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# Induced Cosmological Constant In 2-Brane Models With a Compact Dimension

H.G. Sargsyan

*Department of Physics, Yerevan State University, 1 Alex Manoogian street, 0025 Yerevan, Armenia*
E-mail: sargsyan.hayk@ysu.am

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**Abstract.** The vacuum expectation value of the surface energy-momentum tensor of a charged scalar field is investigated for two parallel branes in anti-de Sitter spacetime with a compact spatial dimension. The existence of a constant gauge field is assumed, which leads to an Aharonov-Bohm type effect because of the nontrivial topology of the considered space. The contribution due to the presence of the hidden brane is extracted from the vacuum expectation value, and it is shown to be a cosmological constant type energy density on the visible brane. This cosmological constant is a periodic function of the magnetic flux through the compact dimension and, depending on the parameters of the problem, can be either positive or negative. For interbrane distances solving the hierarchy problem between the Planck and electroweak energy scales, the cosmological constant on the visible brane is naturally suppressed.

**Keywords:** anti-de Sitter spacetime, brane, scalar field, cosmological constant

## 1. Introduction

Anti-de Sitter (AdS) spacetime is the maximally symmetric solution of the vacuum Einstein equations for the gravitational field with a negative cosmological constant. Because of high symmetry a relatively large number of physical problems are exactly solvable on its background. This has provided an opportunity for discussion of principal questions related to the quantization procedure on curved backgrounds. In contrast to quantum field theory in Minkowski bulk, the field theories in AdS spacetime have several new features such as the lack of global hyperbolicity and the presence of both regular and irregular modes. In recent developments of the gravitational physics and cosmology, the AdS geometry plays a crucial role in braneworld scenarios (for a general review see [1]). Well known examples of the latter are the Randall-Sundrum braneworld models [2]. They provide an interesting alternative to the standard Kaluza-Klein compactification of the extra dimensions and were initially introduced to resolve the hierarchy problem between the energy scales for gravitational and electroweak interactions. The AdS spacetime appears as the background geometry in these models. The main idea to resolve the large hierarchy is that the small coupling of 4-dimensional gravity is generated by the large physical volume of extra dimensions.

The background geometry in the original Randall-Sundrum models is a slice of $(4+1)-$dimensional AdS spacetime that is defined by a $S_1/Z_2$ orbifold with respect to a single or two fixed points. The latter correspond to the locations of $(3+1)-$dimensional branes. In the RS1 model with two branes the negative tension (visible) brane contains the standard model fields and corresponds to our Universe. Depending on the specific model, in addition to those fields, bulk fields can be present for which the extra dimensions are accessible. The presence of the branes induces boundary conditions on bulk fields. In quantum field theory the boundary conditions modify the spectrum of vacuum fluctuations and lead to the Casimir effect (for a review see [3]). In particular, the Casimir forces acting on the branes may provide a mechanism for the stabilization of the distance between the branes (radion field). Related to that possibility, the Casimir effect in braneworld models with AdS bulk has been investigated in a large number of papers (see, for example, references given in [4]). From the point of view of the back reaction of quantum effects on the bulk geometry, an important characteristic of the vacuum state is the vacuum expectation value (VEV) of the energy-momentum tensor. In braneworld models on AdS bulk that VEV has been investigated in [5]-[10] for scalar, fermionic and electromagnetic fields. In manifolds with boundaries the energy-momentum tensor in addition to the bulk part contains a contribution located on the boundary [11]. In braneworld models the boundaries are realized by the branes. For a real scalar field, the VEV of the surface energy-momentum tensor on the branes was investigated in [12]-[14]. In the present paper we investigate the effects of nontrivial topology of the background geometry on the VEV of the surface energy-momentum tensor for a charged scalar field in braneworld models with two branes and with locally AdS spacetime.

The paper is structured as follows. Section 2 contains the detailed description of the background geometry and the periodicity and boundary conditions for the field under consideration. In section 3 the VEV of the surface energy-momentum tensor is evaluated. It is divided into two parts: the first one corresponding to the model with a single brane and the second one being the contribution induced by the second brane. It is shown that the contribution in VEV due to the presence of the hidden brane corresponds to a cosmological constant type energy density on the visible brane. In section 4 we consider the Planck scales and the cosmological constants on the branes. The asymptotic behavior of these quantities is investigated for various regions of the parameters in the problem. It is shown that at separations between the branes solving the hierarchy problem between the Planck and electroweak energy scales, the induced cosmological constant on the visible brane is naturally suppressed without an additional fine tuning.

## 2. The problem setup

The AdS spacetime is a solution of vacuum Einstein equation in the presence of a negative cosmological constant. If $D$ is the number of spatial dimensions, the corresponding metric tensor in Poincaré coordinates is given by the line element

$$ds^2 = g_{ik}dx^i dx^k = e^{-2y/a}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2, \tag{2.1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,\ldots,-1)$, with $\mu,\nu = 0,1,\ldots,D-1$, being the metric tensor for the $D$-dimensional Minkowski spacetime and the indices $i,k$ run over $0$ to $D$. The parameter $a$ determines the Ricci scalar $R$ and the cosmological constant $\Lambda$ through the relations $R = -D(D+1)/a^2$ and $\Lambda = -D(D-1)/(2a^2)$. In what follows, unless noted otherwise, we will use lowercase Latin letters to denote indices running from $0$ through $D$ and lowercase Greek letters to denote indices running from $0$ to $D-1$. In addition to the coordinate $y$, $-\infty < y < +\infty$, we define the coordinate $z$ in accordance with $z = ae^{y/a}$, $0 \le z < \infty$, in terms of which the metric tensor is written in a manifestly conformally flat form: $g_{ik} = (a/z)^2 \eta_{ik}$. One of the key features of the AdS spacetime is the lack of global hyperbolicity due to the timelike boundary $z = 0$ in the conformal infinity (AdS boundary). The hypersurface $z = \infty$ corresponds to the AdS horizon.

In the present paper the local geometry of the spacetime coincides with that for the AdS and is described by the line element (2.1), however, the global properties will be different. Namely, it will be assumed that the spatial dimension $x^{D-1}$ is compactified on a circle of the length $L$, $0 \le x^{D-1} \le L$, whereas for the remaining dimensions $x^i$, $i = 1, \ldots, D-2$, as usual, one has $-\infty < x^i < +\infty$. The proper length of the compact dimension measured by an observer residing on fixed $y$ is given by $L_{(p)} = (a/z)L = e^{-y/a}L$. The proper length decreases with increasing $y$.

Having described the background geometry, we consider a charged scalar field $\varphi(x)$ with $x$ standing for a spacetime point. Assuming the presence of an external gauge field $A_i$, the corresponding field equation reads

$$\left(g^{ik} D_i D_k + m^2 + \xi R\right)\varphi(x) = 0, \tag{2.2}$$

where $D_k = \nabla_k + ieA_k$ is the gauge extended covariant derivative operator and $\xi$ is the curvature coupling parameter. The topology is nontrivial and, in addition, the periodicity condition must be specified along the compact dimension. As such we shall consider the condition

$$\varphi(t, x^1, \ldots, x^{D-1} + L, z) = e^{i\alpha}\varphi(t, x^1, \ldots, x^{D-1}, z), \tag{2.3}$$

with a constant phase $\alpha$. In what follows we consider a gauge field configuration with constant vector potential $A_i$. In this case the corresponding field strengths are zero, however due to the nontrivial topology of the spacetime the constant field $A_i$ gives rise to Aharonov-Bohm like effects. Those effects for the VEV of the current density in braneworlds have been discussed in [4], [15-18]. In the case of a constant $A_i$, with the gauge transformation $\varphi(x) = e^{-ie\chi(x)}\varphi'(x)$, $A_i = A_i' + \partial_i \chi(x)$, where $\chi(x) = A_i x^i$, one can pass to a new gauge, with the zero vector potential, $A_i' = 0$. The periodicity condition for the new field is transformed to

$$\varphi'(t, x^1, \ldots, x^{D-1} + L, z) = e^{i\tilde{\alpha}}\varphi'(t, x^1, \ldots, x^{D-1}, z), \tag{2.4}$$

were the new phase is defined as $\tilde{\alpha} = \alpha + eA_{D-1}L$. The physical quantities do not depend on the gauge and in the discussion below we will work in the gauge $(\varphi'(x), A_i' = 0)$ omitting the primes for the sake of simplicity. In this gauge, one has $D_k = \nabla_k$ in the equation of motion

(2.2).

Here we are interested in the VEV of the surface energy-momentum tensor for the complex scalar field $\varphi(x)$ in the geometry of two parallel branes localized at $y = y_1$ ($z = z_1$) and $y = y_2$ ($z = z_2$). It is assumed, that the field obeys the Robin boundary conditions on the branes:

$$(1+\beta^{(j)}n^i\nabla_i)\varphi(x) = 0,\ z = z_j,\ j = 1,2, \tag{2.5}$$

where $\beta^{(j)}$ are constants and $n^i$ is the normal directed to the region under consideration. The VEVs in the regions $z \le z_1$ and $z \ge z_2$ are the same as those in the geometry of a single brane at $z = z_1$ and $z = z_2$ when the second brane is absent. The corresponding VEV for the surface energy-momentum tensor has been investigated in [14] and here we shall consider the region between the branes, $z_1 \le z \le z_2$. In that region one has $n^i|_{z=z_j} = n^{(j)}\delta^i_D$, where $n^{(1)} = 1$ and $n^{(2)} = -1$. For the evaluation of the VEV the complete set of positive and negative energy solutions to the field equation (2.2) obeying the periodicity condition (2.4) and the boundary conditions (2.5) is required. We shall denote that set by $\{\varphi^{(+)}_\sigma(x)\,\varphi^{(-)}_\sigma(x)\}$, where $\sigma$ is the set of quantum numbers specifying the solutions.

For the background geometry described above the positive and negative energy solutions are presented in the following factorized form

$$\varphi^{(\pm)}_\sigma(x) = z^{D/2}Z_\nu(\lambda z)e^{ik_rx^r \mp i\omega t}, \tag{2.6}$$

where $Z_\nu(\lambda z)$ is a linear combination of cylinder functions of the order

$$\nu = \sqrt{D^2/4 - D(D+1)\xi + m^2a^2}, \tag{2.7}$$

and $\omega = \sqrt{\lambda^2 + k^2}$, $k^2 = \sum_{i=1}^{D-1} k_i^2$. If the order $\nu$ of the cylinder function is imaginary the corresponding vacuum state becomes unstable [19-21]. For that reason, we shall consider only the values for the parameters in the problem for which $\nu \ge 0$ (Breitenlohner-Freedman bound). The components of momentum are determined as follows

$$\begin{aligned} &-\infty < k_i < +\infty, && i = 1,\dots,D-2 \\ &k_{D-1} = p_l \equiv \frac{2\pi l + \tilde{\alpha}}{L}, && l = 0,\pm 1,\pm 2,\dots \end{aligned} \tag{2.8}$$

The complete set of quantum numbers $\sigma$ consists of the components of momentum $k_r$, $r = 1,2,\dots,D-1$ and the "radial" quantum number $\lambda$.

In the region between the branes, $z_1 \le z \le z_2$, by making use the boundary condition (2.5) on the brane $z = z_1$, for the function $Z_\nu(\lambda z)$ one gets (see also [12,13])

$$Z_\nu(\lambda z) = C_\sigma g_\nu^{(1)}(\lambda z_1, \lambda z), \tag{2.9}$$

where

$$g_\nu^{(j)}(u,v) = \bar{Y}_\nu^{(j)}(u) J_\nu(v) - \bar{J}_\nu^{(j)}(u) Y_\nu(v). \tag{2.10}$$

Here $J_\nu(v)$ and $Y_\nu(v)$ are the Bessel and Neumann functions and the notation with the bar is defined as

$$\bar{F}^{(j)}(x) = A_j F(x) + B_j x F'(x),\ A_j = 1 + \frac{n^{(j)} D}{2} \frac{\beta^{(j)}}{a},\ B_j = n^{(j)} \frac{\beta^{(j)}}{a}. \tag{2.11}$$

From the boundary condition on the second brane it follows that the allowed values of the quantum number $\lambda$ are the roots of the following equation

$$g_\nu(\lambda z_1, \lambda z_2) = 0, \tag{2.12}$$

where $g_\nu(u,v) = \bar{J}_\nu^{(1)}(u)\bar{Y}_\nu^{(2)}(v) - \bar{Y}_\nu^{(1)}(u)\bar{J}_\nu^{(2)}(v)$. We shall denote by $\lambda_n$, $n = 1, 2, \ldots$, the positive roots of the equation (2.12) with respect to $\lambda$. The constant $C_\sigma$ in (2.9) is determined from the standard normalization condition and is given by the expression

$$|C_\sigma|^2 = \frac{(2\pi)^{3-D}\lambda}{4\omega a^{D-1} L} \frac{\bar{Y}_\nu^{(2)}(\lambda z_2)/\bar{Y}_\nu^{(1)}(\lambda z_1)}{\partial_\lambda g_\nu(\lambda z_1, \lambda z_2)},\ \lambda = \lambda_n. \tag{2.13}$$

We are interested in the VEV of the surface energy-momentum tensor. For a manifold with a boundary $\partial M_s$, the VEV can be presented in terms of the VEV of the field squared as [11,14]

$$\langle 0 | T_\mu^{(s)\nu} | 0 \rangle = -\frac{c_j}{a} \delta(x; \partial M_s) \delta_\mu^\nu \langle 0 | \varphi\varphi^\dagger | 0 \rangle, \tag{2.14}$$

where the delta function $\delta(x; \partial M_s)$ locates the VEV on the boundary (branes in the problem under consideration) and we have defined

$$c_j = \xi n^{(j)} - \frac{2\xi - 1/2}{\beta^{(j)}} a. \tag{2.15}$$

Hence, the problem is reduced to the evaluation of the VEV $\langle 0 | \varphi\varphi^\dagger | 0 \rangle$ on the branes. For the brane at $z = z_j$ the VEV (2.14) is presented as

$$\langle 0 | T_\mu^{(s)\nu} | 0 \rangle = \delta(x; \partial M_s) \mathrm{diag}(\varepsilon_j, -p_j, \ldots, -p_j), \tag{2.16}$$

with $\partial M_s$ corresponding to $z = z_j$ and the energy density $\varepsilon_j$ and pressure $p_j = -\varepsilon_j$. For an observer living on the brane, the VEV (2.16) corresponds to a gravitational source of the cosmological constant type.

## 3. Surface energy density

Having the complete set of the mode functions, the VEV of the field squared is evaluated by using the mode sum formula

$$\langle 0|\varphi\varphi^{\dagger}|0\rangle = \sum_{\sigma} \varphi_{\sigma}^{(-)}(x)\varphi_{\sigma}^{(-)}(x), \tag{2.17}$$

where a summation is understood for discrete quantum numbers and an integration for continuous ones. Substituting the mode functions, the VEV of the field squared on the brane at $z = z_j$ is presented as

$$\begin{aligned}\langle 0|\varphi\varphi^{\dagger}|0\rangle_{z=z_j} = {} & \frac{2^{3-D} B_j a^{1-D} z_j^D}{\pi^{D/2-1}\Gamma(D/2-1)L} \sum_{l=-\infty}^{\infty} \int_0^{\infty} dK K^{D-3} \\ & \times \sum_{n=1}^{\infty} \frac{\lambda}{\sqrt{\lambda^2 + K^2 + p_l^2}} \left. \frac{g_{\nu}^{(j')}(\lambda z_{j'}, \lambda z_j)}{\partial_{\lambda} g_{\nu}(\lambda z_1, \lambda z_2)} \right|_{\lambda=\lambda_n},\end{aligned} \tag{2.18}$$

where the expressions with $j' = 1$ and $j' = 2$ provide two equivalent forms of the representation. The expression on the right-hand side is divergent and a regularization is needed. For the purpose of regularization, as in [14], we make the replacement $\sqrt{\lambda^2 + K^2 + p_l^2} \to (\lambda^2 + K^2 + p_l^2)^{-s/2}$ and consider the values of the complex parameter $s$ in the range where the expression corresponding to (2.18) is convergent. In addition, to keep the dimensions of the right-hand side, an arbitrary mass scale $\mu$ will be introduced. The function obtained in this way from (2.18) will be denoted by $\Phi_j(s)$. Evaluating the integral over $K$, the corresponding expression is presented as

$$\Phi_j(s) = \frac{a^{1-D} B_j z_j^D}{(4\pi)^{D/2-1} L} \sum_{l=-\infty}^{\infty} \zeta_{jl}(s), \tag{2.19}$$

where the generalized partial zeta function is introduced in accordance with

$$\zeta_{jl}(s) = \frac{\Gamma(-\alpha_s)}{\Gamma(-s/2)\mu^{s+1}} \sum_{n=1}^{\infty} \lambda \left(\lambda^2 + p_l^2\right)^{\alpha_s} \left. \frac{g_{\nu}^{(j')}(\lambda z_{j'}, \lambda z_j)}{\partial_{\lambda} g_{\nu}(\lambda z_1, \lambda z_2)} \right|_{\lambda=\lambda_n}, \tag{2.20}$$

and $\alpha_s = (D+s)/2 - 1$. The renormalized value of $\langle 0|\varphi\varphi^{\dagger}|0\rangle_{z=z_j}$ is obtained by the analytic continuation of $\Phi_j(s)$ at $s=-1$.

The computation of the VEV of the surface energy-momentum tensor requires the analytic continuation of the function $\Phi_j(s)$ to the value $s=-1$,

$$\varepsilon_j = -(c_j/a)\Phi_j(s)|_{s=-1}. \tag{2.21}$$

In accordance with (2.19) the analytic continuation of the function $\Phi_j(s)$ is reduced to the one for the function $\zeta_{jl}(s)$. This is done in a way similar to that used in [12,13] by using the residue theorem. As a result the surface energy density on the brane at $z=z_j$ is decomposed as

$$\varepsilon_j = \varepsilon_{(1)j} + \Delta\varepsilon_j, \tag{2.22}$$

where $\varepsilon_{(1)j}$ is the surface energy density in the geometry of a single brane at $z=z_j$ when the second brane at $z=z_{j'}$, $j' \neq j$, is absent. The part $\Delta\varepsilon_j$ is induced on the brane $z=z_j$ by the brane $z=z_{j'}$. This second brane induced contribution is given by the expression

$$\Delta\varepsilon_j = \frac{2(4\pi)^{(1-D)/2} c_j B_j^2 z_j^D}{\Gamma\left((D-1)/2\right) L a^D} \sum_{l=-\infty}^{\infty} \int_{|p_l|}^{\infty} dx \frac{x\left(x^2-p_l^2\right)^{\frac{D-3}{2}} F_{(j)\nu}(xz_1, xz_2)}{\bar{K}_\nu^{(1)}(xz_1)\bar{I}_\nu^{(2)}(xz_2) - \bar{I}_\nu^{(1)}(xz_1)\bar{K}_\nu^{(2)}(xz_2)}, \tag{2.23}$$

where $p_l$ is given by (2.8) and we have defined the functions

$$F_{(1)\nu}(u,v) = \frac{\bar{K}_\nu^{(2)}(v)}{\bar{K}_\nu^{(1)}(u)}, F_{(2)\nu}(u,v) = \frac{\bar{I}_\nu^{(1)}(u)}{\bar{I}_\nu^{(2)}(v)}. \tag{2.24}$$

Here, $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions and the notation (2.11) is used. The additional energy densities (2.23) are induced on the sides $z=z_1+0$ and $z=z_2-0$ of the first and second branes, respectively. The surface energy densities on the sides $z=z_1-0$ and $z=z_2+0$ are the same as those for the corresponding geometries with single branes and $\Delta\varepsilon_j=0$ for $z=z_1-0$ and $z=z_2+0$. The single brane part $\varepsilon_{(1)j}$ is investigated in [14] and in what follows we shall be concerned with the contribution $\Delta\varepsilon_j$. First of all, by the redefinition of the summation variable $l$ in (2.23) we see that $\Delta\varepsilon_j$ is a periodic function of $\tilde{\alpha}$ with the period $2\pi$. So, we can take $\tilde{\alpha}$ in the region $-\pi \le \tilde{\alpha} \le \pi$. For $\tilde{\alpha}=0$ the mode $l=0$ corresponds to the zero Kaluza-Klein mode.

In order to clarify the behavior of the induced energy density we consider limiting cases. For large values of the AdS curvature radius compared to the interbrane distance, $(y_2-y_1)/a \ll 1$, the order of the Bessel functions is large and we use the uniform asymptotic expansions of the Bessel functions for large orders [22]. As a result, for the energy density we obtain

$$\begin{aligned}\Delta\varepsilon_j \approx \quad & (1-4\xi)\frac{2(4\pi)^{(1-D)/2}\beta^{(j)}}{\Gamma\left((D-1)/2\right)L}\sum_{l=-\infty}^{\infty}\int_{u_l}^{\infty}du u^2\,\frac{\left(u^2-u_l^2\right)^{\frac{D-3}{2}}}{1-\beta^{(j)2}u^2} \\ & \times\left[e^{2u(y_2-y_1)}\frac{1-\beta^{(1)}u}{1+\beta^{(1)}u}\frac{1-\beta^{(2)}u}{1+\beta^{(2)}u}-1\right]^{-1},\end{aligned} \tag{2.25}$$

where $u_l=\sqrt{p_l^2+m^2}$ . The expression on the right hand side of (2.25) corresponds to the energy density in flat background spacetime. For large distances between the branes we consider two cases. The first one corresponds to $p_l z_2 \gg 1$ and $p_l z_1 \lesssim 1$. By using the asymptotic expressions for the modified Bessel functions with the arguments $\lambda z_2$ we can see that in (2.23) the separate terms with given $l$ decay as $e^{-2|p_l|z_2}$. In the second case we assume that $p_l z_1 \ll 1$ for a fixed $p_l z_2$. Replacing the modified Bessel functions with the arguments $\lambda z_1$ by the corresponding asymptotic for small arguments it can be seen that $\Delta\varepsilon_1$ is suppressed by the factor $\left(z_1/z_2\right)^{D+2\nu}$ and $\Delta\varepsilon_2$ is suppressed by $\left(z_1/z_2\right)^{2\nu}$.

Now let us consider the limiting values for the length of the compact dimension. For $L \gg z_2$ the dominant contribution to the series over $l$ in (2.23) comes from large values of $|l|$ and, in the leading order, we can replace the corresponding summation by the integration. This gives

$$\Delta\varepsilon_j \approx \frac{2(4\pi)^{(1-D)/2}c_j B_j^2 z_j^D}{\pi\Gamma\left((D-1)/2\right)a^D}\int_0^{\infty}du\int_0^{\infty}dw\frac{w^{D-2}F_{(j)\nu}(xz_1,xz_2)}{\bar{K}_\nu^{(1)}(xz_1)\bar{I}_\nu^{(2)}(xz_2)-\bar{I}_\nu^{(1)}(xz_1)\bar{K}_\nu^{(2)}(xz_2)}. \tag{2.26}$$

where $x=\sqrt{w^2+u^2}$ . Now we introduce polar coordinates $(x,\theta)$ in the plane $(u,w)$. After the integration over the angular coordinate we get

$$\Delta\varepsilon_j \approx \frac{2(4\pi)^{-D/2}c_j B_j^2 z_j^D}{\Gamma\left(D/2\right)a^D}\int_0^{\infty}dx\frac{x^{D-1}F_{(j)\nu}(xz_1,xz_2)}{\bar{K}_\nu^{(1)}(xz_1)\bar{I}_\nu^{(2)}(xz_2)-\bar{I}_\nu^{(1)}(xz_1)\bar{K}_\nu^{(2)}(xz_2)}. \tag{2.27}$$

The expression in the right-hand side coincides with the induced energy density in the model where the dimension $x^{D-1}$ is decompactified. For a real scalar field, the latter was investigated in [12]. Note that the condition $L \gg z_2$ is written as $Le^{-y_2/a} \gg a$. This means that in the limit under consideration the physical length of the extra dimension on the brane at $y=y_2$ is much larger than the curvature radius.

For $L \ll z_2$ for all the terms with $l \neq 0$ one has $|p_l|z_2 \gg 1$ and it can be seen that for the modes with given $l$ the corresponding contributions to $\Delta\varepsilon_j$ are suppressed by $e^{-2|p_l|z_2}$. For $0<|\tilde{\alpha}|<\pi$ the dominant contribution comes from the mode $l=0$ and the induced energy densities are suppressed by the factor $e^{-2|\tilde{\alpha}|z_2/L}$. For $\tilde{\alpha}=0$ the dominant contribution comes from the zero mode with $p_l = 0$ and one gets

$$\Delta\varepsilon_j \approx \frac{2(4\pi)^{(1-D)/2} c_j B_j^2 z_j^D}{\Gamma\left((D-1)/2\right) L a^D} \int_0^\infty dx \frac{x^{D-2} F_{(j)\nu}(xz_1, xz_2)}{\bar{K}_\nu^{(1)}(xz_1)\bar{I}_\nu^{(2)}(xz_2) - \bar{I}_\nu^{(1)}(xz_1)\bar{K}_\nu^{(2)}(xz_2)}. \quad (2.28)$$

This limit corresponds to small physical lengths of the compact dimension on the brane $y = y_2$ with respect to the curvature scale, $Le^{-y_2/a} \ll a$. In the limit $L \ll z_1$ and for $0 < |\tilde{\alpha}| < \pi$ the arguments of the modified Bessel functions in (2.23) are large and by using the corresponding asymptotic expressions we get

$$\Delta\varepsilon_j \approx \frac{4(4\pi)^{(1-D)/2} c_j B_j^2 z_j^{D+1}}{\Gamma\left((D-1)/2\right) L a^D} \sum_{l=-\infty}^{\infty} \int_{|p_l|}^\infty dx \frac{x^2 \left(x^2 - p_l^2\right)^{\frac{D-3}{2}}}{\left(A_j - n^{(j)} B_j x z_j\right)^2} \frac{A_{j'} + n^{(j')} B_{j'} x z_{j'}}{A_{j'} - n^{(j')} B_{j'} x z_{j'}} e^{-2x(z_2 - z_1)}, \quad (2.29)$$

where $j, j' = 1, 2$, $j' \neq j$. If, in addition, $L \ll (z_2 - z_1)$, the dominant contribution comes from the mode with $l = 0$ and the induced energy densities are suppressed by the factor $e^{-2|\tilde{\alpha}|(z_2 - z_1)/L}$. For $\tilde{\alpha} = 0$ and for $L \ll (z_2 - z_1)$, the dominant contribution comes from the zero mode with $p_l = 0$ and, to the leading order, the energy densities are given by (2.28).

## 4. Planck scales and the cosmological constants on the branes

Let us denote by $G_{D+1}$ the fundamental $(D+1)$-dimensional Newton's constant. For the corresponding Planck mass one has $M_{D+1} = G_{D+1}^{1/(1-D)}$. After the Kaluza-Klein reduction of the $(D+1)$-dimensional gravitational action, we can see that the effective $(D-1)$-dimensional Newton's constant $G_{D-1}^{(j)}$ on the brane at $z = z_j$ is expressed as

$$G_{D-1}^{(j)} = (D-2) \frac{G_{D+1} e^{y_j/a + (D-2)(y_2 - y_j)}}{aL\left[e^{(D-2)(y_2 - y_1)} - 1\right]}. \quad (2.30)$$

For the corresponding $(D-1)$-dimensional effective Planck mass scale one has $M_{D-1}^{(j)} = G_{D-1}^{(j)1/(D-3)}$. For the ratio of the effective and fundamental Planck scales we get

$$\frac{M_{D-1}^{(j)}}{M_{D+1}} = \frac{z_j}{z_1} \left[\frac{1 - (z_1/z_2)^{D-2}}{D-2} M_{D+1}^2 L z_1\right]^{\frac{1}{D-3}}. \quad (2.31)$$

Note that the length of the compact dimension measured by an observer on the brane $z = z_j$ is given by $Le^{-y_j/a} = aL/z_j$. If the interbrane distance is large, from (2.30) one gets $G_{D-1}^{(1)} \sim G_{D+1}e^{y_1/a}/(aL)$, $G_{D-1}^{(2)} \sim G_{D+1}e^{(2-D)(y_2-y_1)/a}e^{y_2/a}/(aL)$. These results show that in the limit under consideration the gravitational interaction on the brane $y = y_2$ is exponentially suppressed. This feature provides a geometrical solution of the hierarchy problem between the gravitational and electroweak energy scales and has been used in the Randall-Sundrum models.

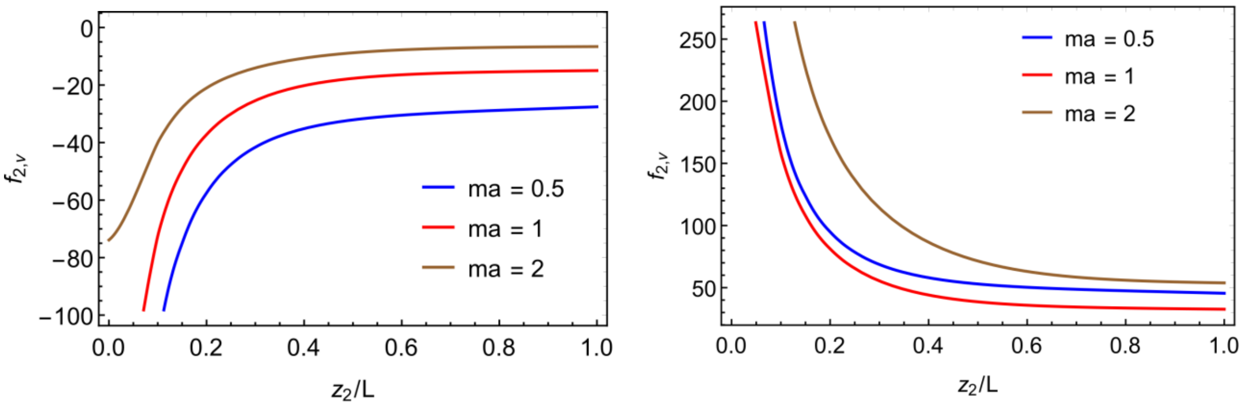



***Fig. 1:*** *The dependence of the quantity defined by (2.38) on* $z_2/L$ *for a minimally coupled scalar field for* $\beta_1/a = -1$ *(left picture) and* $\beta_1/a = 1$ *(right picture) with* $\beta_2/a = 0.2$, $D = 5$, $\tilde{\alpha} = 0.3$ *and for three different values of the mass of the scalar field.*

For an observer living on the brane at $y = y_j$ the corresponding effective $(D-1)$-dimensional cosmological constant (integrated over the compact dimension) is determined by the relation

$$\Lambda_{D-1}^{(j)} = 8\pi G_{D-1}^{(j)} Le^{-y_j/a}\Delta\varepsilon_j = \frac{8\pi Le^{-y_j/a}\Delta\varepsilon_j}{M_{D-1}^{(j)D-3}}. \tag{2.32}$$

For the ratio of the induced cosmological constant (2.32) to the corresponding Planck scale quantity in the brane universe one gets

$$h_j \equiv \frac{\Lambda_{D-1}^{(j)}}{8\pi G_{D-1}^{(j)} M_{D-1}^{(j)D-1}} = \frac{Le^{-y_j/a}\Delta\varepsilon_j}{M_{D-1}^{(j)D-1}}. \tag{2.33}$$

By using the expression for $M_{D-1}^{(j)}$, this ratio can be written as

$$h_j = \left[ \frac{(D-2) z_1 / L}{1-(z_1/z_2)^{D-2}} \left( a M_{D+1} \right)^{1-D} \right]^{\frac{D-1}{D-3}} b_j, \tag{2.34}$$

where we have defined the dimensionless quantities

$$b_j = \frac{2(4\pi)^{(1-D)/2} c_j B_j^2}{\Gamma\left((D-1)/2\right)} \sum_{l=-\infty}^{\infty} \int_{|p_l|z_1}^{\infty} dx \frac{x\left(x^2 - p_l^2 z_1^2\right)^{\frac{D-3}{2}} F_{(j)\nu}(x, xz_2/z_1)}{\bar{K}_\nu^{(1)}(x)\bar{I}_\nu^{(2)}(xz_2/z_1) - \bar{I}_\nu^{(1)}(x)\bar{K}_\nu^{(2)}(xz_2/z_1)}. \tag{2.35}$$

Note that one has the relation

$$\Delta\varepsilon_j = \frac{(z_j/z_1)^{D-1}}{Le^{-y_j/a} a^{D-1}} b_j. \tag{2.36}$$

For large interbrane distances, $z_2/z_1 \gg 1$, for the ratios (2.31) and (2.33) one has

$$\frac{M_{D-1}^{(j)}}{M_{D+1}} \approx \frac{z_j}{z_1} \left( \frac{L z_1}{D-2} M_{D+1}^2 \right)^{\frac{1}{D-3}}, \; h_j \approx \left[ \frac{(D-2) z_1 / L}{\left(a M_{D+1}\right)^{D-1}} \right]^{\frac{D-1}{D-3}} b_j. \tag{2.37}$$

In this limit, for the quantities $b_j$ two qualitatively different cases should be considered. For the first one $z_2/L \gtrsim 1$ and to the leading order we get $b_j \approx f_{j,\nu}(z_2/L)\left(z_1/z_2\right)^{D+2\nu-1}$, with the functions $f_{j,\nu}(z_2/L)$ defined as

$$f_{j,\nu}(z_2/L) = \frac{2^{2-2\nu}(4\pi)^{(1-D)/2} c_j B_j^2}{\Gamma\left((D-1)/2\right)\Gamma^2(\nu)\left(A_1 - B_1\nu\right)^2} \sum_{l=-\infty}^{\infty} \int_{|p_l|z_2}^{\infty} dx\, x^{1+2\nu} \left(x^2 - p_l^2 z_2^2\right)^{\frac{D-3}{2}} g_{j,\nu}(x), \tag{2.38}$$

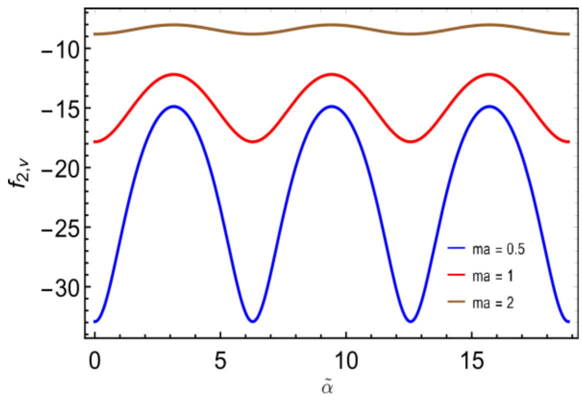
where

$$g_{1,\nu}(x) = 2\frac{\bar{K}_\nu^{(2)}(x)}{\bar{I}_\nu^{(2)}(x)}, \; g_{2,\nu}(x) = \frac{A_1^2 - B_1^2\nu^2}{\nu \bar{I}_\nu^{(2)2}(x)}. \tag{2.39}$$

For $h_j$ this gives

$$h_j \approx \left(z_1/z_2\right)^{(D-1)\frac{D-2}{D-3}+2\nu} \left[ \frac{(D-2) z_2 / L}{\left(a M_{D+1}\right)^{D-1}} \right]^{\frac{D-1}{D-3}} f_{j,\nu}(z_2/L). \tag{2.40}$$

By taking into account that $z_2/L = a/Le^{-y_2/a}$, we see that in this case the proper length of the compact dimension for an observer on the brane $y = y_2$ is of the order of the curvature radius.



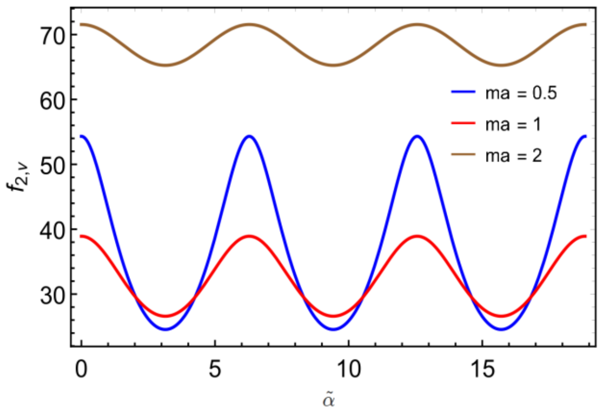




***Fig. 2:*** *The dependence of the quantity defined by (2.38) on $\tilde{\alpha}$ for a minimally coupled scalar field for $\beta_1/a=-1$ (left picture) and $\beta_1/a=1$ (right picture) with $\beta_2/a=0.2$, $D=5$, $z_2/L=0.5$ and for three different values of the mass of the scalar field.*

In Figures 1 and 2 we have depicted the dependence of the quantity defined by the equation (2.38) on the location of the visible brane and on the phase of the periodicity condition along the compact dimension. The dependence on the phase is periodic with period $2\pi$, as expected, and depending on the parameters in the problem the quantity can be either positive or negative.

In the second case we assume that $z_2/z_1 \gg 1$ and $L \ll z_2$. Now the dominant contribution to $b_j$ in (2.35) comes from the mode with $l=0$. For $0<|\tilde{\alpha}|<\pi$ there is no zero mode and for $B_j \neq 0$ the functions $b_j$ are approximated by

$$\begin{aligned} b_1 &\approx -\frac{c_1 B_1^2 (L/z_2)^{\frac{D-1}{2}}}{4(4\pi)^{(D-3)/2}} \frac{(z_1/L)^{D-1}|\tilde{\alpha}|^{\frac{D-1}{2}}}{e^{2|\tilde{\alpha}|z_2/L}\bar{K}_\nu^{(1)2}(|\alpha|z_1/L)}, \\ b_2 &\approx \frac{c_2 (L/z_2)^{\frac{D+1}{2}}}{2(4\pi)^{(D-3)/2}} \frac{(z_1/L)^{D-1}|\tilde{\alpha}|^{\frac{D-3}{2}}}{e^{2|\tilde{\alpha}|z_2/L}} \frac{\bar{I}_\nu^{(1)}(|\tilde{\alpha}|z_1/L)}{\bar{K}_\nu^{(1)}(|\tilde{\alpha}|z_1/L)}. \end{aligned} \tag{2.41}$$

In this case we have an exponential suppression by the factor $e^{-2|\tilde{\alpha}|z_2/L}$. For $\tilde{\alpha}=0$ the contribution of the zero mode with $p_l=0$ dominates and the leading term coincides with the $l=0$ term in (2.35) putting $p_l=0$. Expanding with respect to $z_1/z_2$ one gets

$$b_j \approx \frac{2^{2-2\nu}(4\pi)^{(1-D)/2} c_j B_j^2 \left(z_1/z_2\right)^{D+2\nu-1}}{\Gamma\left((D-1)/2\right)\Gamma^2(\nu)\left(A_j - B_j\nu\right)^2} \int_0^\infty dx x^{D+2\nu-2} g_{j,\nu}(x),$$

with the functions (2.39).

Let us apply the results given above to a generalized $D=5$ Randall-Sundrum type model with an additional compact dimension. In these models it is assumed that the energy scales $M_{D+1}$ and $1/a$ are of the same order. The brane $y=y_2$ corresponds to the visible brane on which the standard model fields are localized. For the corresponding Planck scale one has

$M_{D-1}^{(2)} \sim 10^{16}$ TeV. We assume that the fundamental Planck scale is of the order $M_{D+1} \sim 1$ TeV. For the case $z_2/z_1 \gg 1$ and $z_2/L \gtrsim 1$, from (2.37) and (2.40) we get

$$\frac{z_2}{z_1} \approx 10^{32/3} \frac{(3z_2/L)^{1/3}}{(z_1 M_{D+1})^{2/3}},\ h_2 \approx 10^{-64(1+\nu/3)} \frac{(z_1/a)^{4(1+\nu/3)}}{(aM_{D+1})^{4(1-\nu/3)}} \frac{f_{2,\nu}(z_2/L)}{(3z_2/L)^{2\nu/3}}, \tag{2.42}$$

where $\nu = \sqrt{25/4 - 30\xi + m^2 a^2}$. As seen, for $z_2/L$ and $z_1 M_{D+1}$ of the order of 1, the hierarchy problem is solved for $(y_2 - y_1)/a \approx 25$. In the standard Randall-Sundrum model without compact dimensions (with $D = 4$) for the solution of the hierarchy problem between the Planck and electroweak energy scales one needs $(y_2 - y_1)/a \approx 37$. From the estimate (2.42) for $h_2$ we see that for $a$ of the order of $1/M_{D+1}$ one has a natural suppression for the induced cosmological constant with respect to the Planck scale on the visible brane. In the case $z_2/z_1 \gg 1$, $L \ll z_2$, and $\tilde{\alpha} = 0$ one has

$$\begin{aligned}\frac{z_2}{z_1} &\approx \frac{10^{16}}{\sqrt{3z_1/L}} \frac{1}{z_1 M_{D+1}},\\ h_2 &\approx \frac{10^{-64(1+\nu/2)}}{(aM_{D+1})^{4-2\nu}} \frac{(z_1/a)^{4+2\nu}}{(3z_1/L)^{\nu}} \frac{c_2 B_2^2 (A_2 - B_2\nu)^{-2}}{2^{2+2\nu}\pi^2\Gamma^2(\nu)} \int_0^\infty dx x^{3+2\nu} g_{2,\nu}(x).\end{aligned} \tag{2.43}$$

Assuming that $z_1$ and $L$ are of the order of $a$, the hierarchy problem is solved for $(y_2 - y_1)/a \approx 37$. Again, for that values of the interbrane distance we have a natural suppression of the ratio $h_2$ by the factor $10^{-64(1+\nu/2)}$. And finally, in the case $z_2/z_1 \gg 1$, $L \ll z_2$, $\tilde{\alpha} \neq 0$ the estimate for $z_2/z_1$ remains the same as in (2.43) and for $z_1$ and $L$ of the order of $a$ the ratio $h_2$ is suppressed by the factor $e^{-2|\tilde{\alpha}|z_2/L}$. Here the suppression is much stronger than in the previous two cases.

## 5. Conclusion

In the present paper we have investigated the VEV of the surface energy-momentum tensor for a charged scalar field in the problem with two parallel branes on background of the locally AdS spacetime with a compact dimension. The branes are parallel to the AdS boundary and the field operator obeys the Robin boundary conditions on them. The surface energy densities on the branes are decomposed into two contributions. The first one corresponds to the energy density on a single brane when the second brane is absent and the other contribution is induced by the presence of the second brane. Unlike the single brane part, the second brane contribution does not require an additional renormalization and is given by (2.23) with the functions $F_{(j)\nu}(u,v)$ defined by (2.24) for the first and second branes. For large values of the length of the compact dimension we have shown that the leading order term coincides with the induced energy density in the geometry where the compact dimension is decompactified. For small values of the length of the compact dimension the behavior of the surface energy density is essentially different for the cases $\tilde{\alpha}=0$ and $\tilde{\alpha}\neq 0$. In the first case there is a zero mode and the dominant contribution comes from that mode. The leading order term is given by (2.28). In the case $\tilde{\alpha}\neq 0$ and for small values of $L$ the surface energy density is suppressed by the factor $e^{-2|\tilde{\alpha}|z_2/L}$.

We have considered the Planck mass scales and the cosmological constants on the branes, given by (2.31) and (2.32), respectively. For the ratio of the induced cosmological constant on the brane to the corresponding Planck scale quantity one has the expression (2.34). Among the main motivations of the Randall-Sundrum type braneworld models on the AdS bulk was the possibility for the generation of large hierarchy between the Planck and electroweak energy scales for moderate values of the distance between the branes in units of the AdS curvature radius. For those values of the interbrane distance we have estimated the ratio $h_2$ of the cosmological constant induced by the hidden brane on the visible brane to the Planck energy scale on the visible brane. The behavior of that ratio depends on the length of the compact dimension. For $D=5$ and for $z_2/z_1 \gg 1$, $z_2/L \gtrsim 1$ the ratio $h_2$ is estimated by (2.42) and, assuming that $z_2/L$, $z_1 M_{D+1}$, $aM_{D+1}$ are of the order of 1, it is suppressed by the factor $10^{-64(1+\nu/3)}$. For $z_2/z_1 \gg 1$, $z_2/L \gg 1$ and $\tilde{\alpha}=0$ the ratio $h_2$ is of the order $10^{-64(1+\nu/2)}$ (assuming that $aM_{D+1}, L/a, z_1/a$ are of the order of 1). In both these cases we have a naturally suppressed cosmological constant on the visible brane. For $z_2/z_1 \gg 1$, $L \ll z_2$, $\tilde{\alpha}\neq 0$ and for $z_1$ and $L$ of the order of $a$ the ratio $h_2$ is suppressed by the factor $e^{-2|\tilde{\alpha}|z_2/L}$. In this case the suppression is much stronger.

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**Conflict of Interest**

The author declares no conflict of interest.

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