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ON THE INTEGRABILITY WITH WEIGHT OF TRIGONOMETRIC SERIES

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Abstract. In this paper we have found the necessary and sufficient conditions for the power integrability with a weight of the sum of the sine and cosine series whose coefficients belong to a subclass of $\gamma RBVS$ class.

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1. Introduction

Were Young [15], Boas [1], and then Haywood [3] who have studied the integrability of the formal series

$$g(x) := \sum_{n=1}^{\infty} a_n \sin nx \tag{1.1}$$

and

$$f(x) := \sum_{n=1}^{\infty} b_n \cos nx \tag{1.2}$$

imposing certain conditions on the coefficients a_n and b_n respectively (we denote λ_n either a_n or b_n).

Their results deal with above mentioned trigonometric series whose coefficients are monotone decreasing. Lately, the monotonicity condition on the sequence $\{\lambda_n\}$ was replaced by Leindler [5] to a more general ones $\{\lambda_n\} \in R_0^+ BVS$.

A sequence $c := \{c_n\}$ of positive numbers tending to zero is of rest bounded variation, or briefly $R_0^+ BVS$, if it possesses the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(c) c_m \tag{1.3}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

Later on, Németh [8] considered weight functions more general than power one and obtained some sufficient conditions for the integrability of the sine series with such weights. Namely, he used the so-called almost increasing (decreasing) sequences.

A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called almost increasing (decreasing) if there exists a constant $C := C(\gamma) \geq 1$ such that

$$C\gamma_n \geq \gamma_m \quad (\gamma_n \leq C\gamma_m)$$

holds for any $n \geq m$.

Here and in the sequel, a function $\gamma(x)$ is defined by the sequence γ in the following way: $\gamma\left(\frac{\pi}{n}\right) := \gamma_n$, $n \in \mathbb{N}$ and there exist positive constants C_1 and C_2 such that $C_1\gamma_n \leq \gamma(x) \leq C_2\gamma_{n+1}$ for $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right)$.
In 2005 S. Tikhonov [11] has proved two theorems providing necessary and sufficient conditions for the $p-$th power integrability of the sums of sine and cosine series with weight γ. His results refine the assertions of such results presented earlier by others which show that such conditions depend on the behavior of the sequence γ.

We present Tikhonov's results below.

Theorem 1.1 ([11]). *Suppose that $\{\lambda_n\} \in R_0^+BVS$ and $1 \leq p < \infty$.*

(A) *If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_1 > 0$ such that the sequence $\{\gamma_n n^{-p-1+\varepsilon_1}\}$ is almost decreasing, then the condition*

$$\sum_{n=1}^{\infty} \gamma_n n^{p-2} \lambda_n^p < \infty \tag{1.4}$$

is sufficient for the validity of the condition

$$\gamma(x)|g(x)|^p \in L(0, \pi). \tag{1.5}$$

(B) *If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_2 > 0$ such that the sequence $\{\gamma_n n^{p-1-\varepsilon_2}\}$ is almost increasing, then the condition (1.4) is necessary for the validity of condition (1.5).*

Theorem 1.2 ([11]). *Suppose that $\{\lambda_n\} \in R_0^+BVS$ and $1 \leq p < \infty$.*

(A) *If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_3 > 0$ such that the sequence $\{\gamma_n n^{-1+\varepsilon_3}\}$ is almost decreasing, then the condition*

$$\sum_{n=1}^{\infty} \gamma_n n^{p-2} \lambda_n^p < \infty \tag{1.6}$$

is sufficient for the validity of the inclusion

$$\gamma(x)|f(x)|^p \in L(0, \pi). \tag{1.7}$$

(B) *If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_4 > 0$ such that the sequence $\{\gamma_n n^{p-1-\varepsilon_4}\}$ is almost increasing, then the condition (1.6) is necessary for the validity of condition (1.7).*

Some new results pertaining to related problems, with those we mentioned above, one can find for example in [12] when

$$\{\lambda_n\} \in \left\{ \{c_k\} : \sum_{k=m}^{2m-1} |c_k - c_{k+1}| \le K(c)\, c_n \right\},$$

in [13] when

$$\{\lambda_n\} \in \left\{ \{c_k\} : \sum_{k=m}^{2m-1} |c_k - c_{k+1}| \le K(c) \sum_{k=[m/a]}^{[am]} \frac{c_k}{k} \right\}$$

for some $a > 1$, in [2] when

$$\{\lambda_n\} \in \left\{ \{c_k\} : \sum_{k=m}^{\infty} |c_k - c_{k+1}| \le K(c)\, m^{\theta-1} \sum_{k=[m/a]}^{\infty} \frac{c_k}{k^\theta} \right\}$$

for some $a > 1$ and $\theta \in (0, 1]$, in [10] when

$$\{\lambda_n\} \in \left\{ \{c_k\} : \sum_{k=m}^{\infty} |c_k - c_{k+r}| \le K(c)\, m^{\theta-1} \sum_{k=[m/a]}^{\infty} \frac{c_k}{k^\theta} \right\}$$

for some $a > 1$ and $\theta \in (0, 1]$ and $r \in \mathbb{N}$, in [4] when

$$\{\lambda_n\} \in \left\{ \{c_k\} : \sum_{k=2m}^{\infty} k|c_k - c_{k+2}| \le \frac{K(c)}{m} \sum_{k=m}^{2m-1} k|c_k - c_{k+2}| \right\},$$

where $K(c)$ is a positive constant depending only on a nonnegative sequence $c = \{c_k\}$.

Now, for further investigations we recall an another class of sequences. Namely, was again Leindler [6] who introduced a new class of sequences which is a wider class than the class R_0^+BVS.

Definition 1.1. *A sequence $c := \{c_k\}$ of nonnegative numbers tending to zero belongs to $RBVS_+^{r,\delta}$, if it has the property*

$$\sum_{k=m}^{\infty} |c_k - c_{k+1}| \le \frac{K(c)}{m^{r+1+\delta}} \sum_{n=1}^{m} n^{r+1} c_n$$

for all natural numbers m, where $r, \delta \in \mathbb{R}$ and $K(c)$ is a positive constant depending only on the sequence c.

As is pointed out by Leindler [6], if $0 < \delta \le 1$ and $c \in R_0^+BVS$, then $c \in RBVS_+^{r,\delta}$ also holds true. Indeed,

$$c_m \le m^{1-\delta} c_m \le K(c) m^{-r-1-\delta} \sum_{n=1}^{m} n^{r+1} c_n.$$

Subsequently, the embedding relation $R_0^+BVS \subset RBVS_+^{r,\delta}$ holds true as well. Moreover, it is clear that for a nonnegative sequence $\{c_k\}$ and $m \in \mathbb{N}$

$$\frac{1}{m^{r+1+\delta_1}}\sum_{k=1}^{m} k^{r+1}c_k \le \frac{1}{m^{r+1+\delta_2}}(m)^{\delta_2-\delta_1}\sum_{k=1}^{m} k^{r+1}c_k \le \frac{1}{m^{r+1+\delta_2}}\sum_{k=1}^{m} k^{r+1}c_k,$$

when $\delta_2 \le \delta_1$, $r \in \mathbb{R}$ and

$$\frac{1}{m^{r_1+1+\delta}}\sum_{k=1}^{m} k^{r_1+1}c_k \le \frac{1}{m^{r_1+1+\delta}}(m)^{r_1-r_2}\sum_{k=1}^{m} k^{r_2+1}c_k = \frac{1}{m^{r_2+1+\delta}}\sum_{k=1}^{m} k^{r_2+1}c_k,$$

when $r_2 \le r_1$, $\delta \in \mathbb{R}$. Hence

$$RBVS_+^{r,\delta_1} \subseteq RBVS_+^{r,\delta_2} \quad (\delta_2 \le \delta_1)$$

and

$$RBVS_+^{r_1,\delta} \subseteq RBVS_+^{r_2,\delta} \quad (r_2 \le r_1).$$

Therefore in this paper we are concerned about finding the necessary and sufficient conditions on the sequence $\{\lambda_n\} \in RBVS_+^{r,\delta}$ so that $\gamma(x)|f(x)|^p \in L(0,\pi)$ and $\gamma(x)|g(x)|^p \in L(0,\pi)$, which indeed is the aim of this paper.

To achieve this goal we need some helpful statements given in next section.

2. Auxiliary lemmas

Lemma 2.1 ([7]). *Let $\lambda_n > 0$ and $a_n \ge 0$. Then*

$$\sum_{n=1}^{\infty}\lambda_n\left(\sum_{\nu=1}^{n}a_\nu\right)^p \le p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}a_n^p\left(\sum_{\nu=n}^{\infty}\lambda_\nu\right)^p, \quad p \ge 1.$$

Lemma 2.2 ([9]). *Let $\lambda_n > 0$ and $a_n \ge 0$. Then*

$$\sum_{n=1}^{\infty}\lambda_n\left(\sum_{\nu=n}^{\infty}a_\nu\right)^p \le p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}a_n^p\left(\sum_{\nu=1}^{n}\lambda_\nu\right)^p, \quad p \ge 1.$$

3. Main results

At first, we prove the following.

Theorem 3.1. *Suppose that $\{\lambda_n\} \in RBVS_+^{r,\delta}$, $r \ge 0$, $0 < \delta \le 1$ and $1 \le p < \infty$. If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_1 > 0$ such that the sequence $\{\gamma_n n^{\varepsilon_1-1-\delta p}\}$ is almost decreasing, then the condition*

$$\sum_{n=1}^{\infty}\gamma_n n^{p(2-\delta)-2}\lambda_n^p < \infty \tag{3.1}$$

is sufficient for the validity of the condition

$$\gamma(x)|g(x)|^p \in L(0,\pi). \tag{3.2}$$

Proof. First we denote

$$\widetilde{D}_n(x)=\sum_{k=1}^{n}\sin kx,\ n\in\mathbb{N}.$$

Using Abel's transformation, $\lambda_n\to 0$, and the well-known estimate $|\widetilde{D}_n(x)|=\mathcal{O}(1/x)$ we have

$$\begin{aligned}\sum_{n=m+1}^{\infty}\lambda_n\sin nx &= \lim_{N\to\infty}\left(\sum_{n=m+1}^{N-1}(\lambda_n-\lambda_{n+1})\widetilde{D}_n(x)+\lambda_N\widetilde{D}_N(x)-\lambda_{m+1}\widetilde{D}_m(x)\right)\\ &= \sum_{n=m+1}^{\infty}(\lambda_n-\lambda_{n+1})\widetilde{D}_n(x)-\sum_{n=m+1}^{\infty}(\lambda_n-\lambda_{n+1})\widetilde{D}_m(x).\end{aligned}$$

Whence, for $x\in\left(\frac{\pi}{n+1},\frac{\pi}{n}\right]$, since $|\sin nx|\le nx$, $|\widetilde{D}_n(x)|\le\frac{C}{x}$, and $\{\lambda_n\}\in RBVS_+^{r,\delta}$ we obtain

$$\begin{aligned}|g(x)| &\le C\left(x\sum_{k=1}^{n}k\lambda_k+n\sum_{k=n}^{\infty}|\lambda_k-\lambda_{k+1}|\right)\\ &\le C\left(\frac{1}{n}\sum_{k=1}^{n}k\lambda_k+\frac{1}{n^{r+\delta}}\sum_{k=1}^{n}k^{r+1}\lambda_k\right)\\ &\le C\left(\frac{1}{n}\sum_{k=1}^{n}k\lambda_k+\frac{1}{n^{\delta}}\sum_{k=1}^{n}k\lambda_k\right)\le\frac{C}{n^{\delta}}\sum_{k=1}^{n}k\lambda_k.\end{aligned}$$

Here and elsewhere, C denotes positive constant, which may be different in different cases. So, we get

$$\int_0^{\pi}\gamma(x)|g(x)|^p dx=\sum_{n=1}^{\infty}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}}\gamma(x)|g(x)|^p dx\le C\sum_{n=1}^{\infty}\frac{\gamma_n}{n^{2+\delta p}}\left(\sum_{k=1}^{n}k\lambda_k\right)^p.$$

The use of Lemma 2.1 implies

$$\int_0^{\pi}\gamma(x)|g(x)|^p dx\le C\sum_{n=1}^{\infty}\left(\frac{\gamma_n}{n^{2+\delta p}}\right)^{1-p}(n\lambda_n)^p\left(\sum_{k=n}^{\infty}\frac{\gamma_k}{k^{2+\delta p}}\right)^p.$$

Since $\{m^{\varepsilon_1-\delta p-1}\gamma_m\}$ is almost decreasing sequence, then we get

$$\sum_{k=n}^{\infty}\frac{\gamma_k}{k^{2+\delta p}}=\sum_{k=n}^{\infty}\frac{\gamma_k}{k^{1+\delta p-\varepsilon_1}k^{1+\varepsilon_1}}\le C\frac{\gamma_n}{n^{1+\delta p-\varepsilon_1}}\sum_{k=n}^{\infty}\frac{1}{k^{1+\varepsilon_1}}\le C\frac{\gamma_n}{n^{1+\delta p}}.$$

Thus, we obtain

$$\int_0^{\pi}\gamma(x)|g(x)|^p dx\le C\sum_{n=1}^{\infty}\gamma_n n^{p(2-\delta)-2}\lambda_n^p.$$

The proof is completed. □

Theorem 3.2. *Suppose that* $\{\lambda_n\}\in RBVS_+^{r,\delta}$, $r\ge 0$, $0<\delta\le 1$ *and* $1\le p<\infty$. *If the sequence* $\{\gamma_n\}$ *satisfies the condition: there exists an* $\varepsilon_2>0$ *such that the* *sequence* $\{\gamma_n n^{p-1-\varepsilon_2}\}$ *is almost increasing, then the condition*

$$\sum_{n=1}^{\infty} \gamma_n n^{p\delta-2} \lambda_n^p < +\infty \tag{3.3}$$

is necessary for the validity of condition (3.2).

Proof. Let us show first that $g(x) \in L(0,\pi)$. Namely, if $1 < p < +\infty$ and $p+q = pq$, then applying Hölder's inequality, we get

$$\int_0^{\pi} |g(x)|dx \leq \left(\int_0^{\pi} \gamma(x)|g(x)|^p dx\right)^{1/p} \left(\int_0^{\pi} (\gamma(x))^{-q/p} dx\right)^{1/q}.$$

Now using the estimation (see [11], page 440)

$$\int_0^{\pi} (\gamma(x))^{-q/p} dx < C,$$

we have

$$\int_0^{\pi} |g(x)|dx \leq C \left(\int_0^{\pi} \gamma(x)|g(x)|^p dx\right)^{1/p} < +\infty.$$

Let $p = 1$. Then we can set up that $\{\gamma_n\}$ is almost increasing, and whence

$$\begin{aligned} \int_0^{\pi} |g(x)|dx &\leq \sum_{n=1}^{\infty} \frac{1}{C\gamma_n} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \gamma(x)|g(x)|dx \\ &\leq \frac{1}{C\gamma_1} \int_0^{\pi} \gamma(x)|g(x)|dx < +\infty. \end{aligned}$$

Therefore, for all $p \in [1, +\infty)$ we showed that $g(x) \in L(0,\pi)$. Using this fact we can integrate the function $g(x)$ so that we have

$$F(x) := \int_0^x g(t)dt = \sum_{n=1}^{\infty} \lambda_n \int_0^x \sin nt dt = 2\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \sin^2 \frac{nx}{2}.$$

Denoting

$$d_\nu := \int_{\frac{\pi}{\nu+1}}^{\frac{\pi}{\nu}} |g(x)|dx, \quad \nu \in \mathbb{N},$$

and taking into account that $\{\lambda_n\} \in RBVS_+^{r,\delta}$

$$\begin{aligned} F(\pi/m) &\geq C\sum_{n=1}^{m} \frac{\lambda_n}{n} \left(\frac{n}{m}\right)^2 = \frac{C}{m^2} \sum_{n=1}^{m} n\lambda_n \\ &\geq \frac{C}{m^{r+2}} \sum_{n=1}^{m} n^{r+1}\lambda_n = Cm^{\delta-1} \frac{1}{m^{r+\delta+1}} \sum_{n=1}^{m} n^{r+1}\lambda_n \\ &\geq \frac{Cm^{\delta-1}}{K(\lambda)} \sum_{n=m}^{\infty} |\lambda_n - \lambda_{n+1}| \geq \frac{Cm^{\delta-1}\lambda_m}{K(\lambda)} \end{aligned}$$

then

$$\lambda_n \leq Cn^{1-\delta} F(\pi/n) \leq Cn^{1-\delta} \sum_{\nu=n}^{\infty} d_\nu.$$

So we have

$$I := \sum_{n=1}^{\infty} \gamma_n n^{p\delta-2} \lambda_n^p \leq C \sum_{n=1}^{\infty} \gamma_n n^{p-2} \left(\sum_{\nu=n}^{\infty} d_\nu \right)^p .$$

The use of Lemma 2.2 gives

$$I \leq C \sum_{n=1}^{\infty} d_n^p \left(\gamma_n n^{p-2}\right)^{1-p} \left(\sum_{\nu=1}^{n} \gamma_\nu \nu^{p-2} \right)^p .$$

The sequence $\{\gamma_n n^{p-1-\varepsilon_2}\}$ is almost increasing, by assumption, which implies

$$\begin{aligned} I &\leq C \sum_{n=1}^{\infty} d_n^p \left(\gamma_n n^{p-2}\right)^{1-p} \left(\sum_{\nu=1}^{n} \frac{\gamma_\nu \nu^{p-1-\varepsilon_2}}{\nu^{1-\varepsilon_2}} \right)^p \\ &\leq C \sum_{n=1}^{\infty} d_n^p \left(\gamma_n n^{p-2}\right)^{1-p} \left(\gamma_n n^{p-1-\varepsilon_2} \sum_{\nu=1}^{n} \frac{1}{\nu^{1-\varepsilon_2}} \right)^p \\ &\leq C \sum_{n=1}^{\infty} d_n^p \left(\gamma_n n^{p-2}\right)^{1-p} \left(\gamma_n n^{p-1}\right)^p \leq C \sum_{n=1}^{\infty} d_n^p \gamma_n n^{2(p-1)}. \end{aligned}$$

Now, if $1 < p < +\infty$ and $q = \frac{p}{p-1}$, then applying Hölder's inequality, we easily get

$$d_n^p = \left(\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |g(x)| dx \right)^p \leq C n^{2(1-p)} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |g(x)|^p dx.$$

Subsequently, we obtain that

$$\begin{aligned} I &\leq C \sum_{n=1}^{\infty} \gamma_n \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |g(x)|^p dx \\ &\leq C \sum_{n=1}^{\infty} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \gamma(x) |g(x)|^p dx \leq C \int_0^{\pi} \gamma(x) |g(x)|^p dx. \end{aligned}$$

For $p = 1$, we also have

$$I \leq C \sum_{n=1}^{\infty} \gamma_n d_n \leq C \int_0^{\pi} \gamma(x) |g(x)| dx.$$

The proof is completed. □

Theorem 3.3. *Suppose that* $\{\lambda_n\} \in RBVS_+^{r,\delta}$, $r \geq 0$, $0 < \delta \leq 1$ *and* $1 \leq p < \infty$. *If the sequence* $\{\gamma_n\}$ *satisfies the condition: there exists an* $\varepsilon_3 > 0$ *such that the sequence* $\{\gamma_n n^{\varepsilon_3 - 1}\}$ *is almost decreasing, then the condition (3.1) is sufficient for the validity of the condition*

$$\gamma(x) \, |f(x)|^p \in L(0, \pi). \tag{3.4}$$

Proof. Similar as in the proof of Theorem 3.1, we have

$$\begin{aligned} |f(x)| &\leq \left| \sum_{k=1}^{n} \lambda_k \cos kx \right| + \left| \sum_{k=n+1}^{\infty} \lambda_k \cos kx \right| \\ &\leq \sum_{k=1}^{n} \lambda_k + \sum_{k=n}^{\infty} |\lambda_k - \lambda_{k+1}| \, |D_k(x)| + \lambda_{n+1} \, |D_n(x)| , \end{aligned}$$

where

$$D_n(x)=\sum_{k=1}^{n}\cos kx,\quad n\in\mathbb{N}.$$

Hence, for $x\in\left(\frac{\pi}{n+1},\frac{\pi}{n}\right]$ since $|D_n(x)|\le\frac{C}{x}$ and $\{\lambda_n\}\in RBVS_{+}^{r,\delta}$

$$\begin{aligned}|f(x)| &\le C\left(\sum_{k=1}^{n}\lambda_k+n\sum_{k=n}^{\infty}|\lambda_k-\lambda_{k+1}|\right)\\ &\le C\left(\sum_{k=1}^{n}\lambda_k+\frac{1}{n^{r+\delta}}\sum_{k=1}^{n}k^{r+1}\lambda_k\right)\\ &\le C\left(\sum_{k=1}^{n}\lambda_k+\sum_{k=1}^{n}k^{1-\delta}\lambda_k\right)\le C\sum_{k=1}^{n}k^{1-\delta}\lambda_k.\end{aligned}$$

Therefore

$$\int_0^{\pi}\gamma(x)|f(x)|^p dx=\sum_{n=1}^{\infty}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}}\gamma(x)|f(x)|^p dx\le C\sum_{n=1}^{\infty}\frac{\gamma_n}{n^2}\left(\sum_{k=1}^{n}k^{1-\delta}\lambda_k\right)^p.$$

Using Lemma 2.1 and the fact that $\{m^{\varepsilon_3-1}\gamma_m\}$ is almost decreasing sequence, we obtain similar as in the proof of Theorem 3.1 that

$$\begin{aligned}\int_0^{\pi}\gamma(x)|f(x)|^p dx &\le C\sum_{n=1}^{\infty}\left(\frac{\gamma_n}{n^2}\right)^{1-p}\left(n^{1-\delta}\lambda_n\right)^p\left(\sum_{k=n}^{\infty}\frac{\gamma_k}{k^2}\right)^p\\ &\le C\sum_{n=1}^{\infty}\left(\frac{\gamma_n}{n^2}\right)^{1-p}\left(n^{1-\delta}\lambda_n\right)^p\left(\sum_{k=n}^{\infty}\frac{\gamma_k k^{\varepsilon_3-1}}{k^{1+\varepsilon_3}}\right)^p\\ &\le C\sum_{n=1}^{\infty}\left(\frac{\gamma_n}{n^2}\right)^{1-p}\left(n^{1-\delta}\lambda_n\right)^p\left(\gamma_n n^{\varepsilon_3-1}\sum_{k=n}^{\infty}\frac{1}{k^{1+\varepsilon_3}}\right)^p\\ &\le C\sum_{n=1}^{\infty}\left(\frac{\gamma_n}{n^2}\right)^{1-p}\left(n^{1-\delta}\lambda_n\right)^p\left(\gamma_n n^{-1}\right)^p\le C\sum_{n=1}^{\infty}\gamma_n n^{p(2-\delta)-2}\lambda_n^p.\end{aligned}$$

This ends our proof. □

Theorem 3.4. *Suppose that $\{\lambda_n\}\in RBVS_{+}^{r,\delta}$, $r\ge 0$, $0<\delta\le 1$ and $1\le p<\infty$. If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_2>0$ such that the sequence $\{\gamma_n n^{p-1-\varepsilon_2}\}$ is almost increasing, then the condition (3.3) is necessary for the validity of condition (3.4).*

Proof. Similar as in the proof of Theorem 3.2 we can show that the condition (3.4) implies $f(x)\in L(0,\pi)$. Integrating the function f, we write

$$H(x)=\int_0^x f(t)\,dt=\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\sin nx.$$

Now, we prove if $\{\lambda_n\} \in RBVS_+^{r,\delta}$ then $\{\frac{\lambda_n}{n}\} \in RBVS_+^{r,\delta}$. Suppose $\{\lambda_n\} \in RBVS_+^{r,\delta}$. Then for $m \in \mathbb{N}$

$$\begin{aligned}
\sum_{k=m}^{\infty}\left|\frac{\lambda_k}{k}-\frac{\lambda_{k+1}}{k+1}\right| &\leq \sum_{k=m}^{\infty}\frac{1}{k+1}\left|\lambda_k-\lambda_{k+1}\right|+\sum_{k=m}^{\infty}\frac{1}{k\,(k+1)}\lambda_k \\
&\leq \frac{1}{m+1}\sum_{k=m}^{\infty}\left|\lambda_k-\lambda_{k+1}\right|+\sum_{k=m}^{\infty}\frac{1}{k\,(k+1)}\sum_{l=k}^{\infty}\left|\lambda_l-\lambda_{l+1}\right| \\
&\leq \frac{K\,(\lambda)}{m^{2+r+\delta}}\sum_{n=1}^{m}n^{r+1}\lambda_n+\sum_{l=m}^{\infty}\left|\lambda_l-\lambda_{l+1}\right|\sum_{k=m}^{\infty}\frac{1}{k^2} \\
&\leq \frac{K\,(\lambda)}{m^{2+r+\delta}}\sum_{n=1}^{m}n^{r+1}\lambda_n+\frac{C}{m}\sum_{l=m}^{\infty}\left|\lambda_l-\lambda_{l+1}\right| \\
&\leq \frac{(1+C)\,K\,(\lambda)}{m^{2+r+\delta}}\sum_{n=1}^{m}n^{r+1}\lambda_n \leq \frac{(1+C)\,K\,(\lambda)}{m^{1+r+\delta}}\sum_{n=1}^{m}n^{r+1}\frac{\lambda_n}{n},
\end{aligned}$$

whence $\{\frac{\lambda_n}{n}\} \in RBVS_+^{r,\delta}$.

Applying Theorem 3.2 to the function H we obtain

$$\sum_{n=1}^{\infty}\gamma_n^* n^{p\delta-2}\lambda_n^p \leq C\int_0^{\pi}\gamma^*\,(x)\left|H\,(x)\right|^p dx,$$

where $\{\gamma_n^*\}$ satisfies the following condition: there exists $\varepsilon > 0$ such that the sequence $\left\{\gamma_n^* n^{p-1-\varepsilon}\right\}$ is almost increasing. For $\gamma_n^* = \gamma_n n^p$, this condition is obviously satisfied. Then

$$\begin{aligned}
I &:= \sum_{n=1}^{\infty}\gamma_n n^{p\delta-2}\lambda_n^p = \sum_{n=1}^{\infty}\gamma_n n^p n^{p\delta-2}\left(\frac{\lambda_n}{n}\right)^p \\
&= \sum_{n=1}^{\infty}\gamma_n^* n^{p\delta-2}\lambda_n^p \leq C\int_0^{\pi}\frac{\gamma\,(x)}{x^p}\left|H\,(x)\right|^p dx \\
&\leq C\sum_{n=1}^{\infty}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}}\frac{\gamma\,(x)}{x^p}\left(\int_0^{x}\left|f\,(t)\right|dt\right)^p dx \\
&\leq C\sum_{n=1}^{\infty}\gamma_n n^{p-2}\left(\int_0^{\frac{\pi}{n}}\left|f\,(t)\right|dt\right)^p = C\sum_{n=1}^{\infty}\gamma_n n^{p-2}\left(\sum_{v=k}^{\infty}\int_{\frac{\pi}{v+1}}^{\frac{\pi}{v}}\left|f\,(t)\right|dt\right)^p.
\end{aligned}$$

Denoting

$$f_v = \int_{\frac{\pi}{v+1}}^{\frac{\pi}{v}}\left|f\,(t)\right|dt, \quad v \in \mathbb{N}$$

and using Lemma 2.2 we get

$$I \leq C\sum_{n=1}^{\infty}\gamma_n n^{2p-2}\left(f_n\right)^p.$$

Now, if $1 < p < +\infty$ and $q = \frac{p}{p-1}$, then applying Hölder's inequality, we easily get

$$f_n^p = \left(\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |f(x)| dx \right)^p \leq C n^{2(1-p)} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |f(x)|^p dx.$$

Subsequently, we obtain that

$$\begin{aligned} I &\leq C \sum_{n=1}^{\infty} \gamma_n \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |f(x)|^p dx \\ &\leq C \sum_{n=1}^{\infty} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \gamma(x) |f(x)|^p dx \leq C \int_0^{\pi} \gamma(x) |g(x)|^p dx. \end{aligned}$$

For $p = 1$, we also have

$$I \leq C \sum_{n=1}^{\infty} \gamma_n d_n \leq C \int_0^{\pi} \gamma(x) |g(x)| dx,$$

and the proof is completed. □

Remark 3.1. *Since $R_0^+ BVS \subset RBVS_+^{r,1}$, then Theorem 1.1 and Theorem 1.2 are consequences of our results.*

Remark 3.2. *We know that the class of zero monotone decreasing sequences is a subclass of the class $R_0^+ BVS$. Whence, our results also hold true when condition $\{\lambda_n\} \in RBVS_+^{r,\delta}$ is replaced with condition $\{\lambda_n\} \in M := \{\mathbf{c} : c_n \downarrow 0\}$.*

4. Conclusions

The integrability of functions defined by trigonometric series has been attractive for lots of researchers during last six decades. The questions of integrability with weight of such functions, whose coefficients of their trigonometric series belong to various classes of sequences such as the decreasing sequences [3], the power-monotone sequences [8], the quasi-monotone sequences [14] and the general monotone sequences which is very important class for such questions, see [5], [2], [10], [12] and [13], are of the great interest. Here, in the present paper, we go one step further, finding the necessary and sufficient conditions for the power integrability with a weight of the sum of the sine and cosine series whose coefficients belong to the $RBVS_+^{r,\delta}$, $r \geq 0$, class and in the same time covering the results proved previously by others. In our results, $0 < \delta \leq 1$, we assume that the quantities

$$\sum_{n=1}^{\infty} \gamma_n n^{p(2-\delta)-2} \lambda_n^p \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n n^{p\delta-2} \lambda_n^p$$

are finite, which both coincide, $\delta = 1$, with finiteness of the famous quantity

$$\sum_{n=1}^{\infty} \gamma_n n^{p-2} \lambda_n^p.$$

Among this, we have showed as well the embedding relation

$$RBVS_+^{r;\delta_1} \subseteq RBVS_+^{r;\delta_2}, \text{ when } 0 < \delta_2 \leq \delta_1 \leq 1.$$

The $RBVS_+^{r;\delta}$ class seems to be considered here for the second time since it has been introduced and employing it, especially in the proof of our findings, shows that it could also be useful in other topics similar to this already considered here.

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