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INITIAL-VALUE METHODS FOR INTEGRAL EQUATIONS ARISING IN THEORIES OF THE SOLAR ATMOSPHERE

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In theories of the solar chromosphere and prominences, a problem is to determine the intensity of radiation emerging from the surface of the atmosphere,

$$E(x, f)=\alpha(f)\int_0^x e^{-\alpha(f)(x-t)} B(t)dt,$$

where the source function, $B(t)$, satisfies the integral equation

$$B(t)=g(t)+\int_0^x k(|t-t'|)B(t')dt'.$$

Computed line profiles from various models can then be compared with observed profiles.

In this note, a computationally useful initial-value theory for the direct evaluation of $E(x, f)$ is presented. The source function and resolvent may also be determined if desired. These considerations bear significantly on the solution of inverse problems for estimating the structure of the atmosphere, given observed line profiles.

1. *Introduction.* Fredholm integral equations are fundamental equations in transport theory, optimal filtering theory, and operation research. They have been studied from the invariant imbedding viewpoint in the series of papers [1—13]. In the papers of Kostik [14] and Yakovkin—Zel'dinà [15] on the solar chromosphere and prominences, the basic task is to solve an integral equation of the form [16]

$$B(t)=g(t)+\int_0^x k(|t-t'|)B(t')dt', \quad 0\leqslant t\leqslant x, \tag{1}$$

where

$$k(r)=\int_{-\infty}^{\infty} \alpha^2(f)\, E_1[\alpha(f)\, r]\, df, \quad r>0, \tag{2}$$

$$E_1(s)=\int_1^{\infty} e^{-sz}\, dz/z, \quad s>0. \tag{3}$$

Here B is the source function at optical altitude t, x is the optical thickness, and α is a nonnegative given function. The function $g(t)$ is also given; it corresponds to the strength of the emitting sources of radiation. The aim is to determine the intensity of radiation emerging normal to the surface of the atmosphere,

$$E(x,f)=\alpha(f)\int_0^x e^{-\alpha(f)(x-t)}\, B(t)\, dt, \tag{4}$$

for comparison with observations of line profiles.

In the papers [4, 7, 13,], it was shown that the solution of Eq. (1), for a kernel represented in the exponential form

$$k(r)=\int_a^b e^{-ry}\, \rho(y)\, dy, \quad r>0, \tag{5}$$

is the solution of an initial-value problem, and conversely. The kernel of the problem at hand, namely that given by Eq. (2), may be ragarded as having been tabulated as a function of r. A fitting of exponentials to the tabulated data could provide an approximate weighting function $\rho(y)$. The fitting of exponentials may, however, entail numerical complications. Fortunately, it can be avoided in the following manner.

Write the kernel of Eq. (2) in the form of a double integral,

$$k(r)=\int_{-\infty}^{\infty}\int_1^{\infty} \alpha^2(f)\, e^{-\alpha(f)zr}\, (z^{-1})\, dz\, df, \quad r>0. \tag{6}$$

The earlier theory can then be applied with y being considered a vector whose components are z and f.

In this note, an initial-value theory for a slightly more general kernel is presented. It is shown how to determine the emergent intensity E *directly*, rather than by first determining B and then performing the quadrature indicated in the definition of E. It is also not necessary to compute the resolvent, although it may readily be calculated if desired [6]. In this theory, the emergent intensity is the solution of an initial-value problem in which the independent variable is the interval length or x, the optical thickness. The solution is determined as the thickness is varied from $x=0$, when $E\equiv 0$, to $x=$ the desired thickness value. The primary significance of this initial value theory is that it converts the original problem into a form that can be effectively solved by modern analog and digital computers.

Much valuable information is available in [17—20].

2. *Statement of the initial-value problem for the emergence function.* Consider the integral equation

$$u(t, x) = g(t) + \int_0^x k(|t-t'|)\, u(t', x)\, dt', \qquad 0 \leqslant t \leqslant x, \tag{7}$$

where the kernel is represented in the form

$$k(r) = \iint e^{-a(z, f)r}\, w(z, f)\, dz\, df, \quad r > 0. \tag{8}$$

The variables z and f, which may be complex, vary over fixed domains. In Eq. (7), the dependence of the solution on the upper limit, x, is explicitly indicated. The task is to evaluate the "emergence" function

$$e(x, z, f) = \int_0^x e^{-a(z, f)(x-t)} u(t, x)\, dt. \tag{9}$$

The function $e = e(x, z, f)$ satisfies the exact initial-value problem given by the differential equations

$$\begin{aligned} e_x(x, z, f) &= -a(z, f)\, e(x, z, f) + \\ &+ \left[g(x) + \iint e(x, z', f')\, w(z', f')\, dz'df' \right] [X(x, z, f) - 1], \end{aligned} \tag{10}$$

$$X_x(x, z, f) = Y(x, z, f) \iint Y(x, z', f')\, w(z', f')\, dz'df', \tag{11}$$

$$\begin{aligned} Y_x(x, z, f) &= -a(z, f)\, X(x, z, f) + \\ &+ X(x, z, f) \iint Y(x, z', f')\, w(z', f')\, dz'df', \end{aligned} \tag{12}$$

and the initial conditions

$$e(x, z, f)\big|_{x=0} = 0, \tag{13}$$

$$X(x, z, f)\big|_{x=0} = 1, \tag{14}$$

$$Y(x, z, f)\big|_{x=0} = 1. \tag{15}$$

These equations will be derived in Section 3. Later, a numerical method based on the exact equations will be described. Equations (11) and (12) are related to those given by Ivanov [21].

3. *Derivation of the initial-value problem from the integral equation.* In the theory to follow, important roles will be played by the Sobolev function Φ which is the solution of the integral equation [22]

$$\Phi(t, x) = k(x-t) + \int_0^x k(|t-t'|)\,\Phi(t', x)\,dt', \tag{16}$$

and by J, the solution of the auxiliary equation

$$J(t, x, z, f) = e^{-\alpha(z, f)(x-t)} + \int_0^x k(|t-t'|)\,J(t', x, z, f)\,dt'. \tag{17}$$

In view of the fact that the forcing function in Eq. (16) is expressible in the from

$$k(x-t) = \iint e^{-\alpha(z, f)(x-t)}\,w(z, f)\,dz\,df, \tag{18}$$

the Φ function may be expressed as an integral of J,

$$\Phi(t, x) = \iint J(t, x, z, f)\,w(z, f)\,dz\,df. \tag{19}$$

We first obtain an equation for $u_x(t, x)$ (see, for example, Ref. 4). Differentiate both members of the integral Eq. (7) with respect to x keeping t fixed,

$$u_x(t, x) = k(x-t)\,u(x, x) + \int_0^x k(|t-t'|)\,u_x(t', x)\,dt'. \tag{20}$$

Regard this as an integral equation for $u_x(t, x)$. The solution is $u(x, x)$ times the solution when $k(x-t)$ is the forcing function or (see Eq. (16)),

$$u_x(t, x) = u(x, x)\,\Phi(t, x). \tag{21}$$

The function

$$U(x) = u(x, x) \tag{22}$$

must be evaluated. It is, from Eqs. (7) and (8),

$$U(x) = g(x) + \int_0^x \left[\iint e^{-\alpha(z, f)(x-t)} w(z, f)\, dz df \right] u(t, x)\, dt. \tag{23}$$

Interchange the order of integration and make use of the definition of the function $e(x, z, f)$ in Eq. (9) so that

$$U(x) = g(x) + \iint e(x, z, f)\, w(z, f)\, dz\, df. \tag{24}$$

To obtain a differential equation for e, differentiate Eq. (9):

$$e_x(x, z, f) = -\alpha(z, f)\, e(x, z, f) + \int_0^x e^{-\alpha(z, f)(x-t)} u_x(t, x)\, dt. \tag{25}$$

Then use Eq. (21) for $u_x(t, x)$ so that the above equation becomes

$$e_x(x, z, f) = -\alpha(z, f)\, e(x, z, f) + U(x) \int_0^x e^{-\alpha(z, f)(x-t)} \Phi(t, x)\, dt. \tag{26}$$

The integral which appears in this equation must be evaluated.

Consider the two integral equations

$$u_1(t, x) = g_1(t) + \int_0^x k(|t - t'|)\, u_1(t', x)\, dt', \tag{27}$$

$$u_2(t, x) = g_2(t) + \int_0^x k(|t - t'|)\, u_2(t', x)\, dt'. \tag{28}$$

Through cross multiplication, integration on t from 0 to x, and cancellation of like terms, there results the Hopf auxiliary theorem [23]:

$$\int_0^x u_1(t, x)\, g_2(t)\, dt = \int_0^x g_1(t)\, u_2(t, x)\, dt. \tag{29}$$

This equation, together with Eqs. (16) and (17), enables us to identify the integral in Eq. (26) as

$$\int_0^x \Phi(t,\ x)\, e^{-\alpha(z,\ f)(x-t)}\, dt = \int_0^x k(x-t)\, J(t,\ x,\ z,\ f)\, dt = \\ = J(x,\ x,\ z,\ f) - 1 = X(x,\ z,\ f) - 1. \qquad (30)$$

Here, $X = X(x,\ z,\ f,)$ is defined to be J evaluated at $t = x$

$$X(x,\ z,\ f) = J(x,\ x,\ z,\ f). \qquad (31)$$

Let us at this time also introduce the function $Y = Y(x,\ z,\ f)$ which is J evaluated at $t = 0$:

$$Y(x,\ z,\ f) = J(0,\ x,\ z,\ f). \qquad (32)$$

These X and Y functions are generalizations of the X and Y functions [24, 25] or the φ and ψ functions [22, 26] of radiative transfer theory.

Equation (26) may now be written as

$$e_x(x,\ z,\ f) = -\alpha(z,\ f)\, e(x,\ z,\ f) + U(x)\, [X(x,\ z,\ f) - 1], \qquad (33)$$

where $U(x)$ is expressed in terms of e by Eq. (24). This is the desired Eq. (10). We next must find an equation for X.

Put $x-\tau$ in place of t in Eq. (17):

$$J(x-\tau,\ x,\ z,\ f) = e^{-\alpha(z,\ f)\tau} + \int_0^x k(|\tau-\tau'|)\, J(x-\tau',\ x,\ z,\ f)\, d\tau'. \qquad (34)$$

Differentiate with respect to x:

$$(d/dx)\, J(x-\tau,\ x,\ z,\ f) = k(x-\tau)\, J(0,\ x,\ z,\ f) + \\ + \int_0^x k(|\tau-\tau'|)\, (d/dx)\, J(x-\tau',\ x,\ z,\ f)\, d\tau'. \qquad (35)$$

Regard this as an integral equation for $(d/dx)\, J(x-\tau,\ x,\ z,\ f)$; the solution is

$$(d/dx)\, J(x-\tau,\ x,\ z,\ f) = Y(x,\ z,\ f)\, \Phi(\tau,\ x). \qquad (36)$$

Putting $\tau = 0$, we obtain

$$(d/dx)\, J(x,\ x,\ z,\ f) = Y(x,\ z,\ f)\, \Phi(0,\ x), \qquad (37)$$

or

$$X_x(x, z, f) = Y(x, z, f) \iint Y(x, z, f)\, w(z, f)\, dz\, df, \tag{38}$$

by Eqs. (19), (31) and (32). This is the second of the desired equation.

An equation for Y is next required. Differentiate Eq. (17):

$$\begin{aligned} J_x(t, x, z, f) &= -\alpha(z, f)\, e^{-\alpha(z, f)(x-t)} + \\ &+ k(x-t)\, J(x, x, z, f) + \int_0^x k(|t-t'|)\, J_x(t', x, z, f)\, dt'. \end{aligned} \tag{39}$$

The solution of this integral equation is

$$J_x(t, x, z, f) = -\alpha(z, f)\, J(t, x, z, f) + X(x, z, f)\, \Phi(t, x). \tag{40}$$

Put $t = 0$:

$$\begin{aligned} Y_x(x, z, f) &= -\alpha(z, f)\, Y(x, z, f) + \\ &+ X(x, z, f) \iint Y(x, z, f)\, w(z, f)\, dz\, df. \end{aligned} \tag{41}$$

The complete set of differential equations is at hand.

The initial conditions on e, X and Y, when $x = 0$, are readily seen from the definitions to be those given in Eqs. (13)—(15).

The initial-value problem is to simultaneously solve Eqs. (10)—(12) with the initial conditions of Eqs. (13)—(15) imposed at $x = 0$. The value of x (the upper limit of the integral equation) is increased until it attains the desired value. Along the way, solutions are obtained for all x less than this final value. We emphasize that it is *not* necessary to determine $u(t, x)$ in order to evaluate $e(v, x)$.

4. *Initial-value problem for the solution of the integral equation.* In the event that the value of $u(t, x)$ is desired at a fixed point t, this too can be obtained as the solution of an initial-value problem. The equations are Eqs. (10)—(15) together with the equations (Eqs. (21) and (40)),

$$u_x(t, x) = U(x)\, \Phi(t, x), \tag{42}$$

$$J_x(t, x, z, f) = -\alpha(z, f)\, J(t, x, z, f) + X(x, z, f)\, \Phi(t, x), \quad x > t, \tag{43}$$

where

$$U(x) = g(x) + \iint e(x, z, f)\, w(z, f)\, dz\, df, \tag{44}$$

$$\Phi(t, x) = \iint J(t, x, z, f)\, w(z, f)\, dz\, df, \tag{45}$$

The additional initial conditions required are at $x = t$,

$$u(t, t) = g(t) + \iint e(t, z, f)\, w(z, f)\, dz\, df, \tag{46}$$

$$J(t, t, z, f) = X(t, z, f). \tag{47}$$

The procedure is to solve Eqs. (10) — (15) from $x = 0$ to $x = t$, then adjoin Eqs. (42) and (43), imposing the conditions in Eqs. (46) and (47). The enlarged system is simultaneously solved for increasing x until x has the maximum desired value.

Internal intensities may also be determined directly by the method of Refs. [2] and [3].

Matters concerning mathematical rigor are discussed in Ref. [13].

5. *Numerical method.* The computational procedure is based on the ability of modern computing machines to effectively solve large systems of ordinary differential equations, in the order of several thousands, subject to a complete set of initial conditions.

The differential-integral equations of the exact theory are replaced by a system of ordinary differential equations in which the definite integrals are approximated by sums according to a quadrature formula [1, 2, 27, 28]. A suitably chosen quadrature formula can yield a very good approximation. For example, the definite integral in Eq. (10) may be approximated as

$$\iint e(x, z, f)\, w(z, f)\, dz\, df \cong \sum_{i=1}^{M} \sum_{j=1}^{N} e(x, z_i, f_j)\, W_{ij}. \tag{48}$$

The function of three variables, $e(x, z, f)$, is replaced by a matrix function of one variable, $e_{ij}(x) = e(x, z_i, f_j)$. The differential-integral equation reduces to the system of ordinary differential equations

$$\begin{aligned} (d/dx)\, e_{ij}(x) &= -\alpha(z_i, f_j)\, e_{ij}(x) + \\ &+ \left[g(x) + \sum_{m,n} e_{mn}(x)\, W_{mn} \right] [X_{ij}(x) - 1], \end{aligned} \tag{49}$$

and similary for the remaining equations.

Successful numerical calculations have been performed for the case when $e = e(x, z)$, i. e., for a one-parameter family of functions of x (see Ref. [28]). There is little doubt that the proposed method is feasible for the two-parameter case. Numerical experiments are planned.

6. *Discussion.* To apply the general theory to the problem of the solar atmosphere as given in Eqs. (1)—(6), set

$$\alpha(z, f) = \alpha(f) z \tag{50}$$

$$w(z, f) = \alpha^2(f) z^{-1}. \tag{51}$$

The emergent intensity is obtained as

$$E(x, f) = \alpha(f) e(x, 1, f). \tag{52}$$

Note that z corresponds to a direction secant.

One of the basic purposes of creating models of solar and planetary atmospheres is to seek to deduce unknown aspects of the medium or of the sources of energy, based on available observations. For example, Kostik shows that the distribution function $g(t)$ used by him does not fully account for all the major aspects of the observed line profiles [14]. The problem of determining a function $g(t)$, in a certain class of functions, which will best explain the given data is an inverse problem. Inverse problems can be solved using various system identification techniques [11, 28—30]. In Ref. [28], the quasilinearization method is used to estimate the function $g(t)$, given observations of emergent intensity.

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МЕТОДЫ НАЧАЛЬНЫХ УСЛОВИЙ ДЛЯ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ, ВОЗНИКАЮЩИХ В ТЕОРИЯХ СОЛНЕЧНОЙ АТМОСФЕРЫ

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В теориях солнечной хромосферы и протуберанцев возникает задача определения интенсивности излучения, выходящего с поверхности атмосферы

$$E(x, f) = \alpha(f) \int_0^x e^{-\alpha(f)(x-t)} B(t)\, dt,$$

где функция источника $B(t)$ удовлетворяет интегральному уравнению

$$B(t) = g(t) + \int_0^x k(|t-t'|) B(t')\, dt'.$$

Вычисленные профили линий для различных моделей затем могут быть сравнены с наблюдаемыми профилями.

В настоящей статье представлена полезная для вычислений теория начальных значений для прямой оценки $E(x, f)$. При желании могут быть определены также функции источника и резольвента. Эти соображения опираются главным образом на решения обратных задач для определения структуры атмосферы, заданной наблюдаемыми профилями линий.

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