 \neq 0 \\ \left\langle 5, (e)_1, \left\langle k^{(m)}\left((e)_{2,0}\right), \ldots, k^{(m)}\left((e)_{2,lx((e)_2)}\right)\right\rangle\right\rangle & \text{if } R(e,5) \\ \left\langle 6, (e)_1, k^{(m)}((e)_2), k^{(m)}((e)_3)\right\rangle & \text{if } R(e,6) \\ \left\langle 7, (e)_1, k^{(m)}((e)_2)\right\rangle & \text{if } R(e,7) \\ 0 & \text{otherwise} \end{cases}$$

Using the fact that the set of primitive recursive functions is closed under the definition by cases and course-of-value-recursion, it is evident that the function $k^{(m)}$ is primitive recursive. It is easy to see from the construction of $k^{(m)}$ that if $e \in RecIdx$ is an index of functional $F\left[f_0^m, f_1^{m+1}\right](\overline{x}, y)$ then $k^{(m)}(e)$ is an index of functional $F^*\left[f_0^m, f_1^{m+1}\right](\overline{x}, y)$ obtained from original one by replacing all the occurrences of term $f_0^m(U_1, \ldots, U_m)$ with $\overline{sg}(f_0^m(U_1, \ldots, U_m))$. Now if the solution $\mathsf{f}_0, \mathsf{f}_1$ of recursive equation $F\left[f_0^m, f_1^{m+1}\right](\overline{x}, y) \simeq 0$ with respect to the function symbol f_0^m represents the predicate $P(x_1, \ldots, x_m)$ then the solution of $F^*\left[f_0^m, f_1^{m+1}\right](\overline{x}, y) \simeq 0$ do the same for $\overline{P}(x_1, \ldots, x_m)$. This completes the proof of the lemma. □

Lemma 6.16 *There exists a recursive function $r^{(m)}(t)$ such that if an index e of the set W_e^1 is given and for any $z \in W_e^1$ we have that $z \in RecIdx$ and z is an index of a functional of the signature $\langle m+1, \langle m, m+1\rangle\rangle$ so that the solution $\mathsf{f}_0^m, \mathsf{f}_0^{m+1}$ of the system of G.F.R.E. $\Phi_z^{\langle m+1,\langle m,m+1\rangle\rangle}\left[f_0^m, f_0^{m+1}\right](\overline{x}, y) = 0$ with respect to the function symbol f_0^m represents some predicate $P^{(z)}(x_1, \ldots, x_m)$. Then $r^{(m)}(e) \in RecIdx$ is an index of a functional of the signature $\langle m+1, \langle m, m+1\rangle\rangle$ that the solution of the system of G.F.R.E. (5) with respect to the function symbol f_0^m represents the predicate $\bigcup_{z \in W_e^1} P^{(z)}(x_1, \ldots, x_m)$.*

Proof. Let us define the numbers $h_1, \ldots, h_m, v_1, \ldots v_m, t_0, t_1$ and the function $d^{(m)}$ as follows:

$h_1 \overset{def}{\equiv} \ulcorner P_1^{m+1} \urcorner, \ldots, h_{m-1} \overset{def}{\equiv} \ulcorner P_{m-1}^{m+1} \urcorner$,

$h_m \overset{def}{\equiv} \ulcorner c\left(P_m^{m+1}, P_{m+1}^{m+1}\right) \urcorner$,

$v_1 \overset{def}{\equiv} \ulcorner P_1^m \urcorner, \ldots, v_m \overset{def}{\equiv} \ulcorner P_m^m \urcorner$,

$t_0 \overset{def}{\equiv} \ulcorner R\left(f_0^{m+1}\right) \urcorner$

$t_1 \overset{def}{\equiv} \ulcorner L\left(f_0^{m+1}\right) \urcorner$

$d^{(m)}(y, i) \overset{def}{\equiv}$

$$\stackrel{def}{\equiv}\begin{cases} y & \text{if } R(y,1)\vee \\ & \vee R(y,2)\vee \\ & \vee R(y,3) \\ \langle 5,(y)_1,\langle t_0,\langle 2,m+1,i\rangle,h_1,...,h_{m-1},h_m\rangle\rangle & \text{if } R(y,4)\,\& \\ & \&\,(y)_2=0\& \\ & \&\,(y)_1=m+1 \\ \langle 5,(y)_1,\langle t_1,\langle 2,m,i\rangle,v_1,...,v_m\rangle\rangle & \text{if } R(y,4)\,\& \\ & \&\,(y)_2=0\& \\ & \&\,(y)_1=m \\ \left\langle 5,(y)_1,\left\langle d^{(m)}\left((y)_{2,0},i\right),...,d^{(m)}\left((y)_{2,lx((y)_2)},i\right)\right\rangle\right\rangle & \text{if } R(y,5) \\ \left\langle 6,(y)_1,d^{(m)}\left((y)_2,i\right),d^{(m)}\left((y)_3,i\right)\right\rangle & \text{if } R(y,6) \\ \left\langle 7,(y)_1,d^{(m)}\left((y)_2,i\right)\right\rangle & \text{if } R(y,7) \\ 0 & \text{otherwise} \end{cases}$$

From the fact that the set of primitive recursive functions is closed under the definition by cases and course-of-value-recursion it follows that the function $d^{(m)}$ is primitive recursive. From the construction of $d^{(m)}$ it follows that if $z \in RecIdx$ is an index of functional $F\left[f_0^m, f_1^{m+1}\right](\overline{x}, y)$ then $d^{(m)}(i,z)$ is an index of functional $F^*\left[f_1^{m+1}\right](\overline{x}, y)$ obtained from the original one by replacing all the occurrences of terms $f_0^m(U_1,...,U_m)$, $f_1^{m+1}(V_1,...,V_m,V_{m+1})$ with $R\left(f_1^{m+1}(i,U_1,...,U_m)\right)$ and $L\left(f_1^{m+1}(i,V_1,...,c(V_m,V_{m+1}))\right)$ respectively. Now if $F\left[f_0^m, f_1^{m+1}\right](\overline{x}, y) \simeq 0$ with respect to the function symbol f_0^m represents the predicate $P(x_1,...,x_m)$ and for some $\mathfrak{f}_0^m, \mathfrak{f}_1^{m+1}$ $F\left[\mathfrak{f}_0^m, \mathfrak{f}_1^{m+1}\right](\overline{x}, y) \equiv 0$ then for arbitrary $\mathfrak{f}^{m+1}(\overline{x})$ such that

$$\begin{gathered} R\left(\mathfrak{f}^{m+1}(i,x_1,...,x_m)\right)=0 \Leftrightarrow P(x_1,...,x_m) \\ L\left(\mathfrak{f}^{m+1}(i,x_1,...,c(x_m,x_{m+1}))\right) \equiv \mathfrak{f}_1^{m+1}(x_1,...,x_m,x_{m+1}) \end{gathered} \tag{16}$$

$F^*\left[\mathfrak{f}_1^{m+1}\right](\overline{x}, y) \equiv 0$.

Now let an index e of the set W_e be given with the properties stated in the formulation of Lemma **6.16** and let for some $\mathfrak{f}_0^{m,i}, \mathfrak{f}_1^{m+1,i}$ $\Phi_z^{\langle m+1,\langle m,m+1\rangle\rangle}\left[\mathfrak{f}_0^{m,i}, \mathfrak{f}_1^{m+1,i}\right](\overline{x}, y) \equiv 0$ for $z \in W_e$. Then $\Phi_{f(e)}^{\langle 1,0\rangle}(0)$, $\Phi_{f(e)}^{\langle 1,0\rangle}(1)$, ..., $\Phi_{f(e)}^{\langle 1,0\rangle}(i)$, ..., where f is a function with the properties mentioned in Lemma **6.13** , is an enumeration of the set W_e . By means of the function $d^{(m)}$ we define the functional

$F_1\left[f_0^{m+1}\right](e, \overline{x}, y) \stackrel{def}{\equiv} \sum_{i=0}^{L(y)} \Phi_{d^{(m)}\left(\Phi_{f(e)}^{\langle 1,0\rangle}(i),i\right)}^{\langle m+1,\langle m+1\rangle\rangle}\left[f_0^{m+1}\right](\overline{x}, R(y))$ .From (16) it is easy to see that $F_1\left[\mathfrak{f}^{m+1}\right](e, \overline{x}, y) \equiv 0$ holds for some $\mathfrak{f}^{m+1}$ uniquely determined by e with the properties

$$\begin{gathered} R\left(\mathfrak{f}^{m+1}(i,x_1,...,x_m)\right)=0 \Leftrightarrow P^{(i)}(x_1,...,x_m), \\ L\left(\mathfrak{f}^{m+1}(i,x_1,...,c(x_m,x_{m+1}))\right) \equiv \mathfrak{f}_1^{m+1,i}(x_1,...,x_m,x_{m+1}). \end{gathered} \tag{17}$$

Hence $\mathfrak{f}^{m+1}$ is the solution of G.F.R.E. $F_1\left[f^{m+1}\right](e, \overline{x}, y) \simeq 0$ and the functions $\mathfrak{f}_0^{m,i}, \mathfrak{f}_1^{m+1,i}$: $\forall i \in \omega$ are recursive in $\mathfrak{f}^{m+1}$ according to (17).

We define the functional $F^*\left[f_0^m, f_0^{m+1}\right](\overline{x}, y)$ as following:

$F^*\left[f_0^m, f_0^{m+1}\right](e, \overline{x}, y) \stackrel{def}{\equiv} F_1\left[f_0^{m+1}\right](e, \overline{x}, y) + F_2\left[f_0^m, f_0^{m+1}\right](e, \overline{x}, y)$ where the functional F_2 is

$F_1\ f_0^m, f_0^{m+1}\ (\overline{x}, y) \stackrel{def}{\equiv}$

$$\stackrel{def}{\equiv} \begin{cases} \overline{sg}\left(\mu t\left[L\left(f_0^{m+1}(\overline{x},t)\right)=0\right]+1\right) & \text{if} \quad f_0^m(\overline{x})=0 \\ 0 & \text{if} \quad f_0^m(\overline{x})=1\&L\left(f_0^{m+1}(\overline{x},y)\right)=1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

The proof that, $F^*\left[f_0^m, f_0^{m+1}\right](e,\overline{x},y)\simeq 0$ has a unique solution and with respect to the function symbol f_0^m it represents the predicate $\bigcup_{z\in W_e^1} P^{(Z)}(x_1,\ldots,x_m)$, goes almost without changes like that of lemma 6.2 and we leave it as entirely a matter of routine. Now define $r^{(m)}(e)\stackrel{def}{\equiv} S_1^{m+1}(w_0,e)$ where w_0 is an index of functional F^*. □

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