

NON-ABELIAN TENSOR GAUGE FIELDS

ENHANCED SYMMETRIES

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Abstract

We define a group of extended non-Abelian gauge transformations for tensor gauge fields. On this group one can define generalized field strength tensors, which are transforming homogeneously with respect to the extended gauge transformations. The generalized field strength tensors allow to construct two infinite series of gauge invariant quadratic forms. The invariant Lagrangian is a linear sum of these forms and describes interaction of tensor gauge fields of arbitrarily large integer spins $1, 2, \dots$. It does not contain higher derivatives of the tensor gauge fields, and all interactions take place through three- and four-particle exchanges with dimensionless coupling constant. The first term in this sum is the Yang-Mills Lagrangian and the whole construction generalizes the Yang-Mills theory.

The invariance with respect to the extended gauge transformations does not fix some of the coefficients in the Lagrangian. There is a freedom to vary them without breaking the extended gauge symmetry. We demonstrate that by an appropriate tuning of these coupling constants one can achieve an enhancement of the extended gauge symmetry. This leads to highly symmetric equations. We present the explicit form of the free equations for the rank-2 and rank-3 gauge fields. Their relation to the Schwinger free equation for Abelian rank-3 gauge field is discussed.

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1 Introduction

It is well understood, that the concept of local gauge invariance allows to define non-Abelian gauge fields [1], to derive their dynamical field equations and to develop a universal point of view on matter interactions as resulting from the exchange of gauge quanta of different forms. It is appealing to extend the gauge principle so that it will define the interaction of matter fields which carry not only non-commutative internal charges, but also high spins. This extension will induce the interaction of matter fields mediated by a charged gauge quanta carrying a spin larger than one [2, 3, 4].

In this approach the gauge fields are defined as rank- $(s+1)$ tensors [2, 3, 4]

$$A_{\mu\lambda_1\dots\lambda_s}^a, \quad s=1,2,\dots$$

and are totally symmetric with respect to the indices $\lambda_1\dots\lambda_s$. A priori the tensor fields have no symmetries with respect to the first index μ . The index s runs from zero to infinity. The first member of this family of the tensor gauge bosons is the Yang-Mills vector boson A_μ^a . This is an essential departure from the previous considerations, in which the higher-rank tensors were totally symmetric [8, 9, 10, 11, 12, 13, 14, 15, 17, 18]. The literature devoted to the high spin field theory is extensive [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. On recent development of the high spin field theory associated with the string theory can be found in references [36, 37, 38, 39, 41, 44, 45, 46, 47, 48].

The extended non-Abelian gauge transformation of the tensor gauge fields [2, 3, 4]

$$\delta_\zeta A_{\mu\lambda_1\lambda_2\dots\lambda_s}^a$$

is defined by the equation (4) and comprise a closed algebraic structure, because the commutator of two transformations can be expressed in the form

$$[\delta_\eta, \delta_\zeta] A_{\mu\lambda_1\lambda_2\dots\lambda_s}^a = -ig \delta_\zeta A_{\mu\lambda_1\lambda_2\dots\lambda_s}^a$$

where the gauge parameters $\{\zeta\}$ are given by the matrix commutators (6). This allows to define generalized field strength tensors (7) [2, 3, 4]

$$G_{\mu\nu,\lambda_1,\dots,\lambda_s}^a$$

which are *transforming homogeneously* (8) with respect to the extended gauge transformations (4).

The field strength tensors $G_{\mu\nu,\lambda_1,\dots,\lambda_s}^a$ are used to construct two infinite series of gauge invariant quadratic forms [2, 3, 4, 5, 6]

$$L_s \quad s = 1, 2, 3, \dots$$

and

$$L'_s \quad s = 2, 3, \dots$$

Each term of these infinite series is separately gauge invariant with respect to the generalized gauge transformations (4). These forms contain quadratic kinetic terms and nonlinear terms describing nonlinear interaction of the Yang-Mills type. In order to make all tensor gauge fields dynamical one should add all these forms together. Thus the gauge invariant Lagrangian describing dynamical tensor gauge bosons of all ranks has the form

$$L = \sum_{s=1}^{\infty} g_s L_s + \sum_{s=2}^{\infty} g'_s L'_s, \quad (1)$$

where $L_1 \equiv L_{YM}$ is the Yang-Mills Lagrangian.

It is important that: i) *the Lagrangian does not contain higher derivatives of tensor gauge fields* ii) *all interactions take place through the three- and four-particle exchanges with dimensionless coupling constant g* iii) *the complete Lagrangian contains all higher-rank tensor gauge fields and should not be truncated* iv) *the invariance with respect to the extended gauge transformations does not fix the coupling constants g_s and g'_s .*

The coupling constants g_s and g'_s remain arbitrary because every term of the sum is separately gauge invariant and the extended gauge symmetry alone does not fix them. This means that there is a freedom to vary these constants without breaking the initial gauge symmetry. The important question to which we should address ourselves here is the following: *Can we achieve the enhancement of the initial gauge symmetry properly tuning the coupling constants g_s and g'_s ?*

Let us consider a simple example: the sum of two Z_2 invariant forms $gx^2 + g'y^2$ exhibits the $U(1)$ invariance if we choose $g = g'$ so that the initial symmetry is elevated to a one parameter family of continuous transformations. A less trivial example is a linear sum of Poincaré invariant forms comprising a SUSY invariant Lagrangian. One can find other examples of the same phenomena when a linear sum of invariant forms of the initial group G exhibits a symmetry with

respect to a larger group $G \supset G$ when the coefficients are properly tuned. A similar phenomena appears in our system.

Indeed let us consider a linear sum of two gauge invariant forms in (1)

$$g_2 L_2 + g_2' L_2'$$

which describes the rank-2 tensor gauge field $A_{\mu\lambda}^a$. As we have found in [2, 3, 4] one can chose the coupling constants g_2 and g_2' so that the sum $g_2 L_2 + g_2' L_2'$ exhibits invariance with respect to a bigger gauge group¹. This means that in addition to full extended gauge group (4), which we had initially, now we have bigger gauge group with double number of gauge parameters [2, 3, 4, 7]. The explicit form of the free field equation for the rank-2 tensor gauge field is given by equation (?). It was then demonstrated that it describes propagation of two polarizations of helicity-two and helicity-zero massless charged tensor gauge bosons. This result will be recapitulated in the third section.

Our aim now is to extend this construction to the rank-3 tensor gauge field. We shall consider the linear sum

$$g_3 L_3 + g_3' L_3'$$

and shall demonstrate that for an appropriate choice of the coupling constants $c_3 = g_3'/g_3 = 4/3$ the system have an enhanced gauge symmetry. The explicit description of this symmetry together with the corresponding free field equation (51) for the rank-3 tensor gauge field will be given in the forth and fifth sections. Its relation to the Schwinger equation for the symmetric rank-3 tensor gauge field is discussed in the last seventh section.

First let us recapitulate the construction of the general Lagrangian L in (1).

2 Non-Abelian Tensor Fields

The gauge fields are defined as rank- $(s+1)$ tensors [2, 3, 4]

$$A_{\mu\lambda_1\dots\lambda_s}^a(x), \quad s = 0, 1, 2, \dots$$

and are totally symmetric with respect to the indices $\lambda_1\dots\lambda_s$. *A priori* the tensor fields have no symmetries with respect to the first index μ . The index a numerates the generators L^a of the Lie

¹ $c_2 = g_2'/g_2 = 1$.

algebra of a *compact*² Lie group G .

One can think of these tensor fields as appearing in the expansion of the extended gauge field $A_\mu(x)$ over the unite tangent vector e_λ [2, 3, 4]:

$$A_\mu(x) = \sum_{s=0}^{\infty} A_{\mu\lambda_1\dots\lambda_s}^a(x) L_{\lambda_1\dots\lambda_s}^a. \quad (2)$$

The gauge field $A_{\mu\lambda_1\dots\lambda_s}^a$ carries indices $a, \lambda_1, \dots, \lambda_s$ labeling the generators of *extended current algebra* G associated with compact Lie group G . It has infinite many generators $L_{\lambda_1\dots\lambda_s}^a = L^a e_{\lambda_1} \dots e_{\lambda_s}$ and the corresponding algebra is given by the commutator [4]³

$$[L_{\lambda_1\dots\lambda_s}^a, L_{\rho_1\dots\rho_k}^b] = if^{abc} L_{\lambda_1\dots\lambda_s\rho_1\dots\rho_k}^c. \quad (3)$$

The extended non-Abelian gauge transformations of the tensor gauge fields are defined by the following equations [2, 3, 4]:

$$\delta A_\mu^a = (\delta^{ab} \partial_\mu + gf^{acb} A_\mu^c) \xi^b, \quad (4)$$

$$\delta A_{\mu\nu}^a = (\delta^{ab} \partial_\mu + gf^{acb} A_\mu^c) \xi_\nu^b + gf^{acb} A_{\mu\nu}^c \xi^b,$$

$$\delta A_{\mu\nu\lambda}^a = (\delta^{ab} \partial_\mu + gf^{acb} A_\mu^c) \xi_{\nu\lambda}^b + gf^{acb} (A_{\mu\nu}^c \xi_\lambda^b + A_{\mu\lambda}^c \xi_\nu^b + A_{\nu\lambda}^c \xi_\mu^b),$$

.....

where $\xi_{\lambda_1\dots\lambda_s}^a(x)$ are totally symmetric gauge parameters. These extended gauge transformations generate a closed algebraic structure. To see that, one should compute the commutator of two extended gauge transformations δ_η and δ_ξ of parameters η and ξ . The commutator of two transformations can be expressed in the form [2, 3, 4]

$$[\delta_\eta, \delta_\xi] A_{\mu\lambda_1\lambda_2\dots\lambda_s} = -ig \delta_\zeta A_{\mu\lambda_1\lambda_2\dots\lambda_s} \quad (5)$$

and is again an extended gauge transformation with the gauge parameters $\{\zeta\}$ which are given by the matrix commutators

$$\zeta = [\eta, \xi] \quad (6)$$

$$\zeta_{\lambda_1} = [\eta, \xi_{\lambda_1}] + [\eta_{\lambda_1}, \xi]$$

$$\zeta_{\nu\lambda} = [\eta, \xi_{\nu\lambda}] + [\eta_\nu, \xi_\lambda] + [\eta_\lambda, \xi_\nu] + [\eta_{\nu\lambda}, \xi],$$

²The algebra $\tilde{\mathfrak{g}}$ possesses an orthogonal basis in which the structure constants f^{abc} are totally antisymmetric.

³See also the alternative expansions in [10, 11, 19, 20, 21, 37] and the algebras based on diffeomorphisms group in [38, 39, 35, 40].

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The generalized field strengths are defined as [2, 3, 4]

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c, \quad (7)$$

$$G_{\mu\nu,\lambda}^a = \partial_\mu A_{\nu\lambda}^a - \partial_\nu A_{\mu\lambda}^a + gf^{abc} (A_\mu^b A_{\nu\lambda}^c + A_{\mu\lambda}^b A_\nu^c),$$

$$G_{\mu\nu,\lambda\rho}^a = \partial_\mu A_{\nu\lambda\rho}^a - \partial_\nu A_{\mu\lambda\rho}^a + gf^{abc} (A_\mu^b A_{\nu\lambda\rho}^c + A_{\mu\lambda}^b A_{\nu\rho}^c + A_{\mu\rho}^b A_{\nu\lambda}^c + A_{\mu\lambda\rho}^b A_\nu^c),$$

.....

and transform homogeneously with respect to the extended gauge transformations (4). The field strength tensors are antisymmetric in their first two indices and are totally symmetric with respect to the rest of the indices.

The inhomogeneous extended gauge transformation (4) induces the homogeneous gauge transformation of the corresponding field strength (7) of the form [2, 3, 4]

$$\delta G_{\mu\nu}^a = gf^{abc} G_{\mu\nu}^b \xi^c \quad (8)$$

$$\delta G_{\mu\nu,\lambda}^a = gf^{abc} (G_{\mu\nu,\lambda}^b \xi^c + G_{\mu\nu}^b \xi_\lambda^c),$$

$$\delta G_{\mu\nu,\lambda\rho}^a = gf^{abc} (G_{\mu\nu,\lambda\rho}^b \xi^c + G_{\mu\nu,\lambda}^b \xi_\rho^c + G_{\mu\nu,\rho}^b \xi_\lambda^c + G_{\mu\nu}^b \xi_{\lambda\rho}^c)$$

.....,

The field strength tensors are antisymmetric in their first two indices and are totally symmetric with respect to the rest of the indices. The symmetry properties of the field strength $G_{\mu\nu,\lambda_1\dots\lambda_s}^a$ remain invariant in the course of this transformation.

These tensor gauge fields and the corresponding field strength tensors allow to construct two series of gauge invariant quadratic forms. The first series is given by the formula [2, 3, 4]:

$$\begin{aligned} L_{s+1} &= -14 G_{\mu\nu,\lambda_1\dots\lambda_s}^a G_{\mu\nu,\lambda_1\dots\lambda_s}^a + \dots \\ &= -14 \sum_{i=0}^{2s} a_i^s G_{\mu\nu,\lambda_1\dots\lambda_i}^a G_{\mu\nu,\lambda_{i+1}\dots\lambda_{2s}}^a \left(\sum_{p's} \eta^{\lambda_1 \lambda_2 \dots \lambda_{i-1} \lambda_i} \dots \eta^{\lambda_{2s-1} \lambda_{2s}} \right), \end{aligned} \quad (9)$$

where the sum \sum_p runs over all nonequal permutations of i 's, in total $(2s-1)!!$ terms and the numerical coefficients are $a_i^s = s!i!(2s-i)!$.

The second series of gauge invariant quadratic forms is given by the formula [2, 3, 4, 6]:

$$\begin{aligned} L'_{s+1} &= 14 G_{\mu\lambda_1,\lambda_2\dots\lambda_{s+1}}^a G_{\mu\lambda_2,\lambda_1\dots\lambda_{s+1}}^a + \dots \\ &= 18 \sum_{i=1}^{2s+1} a_{i-1}^s G_{\mu\lambda_1,\lambda_2\dots\lambda_i}^a G_{\mu\lambda_{i+1},\lambda_{i+2}\dots\lambda_{2s+2}}^a \left(\sum_{p's} \eta^{\lambda_1 \lambda_2 \dots \lambda_{i-1} \lambda_i} \dots \eta^{\lambda_{2s+1} \lambda_{2s+2}} \right), \end{aligned} \quad (10)$$

where the sum \sum_p' runs over all nonequal permutations of i 's, with exclusion of the terms which contain $\eta^{\lambda_1 \lambda_{i+1}}$.

In order to make all tensor gauge fields dynamical one should add the corresponding kinetic terms. Thus the invariant Lagrangian describing dynamical tensor gauge bosons of all ranks has the form

$$L = \sum_{s=1}^{\infty} g_s L_s + \sum_{s=2}^{\infty} g_s' L_s', \quad (11)$$

where $L_1 \equiv L_{YM}$.

It is important that: i) the Lagrangian does not contain higher derivatives of tensor gauge fields ii) all interactions take place through the three- and four-particle exchanges with dimensionless coupling constant g iii) the complete Lagrangian contains all higher-rank tensor gauge fields and should not be truncated iv) the invariance with respect to the extended gauge transformations does not fix the coupling constants g_s and g_s' .

The coupling constants g_s and g_s' remain arbitrary because every term of the sum is separately gauge invariant and the extended gauge symmetry alone does not fix them. This means that we have a freedom to chose these constants without breaking the initial gauge symmetry. The important question which should be addressed here is the following: *Can we achieve the enhancement of the gauge symmetry tuning the coupling constants g_s and g_s' ?*

Let us consider a linear sum of two gauge invariant forms in (1)

$$g_2 L_2 + g_2' L_2',$$

which describes the rank-2 tensor gauge field $A_{\mu\nu}^a$. As we have found in [2, 3, 4] one can choose the coupling constants g_2 and g_2' so that the sum $g_2 L_2 + g_2' L_2'$ exhibits invariance with respect to a bigger gauge group (see the next section for details, where we will demonstrate that $c_2 = g_2'/g_2 = 1$). This means that in addition to full extended gauge group (4), which we had initially, now we have a bigger gauge group with double number of gauge parameters. It was then demonstrated that the corresponding kinetic term describes propagation of two polarizations of helicity-two and helicity-zero massless charged tensor gauge bosons.

In summary the gauge invariant Lagrangian for the lower-rank tensor gauge fields has the form [2, 3, 4]:

$$L = L_1 + L_2 + L_2' = -14G_{\mu\nu}^a G_{\mu\nu}^a \quad (12)$$

$$\begin{aligned}
& -14G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda}^a - 14G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda}^a \\
& + 14G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu}^a + 14G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda}^a + 12G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda}^a.
\end{aligned}$$

The equations of motion which follow from this Lagrangian are:

$$\begin{aligned}
& \nabla_{\mu}^{ab} G_{\mu\nu}^b + 12\nabla_{\mu}^{ab} (G_{\mu\nu,\lambda\lambda}^b + G_{\nu\lambda,\mu\lambda}^b + G_{\lambda\mu,\nu\lambda}^b) + gf^{acb} A_{\mu\lambda}^c G_{\mu\nu,\lambda}^b \\
& - 12gf^{acb} (A_{\mu\lambda}^c G_{\mu\lambda,\nu}^b + A_{\mu\lambda}^c G_{\lambda\nu,\mu}^b + A_{\lambda\lambda}^c G_{\mu\nu,\mu}^b + A_{\mu\nu}^c G_{\mu\lambda,\lambda}^b) \\
& + 12gf^{acb} (A_{\mu\lambda\lambda}^c G_{\mu\nu}^b + A_{\mu\mu\lambda}^c G_{\nu\lambda}^b + A_{\mu\nu\lambda}^c G_{\lambda\mu}^b) = 0
\end{aligned} \tag{13}$$

and for the second-rank tensor gauge field $A_{\nu\lambda}^a$:

$$\begin{aligned}
& \nabla_{\mu}^{ab} G_{\mu\nu,\lambda}^b - 12(\nabla_{\mu}^{ab} G_{\mu\lambda,\nu}^b + \nabla_{\mu}^{ab} G_{\lambda\nu,\mu}^b + \nabla_{\lambda}^{ab} G_{\mu\nu,\mu}^b + \eta_{\nu\lambda} \nabla_{\mu}^{ab} G_{\mu\rho,\rho}^b) \\
& + gf^{acb} A_{\mu\lambda}^c G_{\mu\nu}^b - 12gf^{acb} (A_{\mu\nu}^c G_{\mu\lambda}^b + A_{\mu\mu}^c G_{\lambda\nu}^b + A_{\lambda\mu}^c G_{\mu\nu}^b + \eta_{\nu\lambda} A_{\mu\rho}^c G_{\mu\rho}^b) = 0.
\end{aligned} \tag{14}$$

The variation of the action with respect to the third-rank gauge field $A_{\nu\lambda\rho}^a$ will give the equations

$$\eta_{\lambda\rho} \nabla_{\mu}^{ab} G_{\mu\nu}^b - 12(\eta_{\nu\rho} \nabla_{\mu}^{ab} G_{\mu\lambda}^b + \eta_{\lambda\nu} \nabla_{\mu}^{ab} G_{\mu\rho}^b) + 12(\nabla_{\rho}^{ab} G_{\nu\lambda}^b + \nabla_{\lambda}^{ab} G_{\nu\rho}^b) = 0. \tag{15}$$

Representing this system of equations in the form

$$\begin{aligned}
& \partial_{\mu} F_{\mu\nu}^a + 12\partial_{\mu} (F_{\mu\nu,\lambda\lambda}^a + F_{\nu\lambda,\mu\lambda}^a + F_{\lambda\mu,\nu\lambda}^a) = j_{\nu}^a \\
& \partial_{\mu} F_{\mu\nu,\lambda}^a - 12(\partial_{\mu} F_{\mu\lambda,\nu}^a + \partial_{\mu} F_{\lambda\nu,\mu}^a + \partial_{\lambda} F_{\mu\nu,\mu}^a + \eta_{\nu\lambda} \partial_{\mu} F_{\mu\rho,\rho}^a) = j_{\nu\lambda}^a \\
& \eta_{\lambda\rho} \partial_{\mu} F_{\mu\nu}^a - 12(\eta_{\nu\rho} \partial_{\mu} F_{\mu\lambda}^a + \eta_{\nu\lambda} \partial_{\mu} F_{\mu\rho}^a) + 12(\partial_{\rho} F_{\nu\lambda}^a + \partial_{\lambda} F_{\nu\rho}^a) = j_{\nu\lambda\rho}^a,
\end{aligned} \tag{16}$$

where $F_{\mu\nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a$, $F_{\mu\nu,\lambda}^a = \partial_{\mu} A_{\nu\lambda}^a - \partial_{\nu} A_{\mu\lambda}^a$, $F_{\mu\nu,\lambda\rho}^a = \partial_{\mu} A_{\nu\lambda\rho}^a - \partial_{\nu} A_{\mu\lambda\rho}^a$, we can find the corresponding conserved currents

$$\begin{aligned}
& j_{\nu}^a = -gf^{abc} A_{\mu}^b G_{\mu\nu}^c - gf^{abc} \partial_{\mu} (A_{\mu}^b A_{\nu}^c) \\
& - 12gf^{abc} A_{\mu}^b (G_{\mu\nu,\lambda\lambda}^c + G_{\nu\lambda,\mu\lambda}^c + G_{\lambda\mu,\nu\lambda}^c) - 12\partial_{\mu} (I_{\mu\nu,\lambda\lambda}^a + I_{\nu\lambda,\mu\lambda}^a + I_{\lambda\mu,\nu\lambda}^a) \\
& - gf^{abc} A_{\mu\lambda}^b G_{\mu\nu,\lambda}^c + 12gf^{abc} (A_{\mu\lambda}^b G_{\mu\lambda,\nu}^c + A_{\mu\lambda}^b G_{\lambda\nu,\mu}^c + A_{\lambda\lambda}^b G_{\mu\nu,\mu}^c + A_{\mu\nu}^b G_{\mu\lambda,\lambda}^c) \\
& - 12gf^{abc} (A_{\mu\lambda\lambda}^b G_{\mu\nu}^c + A_{\lambda\mu\lambda}^b G_{\nu\mu}^c + A_{\mu\lambda\nu}^b G_{\lambda\mu}^c),
\end{aligned} \tag{17}$$

where $I_{\mu\nu,\lambda\rho}^a = gf^{abc} (A_{\mu}^b A_{\nu\lambda\rho}^c + A_{\mu\lambda}^b A_{\nu\rho}^c + A_{\mu\rho}^b A_{\nu\lambda}^c + A_{\mu\lambda\rho}^b A_{\nu}^c)$ and

$$\begin{aligned}
& j_{\nu\lambda}^a = -gf^{abc} A_{\mu}^b G_{\mu\nu,\lambda}^c + 12gf^{abc} (A_{\mu}^b G_{\mu\lambda,\nu}^c + A_{\mu}^b G_{\lambda\nu,\mu}^c + A_{\lambda}^b G_{\mu\nu,\mu}^c + \eta_{\nu\lambda} A_{\mu}^b G_{\mu\rho,\rho}^c) \\
& - gf^{abc} A_{\mu\lambda}^b G_{\mu\nu}^c + 12gf^{abc} (A_{\mu\nu}^b G_{\mu\lambda}^c + A_{\lambda\mu}^b G_{\mu\nu}^c + A_{\mu\mu}^b G_{\lambda\nu}^c + \eta_{\nu\lambda} A_{\mu\rho}^b G_{\mu\rho}^c) \\
& - gf^{abc} \partial_{\mu} (A_{\mu}^b A_{\nu\lambda}^c + A_{\mu\lambda}^b A_{\nu}^c) + 12gf^{abc} [\partial_{\mu} (A_{\mu}^b A_{\lambda\nu}^c + A_{\mu\nu}^b A_{\lambda}^c) + \partial_{\mu} (A_{\lambda}^b A_{\nu\mu}^c + A_{\lambda\mu}^b A_{\nu}^c) \\
& + \partial_{\lambda} (A_{\mu}^b A_{\nu\mu}^c + A_{\mu\mu}^b A_{\nu}^c) + \eta_{\nu\lambda} \partial_{\mu} (A_{\mu}^b A_{\rho\rho}^c + A_{\mu\rho}^b A_{\rho}^c)],
\end{aligned} \tag{18}$$

$$\begin{aligned}
j_{\nu\lambda\rho}^a &= -\eta_{\lambda\rho} g f^{abc} A_\mu^b G_{\mu\nu}^c + 12 g f^{abc} (\eta_{\nu\rho} A_\mu^b G_{\mu\lambda}^c + \eta_{\nu\lambda} A_\mu^b G_{\mu\rho}^c - A_\rho^b G_{\nu\lambda}^c - A_\lambda^b G_{\nu\rho}^c) \\
&- \eta_{\lambda\rho} g f^{abc} \partial_\mu (A_\mu^b A_\nu^c) + 12 g f^{abc} [\partial_\mu (\eta_{\nu\lambda} A_\mu^b A_\rho^c + \eta_{\nu\rho} A_\mu^b A_\lambda^c) - \partial_\lambda (A_\nu^b A_\rho^c) - \partial_\rho (A_\nu^b A_\lambda^c)].
\end{aligned} \tag{19}$$

The conservation of the corresponding currents follows from the fact that we have enhancement of the gauge group and therefore the partial derivatives of the l.h.s. of the equations (16) are equal to zero

$$\begin{aligned}
\partial_\nu j_\nu^a &= 0, \\
\partial_\nu j_{\nu\lambda}^a &= 0, \quad \partial_\lambda j_{\nu\lambda}^a = 0, \\
\partial_\nu j_{\nu\lambda\rho}^a &= 0, \quad \partial_\lambda j_{\nu\lambda\rho}^a = 0, \quad \partial_\rho j_{\nu\lambda\rho}^a = 0.
\end{aligned} \tag{20}$$

Our aim now is to extend this analysis to the case of rank-3 tensor gauge field. We shall consider the linear sum

$$g_3 L_3 + g_3' L_3'$$

and demonstrate that for an appropriate choice of the coupling constants ratio $c_3 = g_3'/g_3 = 4/3$ the system will have an enhanced symmetry. The explicit form of the Lagrangian for the third-rank tensor gauge field $A_{\mu\nu\lambda}^a$ can be obtained from our general formulas (??), (??) and (11) when we substitute $s = 2$. The Lagrangian is:

$$\begin{aligned}
L_3 + c_3 L_3' &= -14 G_{\mu\nu,\lambda\rho}^a G_{\mu\nu,\lambda\rho}^a - 18 G_{\mu\nu,\lambda\lambda}^a G_{\mu\nu,\rho\rho}^a - 12 G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda\rho\rho}^a - 18 G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda\rho\rho}^a + \\
&+ c_3 \{ 14 G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu\rho}^a + 14 G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho\lambda}^a + 14 G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho\rho}^a + \\
&+ 14 G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu\rho\rho}^a + 12 G_{\mu\nu,\lambda}^a G_{\mu\rho,\nu\lambda\rho}^a + 14 G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda\rho\rho}^a + 14 G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda\rho\rho}^a \},
\end{aligned} \tag{21}$$

where $c_3 = g_3'/g_3$ is a constant⁴.

3 Enhanced Symmetry. Rank-2 Gauge Field

As we have seen above there are two invariant forms for the rank-2 tensor gauge field L_2 and L_2' and the general Lagrangian is a linear combination $L_2 + c_2 L_2'$, where $c_2 = g_2'/g_2$ is a constant coefficient. Let us review how this coefficient has been fixed by the requirement of an enhanced

⁴It is not difficult to present the explicit form of the Lagrangian for any higher-rank tensor field using expressions (??), (??) and (11).

symmetry. For that let us consider the situation at the linearized level when the gauge coupling constant g is equal to zero. The free part of the L_2 Lagrangian

$$L_2^{free} = 12A_{\alpha\alpha'}^a(\eta_{\alpha\gamma}\eta_{\alpha'\gamma'}\partial^2 - \eta_{\alpha'\gamma'}\partial_\alpha\partial_\gamma)A_{\gamma\gamma'}^a = 12A_{\alpha\alpha'}^a H_{\alpha\alpha'\gamma\gamma'} A_{\gamma\gamma'}^a,$$

where the quadratic form in the momentum representation has the form

$$H_{\alpha\alpha'\gamma\gamma'}(k) = (-k^2\eta_{\alpha\gamma} + k_\alpha k_\gamma)\eta_{\alpha'\gamma'},$$

is obviously invariant with respect to the gauge transformation $\delta A_{\mu\lambda}^a = \partial_\mu \xi_\lambda^a$, but it is not invariant with respect to the alternative gauge transformations $\tilde{\delta} A_{\mu\lambda}^a = \partial_\lambda \eta_\mu^a$. This can be seen, for example, from the following relations in momentum representation

$$k_\alpha H_{\alpha\alpha'\gamma\gamma'}(k) = 0, \quad k_{\alpha'} H_{\alpha\alpha'\gamma\gamma'}(k) = -(k^2\eta_{\alpha\gamma} - k_\alpha k_\gamma)k_{\gamma'} \neq 0. \quad (22)$$

Let us consider now the free part of the second Lagrangian

$$\begin{aligned} L_2'^{free} &= 14A_{\alpha\alpha'}^a(-\eta_{\alpha\gamma}\eta_{\alpha'\gamma'}\partial^2 - \eta_{\alpha\alpha'}\eta_{\gamma\gamma'}\partial^2 + \eta_{\alpha\gamma}\partial_\alpha\partial_\gamma + \eta_{\alpha'\gamma'}\partial_{\alpha'}\partial_{\gamma'} + \eta_{\alpha\alpha'}\partial_\gamma\partial_{\gamma'} + \\ &+ \eta_{\gamma\gamma'}\partial_\alpha\partial_{\alpha'} - 2\eta_{\alpha\gamma}\partial_{\alpha'}\partial_{\gamma'})A_{\gamma\gamma'}^a = 12A_{\alpha\alpha'}^a H'_{\alpha\alpha'\gamma\gamma'} A_{\gamma\gamma'}^a, \end{aligned} \quad (23)$$

where

$$H'_{\alpha\alpha'\gamma\gamma'}(k) = 12(\eta_{\alpha\gamma}\eta_{\alpha'\gamma'} + \eta_{\alpha\alpha'}\eta_{\gamma\gamma'})k^2 - 12(\eta_{\alpha\gamma}k_\alpha k_\gamma + \eta_{\alpha'\gamma'}k_\alpha k_{\gamma'} + \eta_{\alpha\alpha'}k_\gamma k_{\gamma'} + \eta_{\gamma\gamma'}k_\alpha k_{\alpha'} - 2\eta_{\alpha\gamma}k_{\alpha'}k_{\gamma'}).$$

It is again invariant with respect to the gauge transformation $\delta A_{\mu\lambda}^a = \partial_\mu \xi_\lambda^a$, but it is not invariant with respect to the gauge transformations $\tilde{\delta} A_{\mu\lambda}^a = \partial_\lambda \eta_\mu^a$, as one can see from analogous relations

$$k_\alpha H'_{\alpha\alpha'\gamma\gamma'}(k) = 0, \quad k_{\alpha'} H'_{\alpha\alpha'\gamma\gamma'}(k) = (k^2\eta_{\alpha\gamma} - k_\alpha k_\gamma)k_{\gamma'} \neq 0. \quad (24)$$

As it is obvious from (22) and (24), the total Lagrangian $L_2^{free} + L_2'^{free}$ now poses new enhanced invariance with respect to the larger, eight-parameter, gauge transformations

$$\delta A_{\mu\lambda}^a = \partial_\mu \xi_\lambda^a + \partial_\lambda \eta_\mu^a, \quad (25)$$

where ξ_λ^a and η_μ^a are eight arbitrary functions, because

$$k_\alpha (H_{\alpha\alpha'\gamma\gamma'} + H'_{\alpha\alpha'\gamma\gamma'}) = 0, \quad k_{\alpha'} (H_{\alpha\alpha'\gamma\gamma'} + H'_{\alpha\alpha'\gamma\gamma'}) = 0. \quad (26)$$

Thus our free part of the Lagrangian is

$$\begin{aligned} L_2^{tot\ free} &= -12\partial_\mu A_{\nu\lambda}^a \partial_\mu A_{\nu\lambda}^a + 12\partial_\mu A_{\nu\lambda}^a \partial_\nu A_{\mu\lambda}^a + \\ &+ 14\partial_\mu A_{\nu\lambda}^a \partial_\mu A_{\lambda\nu}^a - 14\partial_\mu A_{\nu\lambda}^a \partial_\lambda A_{\mu\nu}^a - 14\partial_\nu A_{\mu\lambda}^a \partial_\mu A_{\lambda\nu}^a + 14\partial_\nu A_{\mu\lambda}^a \partial_\lambda A_{\mu\nu}^a \\ &+ 14\partial_\mu A_{\nu\nu}^a \partial_\mu A_{\lambda\lambda}^a - 12\partial_\mu A_{\nu\nu}^a \partial_\lambda A_{\mu\lambda}^a + 14\partial_\nu A_{\mu\nu}^a \partial_\lambda A_{\mu\lambda}^a \end{aligned} \quad (27)$$

or, in equivalent form, it is

$$L_2^{tot\ free} = 12A_{\alpha\alpha'}^a \{ (\eta_{\alpha\gamma}\eta_{\alpha'\gamma'} - 12\eta_{\alpha\gamma}\eta_{\alpha'\gamma} - 12\eta_{\alpha\alpha'}\eta_{\gamma\gamma'})\partial^2 - \eta_{\alpha'\gamma'}\partial_\alpha\partial_\gamma - \eta_{\alpha\gamma}\partial_{\alpha'}\partial_{\gamma'} + 12(\eta_{\alpha\gamma'}\partial_{\alpha'}\partial_\gamma + \eta_{\alpha'\gamma}\partial_\alpha\partial_{\gamma'} + \eta_{\alpha\alpha'}\partial_\gamma\partial_{\gamma'} + \eta_{\gamma\gamma'}\partial_\alpha\partial_{\alpha'}) \} A_{\gamma\gamma'}^a \quad (28)$$

and is invariant with respect to the larger gauge transformations $\delta A_{\mu\lambda}^a = \partial_\mu \xi_\lambda^a + \partial_\lambda \eta_\mu^a$, where ξ_λ^a and η_μ^a are eight arbitrary functions. In momentum representation the quadratic form is

$$H_{\alpha\alpha'\gamma\gamma'}^{tot} (k) = (-\eta_{\alpha\gamma}\eta_{\alpha'\gamma'} + 12\eta_{\alpha\gamma}\eta_{\alpha'\gamma} + 12\eta_{\alpha\alpha'}\eta_{\gamma\gamma'})k^2 + \eta_{\alpha\gamma}k_\alpha k_{\gamma'} + \eta_{\alpha'\gamma'}k_\alpha k_\gamma - 12(\eta_{\alpha\gamma'}k_\alpha k_\gamma + \eta_{\alpha'\gamma}k_\alpha k_{\gamma'} + \eta_{\alpha\alpha'}k_\gamma k_{\gamma'} + \eta_{\gamma\gamma'}k_\alpha k_{\alpha'}). \quad (29)$$

Free equations of motion which follow from the Lagrangian (??) will take the form

$$\partial^2 (A_{\nu\lambda}^a - 12A_{\lambda\nu}^a) - \partial_\nu \partial_\mu (A_{\mu\lambda}^a - 12A_{\lambda\mu}^a) - \partial_\lambda \partial_\mu (A_{\nu\mu}^a - 12A_{\mu\nu}^a) + \partial_\nu \partial_\lambda (A_{\mu\mu}^a - 12A_{\mu\mu}^a) + 12\eta_{\nu\lambda} (\partial_\mu \partial_\rho A_{\mu\rho}^a - \partial^2 A_{\mu\mu}^a) = 0 \quad (30)$$

and describe the propagation of massless particles of spin 2 and spin 0. It is also easy to see that for the symmetric tensor gauge fields $A_{\nu\lambda}^a = A_{\lambda\nu}^a$ our equation reduces to the Einstein-Fierz-Pauli-Schwinger-Chang-Singh-Hagen-Fronsdal equation

$$\partial^2 A_{\nu\lambda} - \partial_\nu \partial_\mu A_{\mu\lambda} - \partial_\lambda \partial_\mu A_{\mu\nu} + \partial_\nu \partial_\lambda A_{\mu\mu} + \eta_{\nu\lambda} (\partial_\mu \partial_\rho A_{\mu\rho} - \partial^2 A_{\mu\mu}) = 0,$$

which describes the propagation of massless boson with two physical polarizations, the $s = \pm 2$ helicity states. For the antisymmetric fields it reduces to the equation

$$\partial^2 A_{\nu\lambda} - \partial_\nu \partial_\mu A_{\mu\lambda} + \partial_\lambda \partial_\mu A_{\mu\nu} = 0,$$

and describes the propagation of one physical polarization $s = 0$, the zero helicity state.

The above consideration brings the final form of the gauge invariant Lagrangian for the lower-rank tensor gauge fields to the form (12).

4 Rank-3 Tensor Gauge Field

The Lagrangian $L_1 + g_2(L_2 + L_2')$ contains the third-rank gauge fields $A_{\mu\nu\lambda}^a$, but without corresponding kinetic term. In order to make the fields $A_{\mu\nu\lambda}^a$ dynamical we have to add the corresponding Lagrangian $g_3 L_3 + g_3' L_3'$ presented in (??), so that at this level the total Lagrangian is the sum [2, 3, 4]

$$L = L_1 + g_2(L_2 + L_2') + g_3(L_3 + c_3 L_3') + \dots,$$

where $c_3 = g_3'/g_3$. The Lagrangian L_3 has the form (??):

$$L_3 = -14G_{\mu\nu,\lambda\rho}^a G_{\mu\nu,\lambda\rho}^a - 18G_{\mu\nu,\lambda\lambda}^a G_{\mu\nu,\rho\rho}^a - 12G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda\rho\rho}^a - 18G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda\rho\rho}^a, \quad (31)$$

where the field strength tensors (7) are

$$G_{\mu\nu,\lambda\rho\sigma}^a = \partial_\mu A_{\nu\lambda\rho\sigma}^a - \partial_\nu A_{\mu\lambda\rho\sigma}^a + gf^{abc} \{ A_\mu^b A_{\nu\lambda\rho\sigma}^c + A_{\mu\lambda}^b A_{\nu\rho\sigma}^c + A_{\mu\rho}^b A_{\nu\lambda\sigma}^c + A_{\mu\sigma}^b A_{\nu\lambda\rho}^c + A_{\mu\lambda\rho}^b A_{\nu\sigma}^c + A_{\mu\lambda\sigma}^b A_{\nu\rho}^c + A_{\mu\rho\sigma}^b A_{\nu\lambda}^c + A_{\mu\lambda\rho\sigma}^b A_\nu^c \}$$

and

$$G_{\mu\nu,\lambda\rho\sigma\delta}^a = \partial_\mu A_{\nu\lambda\rho\sigma\delta}^a - \partial_\nu A_{\mu\lambda\rho\sigma\delta}^a + gf^{abc} \{ A_\mu^b A_{\nu\lambda\rho\sigma\delta}^c + \sum_{\lambda\leftrightarrow\rho,\sigma,\delta} A_{\mu\lambda}^b A_{\nu\rho\sigma\delta}^c + \sum_{\lambda,\rho\leftrightarrow\sigma,\delta} A_{\mu\lambda\rho}^b A_{\nu\sigma\delta}^c + \sum_{\lambda,\rho,\sigma\leftrightarrow\delta} A_{\mu\lambda\rho\sigma}^b A_{\nu\delta}^c + A_{\mu\lambda\rho\sigma\delta}^b A_\nu^c \}.$$

The terms in parentheses are symmetric over $\lambda\rho\sigma$ and $\lambda\rho\sigma\delta$ respectively. The Lagrangian L_3 is invariant with respect to the extended gauge transformations (4) of the low-rank gauge fields $A_\mu, A_{\mu\nu}, A_{\mu\nu\lambda}$ and of the fourth-rank gauge field (4)

$$\delta_\xi A_{\mu\nu\lambda\rho} = \partial_\mu \xi_{\nu\lambda\rho} - ig[A_\mu, \xi_{\nu\lambda\rho}] - ig[A_{\mu\nu}, \xi_{\lambda\rho}] - ig[A_{\mu\lambda}, \xi_{\nu\rho}] - ig[A_{\mu\rho}, \xi_{\nu\lambda}] - ig[A_{\mu\nu\lambda}, \xi_\rho] - ig[A_{\mu\nu\rho}, \xi_\lambda] - ig[A_{\mu\lambda\rho}, \xi_\nu] - ig[A_{\mu\nu\lambda\rho}, \xi]$$

and of the fifth-rank tensor gauge field (4)

$$\delta_\xi A_{\mu\nu\lambda\rho\sigma} = \partial_\mu \xi_{\nu\lambda\rho\sigma} - ig[A_\mu, \xi_{\nu\lambda\rho\sigma}] - ig \sum_{\nu\leftrightarrow\lambda\rho\sigma} [A_{\mu\nu}, \xi_{\lambda\rho\sigma}] - ig \sum_{\nu\lambda\leftrightarrow\rho\sigma} [A_{\mu\nu\lambda}, \xi_{\rho\sigma}] - ig \sum_{\nu\lambda\rho\leftrightarrow\sigma} [A_{\mu\nu\lambda\rho}, \xi_\sigma] - ig[A_{\mu\nu\lambda\rho\sigma}, \xi],$$

where the gauge parameters $\xi_{\nu\lambda\rho}$ and $\xi_{\nu\lambda\rho\sigma}$ are totally symmetric rank-3 and rank-4 tensors. The extended gauge transformation of the higher-rank tensor gauge fields induces the gauge transformation of the fields strengths of the form (8)

$$\delta G_{\mu\nu,\lambda\rho\sigma}^a = gf^{abc} (G_{\mu\nu,\lambda\rho\sigma}^b \xi^c + G_{\mu\nu,\lambda\rho}^b \xi_\sigma^c + G_{\mu\nu,\lambda\sigma}^b \xi_\rho^c + G_{\mu\nu,\rho\sigma}^b \xi_\lambda^c + G_{\mu\nu,\lambda}^b \xi_{\rho\sigma}^c + G_{\mu\nu,\rho}^b \xi_{\lambda\sigma}^c + G_{\mu\nu,\sigma}^b \xi_{\lambda\rho}^c + G_{\mu\nu}^b \xi_{\lambda\rho\sigma}^c) \quad (32)$$

and

$$\delta G_{\mu\nu,\lambda\rho\sigma\delta}^a = gf^{abc} (G_{\mu\nu,\lambda\rho\sigma\delta}^b \xi^c + \sum_{\lambda\rho,\sigma\leftrightarrow\delta} G_{\mu\nu,\lambda\rho\sigma}^b \xi_\delta^c + \sum_{\lambda\rho\leftrightarrow\sigma,\delta} G_{\mu\nu,\lambda\rho}^b \xi_{\sigma\delta}^c + \sum_{\lambda\leftrightarrow\rho,\sigma,\delta} G_{\mu\nu,\lambda}^b \xi_{\rho\sigma\delta}^c + G_{\mu\nu}^b \xi_{\lambda\rho\sigma\delta}^c). \quad (33)$$

Using the above homogeneous transformations for the field strengths tensors one can demonstrate

the invariance of the Lagrangian L_3 with respect to the extended gauge transformations (see reference [3] for details).

The second invariant Lagrangian can also be constructed explicitly in terms of the above field strength tensors. The following seven Lorentz invariant quadratic forms can be constructed by the corresponding field strength tensors [2, 3, 4]

$$\begin{aligned} &G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu\rho}^a, \quad G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho\lambda}^a, \quad G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho\rho}^a, \quad G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu\rho\rho}^a, \\ &G_{\mu\nu,\lambda}^a G_{\mu\rho,\nu\lambda\rho}^a, \quad G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda\rho\rho}^a, \quad G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda\rho\rho}^a. \end{aligned} \quad (34)$$

Calculating the variation of each of these terms with respect to the gauge transformation (8), (32) and (33) one can get convinced that the particular linear combination

$$\begin{aligned} L_3' &= 14G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu\rho}^a + 14G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho\lambda}^a + 14G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho\rho}^a \\ &+ 14G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu\rho\rho}^a + 12G_{\mu\nu,\lambda}^a G_{\mu\rho,\nu\lambda\rho}^a + 14G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda\rho\rho}^a + 14G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda\rho\rho}^a. \end{aligned} \quad (35)$$

forms an invariant Lagrangian (see Appendix A). In summary we have the following Lagrangian for the third-rank gauge field $A_{\mu\nu\lambda}^a$

$$\begin{aligned} L_3 + c_3 L_3' &= -14G_{\mu\nu,\lambda\rho}^a G_{\mu\nu,\lambda\rho}^a - 18G_{\mu\nu,\lambda\lambda}^a G_{\mu\nu,\rho\rho}^a - 12G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda\rho\rho}^a - 18G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda\rho\rho}^a + \\ &+ c_3 \{ 14G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu\rho}^a + 14G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho\lambda}^a + 14G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho\rho}^a + \\ &+ 14G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu\rho\rho}^a + 12G_{\mu\nu,\lambda}^a G_{\mu\rho,\nu\lambda\rho}^a + 14G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda\rho\rho}^a + 14G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda\rho\rho}^a \}, \end{aligned} \quad (36)$$

where c_3 is an arbitrary constant. Our intention is to investigate the dependence of symmetries of the system $L_3 + c_3 L_3'$ as a function of constant c_3 . The system is always invariant with respect to the initial, extended gauge group of transformations (8), (32) and (33) for any value of the constant c_3 . We wish to know if there exists a special value of the constant c_3 at which the system will have even higher symmetry, as it happens in the case of the rank-2 gauge field. We shall see that this indeed takes place.

5 Enhanced Symmetry. Rank-3 Gauge Field

The free part of the Lagrangian L_3 comes from the terms

$$-14G_{\mu\nu,\lambda\rho}^a G_{\mu\nu,\lambda\rho}^a - 18G_{\mu\nu,\lambda\lambda}^a G_{\mu\nu,\rho\rho}^a \quad (37)$$

and has the form

$$L_3^{free} = -12\partial_\mu A_{\nu\lambda\rho}^a \partial_\mu A_{\nu\lambda\rho}^a + 12\partial_\mu A_{\nu\lambda\rho}^a \partial_\nu A_{\mu\lambda\rho}^a - 14\partial_\mu A_{\nu\lambda\lambda}^a \partial_\mu A_{\nu\rho\rho}^a + 14\partial_\mu A_{\nu\lambda\lambda}^a \partial_\nu A_{\mu\rho\rho}^a$$

$$\begin{aligned}
&= 12A_{\alpha\alpha'}^a (\eta_{\alpha\gamma} \partial^2 - \partial_\alpha \partial_\gamma) (12\eta_{\alpha\gamma} \eta_{\alpha'\gamma'} + 12\eta_{\alpha\gamma} \eta_{\alpha'\gamma''} + 12\eta_{\alpha\alpha'} \eta_{\gamma\gamma'}) A_{\gamma\gamma'}^a \\
&= 12A_{\alpha\alpha'}^a H_{\alpha\alpha' \gamma\gamma'} A_{\gamma\gamma'}^a,
\end{aligned} \tag{38}$$

where the quadratic form in the momentum representation is

$$H_{\alpha\alpha' \gamma\gamma'}(k) = -12H_{\alpha\gamma} (\eta_{\alpha\gamma} \eta_{\alpha'\gamma''} + \eta_{\alpha\gamma} \eta_{\alpha'\gamma'} + \eta_{\alpha\alpha'} \eta_{\gamma\gamma'}),$$

where $H_{\alpha\gamma} = k^2 \eta_{\alpha\gamma} - k_\alpha k_\gamma$ and by construction is invariant with respect to the gauge transformation

$$\delta A_{\mu\nu\lambda}^a = \partial_\mu \xi_{\nu\lambda}^a$$

because we have

$$k_\alpha H_{\alpha\alpha' \gamma\gamma'}(k) = 0.$$

But it is not invariant with respect to the alternative gauge transformations

$$\tilde{\delta} A_{\mu\nu\lambda}^a = \partial_\nu \zeta_{\mu\lambda}^a + \partial_\lambda \zeta_{\mu\nu}^a,$$

where the gauge parameter $\zeta_{\mu\lambda}^a$ is a totally symmetric tensor. This can be seen from the following relation in momentum representation

$$k_\alpha H_{\alpha\alpha' \gamma\gamma'}(k) = -12H_{\alpha\gamma} (k_\gamma \eta_{\alpha'\gamma''} + k_{\gamma'} \eta_{\alpha'\gamma} + k_{\alpha'} \eta_{\gamma\gamma'}) \neq 0. \tag{39}$$

Let us consider now the free part of the Lagrangian L_3' which comes from the terms

$$14G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu\rho}^a + 14G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho\lambda}^a + 14G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho\rho}^a,$$

thus

$$\begin{aligned}
L_3'^{free} &= +14\partial_\mu A_{\nu\lambda\rho}^a \partial_\mu A_{\lambda\nu\rho}^a - 14\partial_\mu A_{\nu\lambda\rho}^a \partial_\lambda A_{\mu\nu\rho}^a - 14\partial_\nu A_{\mu\lambda\rho}^a \partial_\mu A_{\lambda\nu\rho}^a + 14\partial_\nu A_{\mu\lambda\rho}^a \partial_\lambda A_{\mu\nu\rho}^a \\
&+ 14\partial_\mu A_{\nu\nu\lambda}^a \partial_\mu A_{\rho\rho\lambda}^a - 14\partial_\mu A_{\nu\nu\lambda}^a \partial_\rho A_{\mu\rho\lambda}^a - 14\partial_\nu A_{\mu\nu\lambda}^a \partial_\mu A_{\rho\rho\lambda}^a + 14\partial_\nu A_{\mu\nu\lambda}^a \partial_\rho A_{\mu\rho\lambda}^a \\
&+ 14\partial_\mu A_{\nu\nu\lambda}^a \partial_\mu A_{\lambda\rho\rho}^a - 14\partial_\mu A_{\nu\nu\lambda}^a \partial_\lambda A_{\mu\rho\rho}^a - 14\partial_\nu A_{\mu\nu\lambda}^a \partial_\mu A_{\lambda\rho\rho}^a + 14\partial_\nu A_{\mu\nu\lambda}^a \partial_\lambda A_{\mu\rho\rho}^a \\
&= 12A_{\alpha\alpha'}^a H_{\alpha\alpha' \gamma\gamma'} A_{\gamma\gamma'}^a,
\end{aligned} \tag{40}$$

where one should symmetrize the $H_{\alpha\alpha' \gamma\gamma'}$ over the $\alpha' \leftrightarrow \alpha''$, $\gamma' \leftrightarrow \gamma''$ and the exchange of two sets of indices $\alpha\alpha' \leftrightarrow \gamma\gamma''$, so that the second quadratic form in the momentum representation is (see also Appendix B for derivation)

$$\begin{aligned}
H_{\alpha\alpha' \gamma\gamma'}(k) &= 18\{+(k^2 \eta_{\alpha\alpha'} - k_\alpha k_{\alpha'}) (\eta_{\alpha\gamma} \eta_{\gamma\gamma''} + \eta_{\alpha\gamma} \eta_{\gamma\gamma'} + \eta_{\alpha\alpha'} \eta_{\gamma\gamma'}) \\
&+ (k^2 \eta_{\alpha\alpha''} - k_\alpha k_{\alpha''}) (\eta_{\alpha\gamma} \eta_{\gamma\gamma''} + \eta_{\alpha\gamma} \eta_{\gamma\gamma'} + \eta_{\alpha\gamma} \eta_{\gamma\gamma'})\}
\end{aligned}$$

$$\begin{aligned}
& + (k^2 \eta_{\alpha\gamma} - k_\alpha k_\gamma) (\eta_{\alpha\gamma} \eta_{\alpha\gamma}'' + \eta_{\alpha\gamma}' \eta_{\alpha\gamma}'' + \eta_{\alpha\alpha}'' \eta_{\gamma\gamma}'') \\
& + (k^2 \eta_{\alpha\gamma}'' - k_\alpha k_\gamma'') (\eta_{\alpha\gamma} \eta_{\alpha\gamma}'' + \eta_{\alpha\gamma}' \eta_{\alpha\gamma}'' + \eta_{\alpha\alpha}'' \eta_{\gamma\gamma}'') \} \\
& - 18 \{ + k_\gamma k_\alpha' (\eta_{\alpha\gamma} \eta_{\alpha\gamma}'' + \eta_{\alpha\gamma}' \eta_{\alpha\gamma}'' + \eta_{\alpha\alpha}'' \eta_{\gamma\gamma}'') \\
& + k_\gamma k_\alpha'' (\eta_{\alpha\gamma}' \eta_{\alpha\gamma}'' + \eta_{\alpha\gamma}'' \eta_{\alpha\gamma}') + \eta_{\alpha\alpha}'' \eta_{\gamma\gamma}'') \\
& + k_\gamma k_\gamma' (\eta_{\alpha\alpha}'' \eta_{\alpha\gamma}'' + \eta_{\alpha\alpha}'' \eta_{\alpha\gamma}' + \eta_{\alpha\gamma}'' \eta_{\alpha\alpha}'') \\
& + k_\gamma k_\gamma'' (\eta_{\alpha\alpha}'' \eta_{\alpha\gamma}' + \eta_{\alpha\alpha}'' \eta_{\alpha\gamma}'' + \eta_{\alpha\gamma}'' \eta_{\alpha\alpha}'') \} \\
& + 14 \{ + \eta_{\alpha\gamma} (k_\alpha k_\gamma \eta_{\alpha\gamma}'' + k_\alpha k_\gamma'' \eta_{\alpha\gamma}'' + k_\alpha'' k_\gamma \eta_{\alpha\gamma}'') \\
& + k_\alpha'' k_\gamma \eta_{\alpha\gamma}' + k_\alpha k_\alpha'' \eta_{\gamma\gamma}'' + k_\gamma k_\gamma'' \eta_{\alpha\alpha}'') \}. \tag{41}
\end{aligned}$$

It is again invariant with respect to the transformation $\delta A_{\mu\nu\lambda}^a = \partial_\mu \xi_{\nu\lambda}^a$ because we have

$$k_\alpha H_{\alpha\alpha\alpha\gamma\gamma\gamma}''(k) = 0,$$

but it is not invariant with respect to the transformation $\tilde{\delta} A_{\mu\nu\lambda}^a = \partial_\nu \zeta_{\mu\lambda}^a + \partial_\lambda \zeta_{\mu\nu}^a$, as one can see from the analogous relation (see also Appendix B for derivation)

$$\begin{aligned}
& k_\alpha H_{\alpha\alpha\alpha\gamma\gamma\gamma}''(k) = 18 \{ + H_{\alpha\alpha}'' (k_\gamma \eta_{\gamma\gamma}'' + k_\gamma'' \eta_{\gamma\gamma}') \\
& + H_{\alpha\gamma}'' (k_\gamma \eta_{\alpha\gamma}'' + k_\gamma'' \eta_{\gamma\gamma}'') \\
& + H_{\alpha\gamma}'' (k_\gamma \eta_{\alpha\gamma}' + k_\alpha'' \eta_{\gamma\gamma}') \} \\
& - 14 \{ + k_\gamma k_\alpha'' (k_\gamma \eta_{\alpha\gamma}' + k_\gamma \eta_{\alpha\gamma}'') + k_\gamma k_\gamma'' k_\gamma \eta_{\alpha\alpha}'' - 3 \eta_{\alpha\gamma} k_\alpha'' k_\gamma k_\gamma'' \} \\
& + 14 \{ + H_{\alpha\gamma}'' (k_\gamma \eta_{\alpha\gamma}'' + k_\gamma'' \eta_{\alpha\gamma}' + k_\alpha'' \eta_{\gamma\gamma}') \} \neq 0. \tag{42}
\end{aligned}$$

We have to see now whether the sum

$$k_\alpha (H_{\alpha\alpha\alpha\gamma\gamma\gamma}'' + c_3 H_{\alpha\alpha\alpha\gamma\gamma\gamma}'')$$

can be made equal to zero by an appropriate choice of the coefficient c_3 . For that let us compare the expressions (39) and (??) for divergences. As one can see, only the last term in (??)

$$H_{\alpha\gamma}'' (k_\gamma \eta_{\alpha\gamma}'' + k_\gamma'' \eta_{\alpha\gamma}' + k_\alpha'' \eta_{\gamma\gamma}')$$

and the whole term (39) can cancel each other if we choose $c_3 = 2$, but this will leave the rest of the terms in (??) untouched, thus

$$k_\alpha (H_{\alpha\alpha\alpha\gamma\gamma\gamma}'' + c_3 H_{\alpha\alpha\alpha\gamma\gamma\gamma}'') \neq 0$$

for all values of c_3 . This situation differs from the case of the rank-2 gauge field $A_{\mu\nu}^a$. In the last case we were able to choose the coefficient $c_2 = 1$ so that the divergences (22) and (24) cancel each other and we got (26)

$$k_{\alpha'}(H_{\alpha\alpha'\gamma\gamma'} + H'_{\alpha\alpha'\gamma\gamma'}) = 0.$$

Therefore the Lagrangian $L_2^{free} + L_2'^{free}$ has enhanced invariance with respect to a large gauge group of transformations $\delta A_{\mu\lambda}^a = \partial_\mu \xi_\lambda^a + \partial_\lambda \eta_\mu^a$.

In order to understand the reason why in the case of the rank-3 gauge field it is impossible fully cancel divergences we have to analyze the corresponding field equations. We shall compare the resulting equation with the equation derived by Schwinger [12] for the totally symmetric Abelian rank-3 tensor field in order to get better insight into the problem. It has been proved by Schwinger [12] that it is impossible to derive field equation for the totally symmetric rank-3 tensor which is invariant with respect to the full gauge group of transformations $\delta A_{\mu\nu\lambda} = \partial_\mu \xi_{\nu\lambda}^a + \partial_\nu \xi_{\mu\lambda}^a + \partial_\lambda \xi_{\mu\nu}^a$ without imposing some restriction on the gauge parameters $\xi_{\mu\nu}^a$. As he demonstrated, the gauge parameter should be traceless $\xi_{\mu\mu}^a = 0$. We shall see that similar phenomena take place also in our case, that is, the second gauge parameter $\zeta_{\mu\lambda}^a$ should fulfill constraint which we shall derive below (see equation (44)).

What we would like to prove is that the equation has enhanced invariance with respect to the gauge group of transformations

$$\tilde{\delta} A_{\mu\nu\lambda}^a = \partial_\nu \zeta_{\mu\lambda}^a + \partial_\lambda \zeta_{\mu\nu}^a, \quad (43)$$

but in our case the gauge parameters $\zeta_{\mu\lambda}^a$ should fulfill the following constraint:

$$\partial_\rho \zeta_{\rho\lambda}^a - \partial_\lambda \zeta_{\rho\rho}^a = 0. \quad (44)$$

This takes place when we choose the coefficient $c_3 = 4/3$. Indeed, let us consider the equation of motion. From the first form L_3^{free} we have the following contribution to the field equation:

$$H_{\alpha\alpha'\gamma\gamma'}^{\cdot\cdot\cdot} A_{\gamma\gamma'}^a = \partial^2 A_{\alpha\alpha'}^a - \partial_\alpha \partial_\rho A_{\rho\alpha'}^a + 12\eta_{\alpha\alpha'}^{\cdot\cdot} (\partial^2 A_{\alpha\rho\rho}^a - \partial_\alpha \partial_\rho A_{\rho\lambda\lambda}^a), \quad (45)$$

and from the second one $L_3'^{free}$ we have

$$\begin{aligned} H_{\alpha\alpha'\gamma\gamma'}^{\cdot\cdot\cdot} A_{\gamma\gamma'}^a &= -18\{\partial^2 (A_{\alpha\alpha'}^a + A_{\alpha\alpha}^a + A_{\alpha\alpha'}^a + A_{\alpha\alpha}^a) - \\ &- \partial_\alpha \partial_\rho (A_{\alpha\rho}^a + A_{\alpha\rho}^a + A_{\alpha\rho}^a + A_{\alpha\rho}^a) - \end{aligned}$$

$$\begin{aligned}
& -\partial_{\alpha} \partial_{\rho} (A_{\rho\alpha\alpha}^a + A_{\rho\alpha}^a - A_{\alpha\rho\alpha}^a - A_{\alpha\alpha\rho}^a - A_{\alpha\alpha\rho}^a - A_{\alpha\rho\alpha}^a) - \\
& -\partial_{\alpha} \partial_{\rho} (A_{\rho\alpha\alpha}^a + A_{\rho\alpha}^a - A_{\alpha\rho\alpha}^a - A_{\alpha\alpha\rho}^a - A_{\alpha\alpha\rho}^a - A_{\alpha\rho\alpha}^a) - \\
& -\partial_{\alpha} \partial_{\rho} (A_{\alpha\rho\rho}^a + A_{\rho\alpha\rho}^a + A_{\rho\rho\alpha}^a) - \partial_{\alpha} \partial_{\rho} (A_{\alpha\rho\rho}^a + A_{\rho\alpha\rho}^a + A_{\rho\rho\alpha}^a) + 2\partial_{\alpha} \partial_{\rho} A_{\alpha\rho\rho}^a \} - \\
& -18\{\eta_{\alpha\alpha} [\partial^2 (A_{\alpha\rho\rho}^a + A_{\rho\alpha\rho}^a + A_{\rho\rho\alpha}^a) - \partial_{\alpha} \partial_{\rho} A_{\rho\lambda\lambda}^a - \partial_{\lambda} \partial_{\rho} (A_{\rho\alpha\lambda}^a + A_{\rho\lambda\alpha}^a)] + \\
& + \eta_{\alpha\alpha} [\partial^2 (A_{\alpha\rho\rho}^a + A_{\rho\alpha\rho}^a + A_{\rho\rho\alpha}^a) - \partial_{\alpha} \partial_{\rho} A_{\rho\lambda\lambda}^a - \partial_{\lambda} \partial_{\rho} (A_{\rho\alpha\lambda}^a + A_{\rho\lambda\alpha}^a)] + \\
& + \eta_{\alpha\alpha} [\partial^2 (A_{\rho\alpha\rho}^a + A_{\rho\rho\alpha}^a) - \partial_{\alpha} \partial_{\rho} (A_{\lambda\rho\lambda}^a + A_{\lambda\lambda\rho}^a) - \partial_{\lambda} \partial_{\rho} (A_{\rho\alpha\lambda}^a + A_{\lambda\rho\alpha}^a - 2A_{\alpha\lambda\rho}^a)]\}. \quad (46)
\end{aligned}$$

Summing these two pieces together we shall get the following free field equation of motion for the rank-3 tensor gauge field:

$$\begin{aligned}
& (H_{\alpha\alpha} \cdot \cdot \cdot \cdot + c_3 H_{\alpha\alpha} \cdot \cdot \cdot \cdot) A_{\gamma\gamma}^a = \partial^2 (A_{\alpha\alpha}^a - c_3 4A_{\alpha\alpha}^a - c_3 4A_{\alpha\alpha}^a) - \\
& -\partial_{\alpha} \partial_{\rho} (A_{\rho\alpha\alpha}^a - c_3 4A_{\alpha\alpha\rho}^a - c_3 4A_{\alpha\rho\alpha}^a) - c_3 4\partial_{\alpha} \partial_{\rho} (A_{\alpha\rho\alpha}^a + A_{\alpha\alpha\rho}^a - A_{\rho\alpha\alpha}^a) - \\
& -c_3 4\partial_{\alpha} \partial_{\rho} (A_{\alpha\rho\alpha}^a + A_{\alpha\alpha\rho}^a - A_{\rho\alpha\alpha}^a) + c_3 8\partial_{\alpha} \partial_{\rho} (A_{\alpha\rho\rho}^a + A_{\rho\alpha\rho}^a + A_{\rho\rho\alpha}^a) + \\
& + c_3 8\partial_{\alpha} \partial_{\rho} (A_{\alpha\rho\rho}^a + A_{\rho\alpha\rho}^a + A_{\rho\rho\alpha}^a) - c_3 4\partial_{\alpha} \partial_{\rho} A_{\alpha\rho\rho}^a - \\
& -c_3 8\eta_{\alpha\alpha} (\partial^2 A_{\alpha\rho\rho}^a - \partial_{\alpha} \partial_{\rho} A_{\rho\lambda\lambda}^a + 2\partial^2 A_{\rho\rho\alpha}^a - 2\partial_{\lambda} \partial_{\rho} A_{\rho\lambda\alpha}^a) - \\
& -c_3 8\eta_{\alpha\alpha} (\partial^2 A_{\alpha\rho\rho}^a - \partial_{\alpha} \partial_{\rho} A_{\rho\lambda\lambda}^a + 2\partial^2 A_{\rho\rho\alpha}^a - 2\partial_{\lambda} \partial_{\rho} A_{\rho\lambda\alpha}^a) - \\
& + 12\eta_{\alpha\alpha} (\partial^2 A_{\alpha\rho\rho}^a - \partial_{\alpha} \partial_{\rho} A_{\rho\lambda\lambda}^a - c_3 2\partial^2 A_{\rho\rho\alpha}^a + c_3 2\partial_{\alpha} \partial_{\rho} A_{\lambda\lambda\rho}^a - c_3 2\partial_{\lambda} \partial_{\rho} A_{\alpha\lambda\rho}^a + c_3 2\partial_{\lambda} \partial_{\rho} A_{\lambda\rho\alpha}^a) = 0.
\end{aligned}$$

We shall prove that this equation is invariant with respect to the gauge transformation

$$\tilde{\delta} A_{\mu\nu\lambda}^a = \partial_{\nu} \zeta_{\mu\lambda}^a + \partial_{\lambda} \zeta_{\mu\nu}^a$$

if we choose the coefficient $c_3 = 4/3$. Performing the above gauge transformation of the field one can see that the terms which originate from differential operators ∂^2 , $\partial_{\alpha} \partial_{\rho}$, $\partial_{\alpha} \partial_{\rho}$ and $\partial_{\alpha} \partial_{\rho}$ in the above equation cancel each other if we choose

$$c_3 = 43. \quad (47)$$

The rest of the terms have the following form:

$$\begin{aligned}
& (H_{\alpha\alpha} \cdot \cdot \cdot \cdot + 43H_{\alpha\alpha} \cdot \cdot \cdot \cdot) \tilde{\delta} A_{\gamma\gamma}^a = \\
& + 13\partial_{\alpha} \partial_{\rho} \partial_{\rho} \zeta_{\rho\alpha}^a + 13\partial_{\alpha} \partial_{\rho} \partial_{\rho} \zeta_{\rho\alpha}^a - 43\partial_{\alpha} \partial_{\rho} \partial_{\rho} \zeta_{\rho\alpha}^a + 23\partial_{\alpha} \partial_{\rho} \partial_{\rho} \zeta_{\rho\rho}^a
\end{aligned}$$

$$\begin{aligned}
& -16\eta_{\alpha\alpha}{}^{\cdot} (2\partial_{\rho} \partial^2 \zeta_{\rho\alpha}^a - 4\partial_{\alpha}{}^{\cdot} \partial_{\lambda} \partial_{\rho} \zeta_{\lambda\rho}^a + 2\partial_{\alpha}{}^{\cdot} \partial^2 \zeta_{\rho\rho}^a) \\
& -16\eta_{\alpha\alpha}{}^{\cdot} (2\partial_{\rho} \partial^2 \zeta_{\rho\alpha}^{\cdot} - 4\partial_{\alpha}{}^{\cdot} \partial_{\lambda} \partial_{\rho} \zeta_{\lambda\rho}^{\cdot} + 2\partial_{\alpha}{}^{\cdot} \partial^2 \zeta_{\rho\rho}^{\cdot}) \\
& +13\eta_{\alpha\alpha}{}^{\cdot} (\partial_{\rho} \partial^2 \zeta_{\rho\alpha}^a - \partial_{\alpha} \partial_{\lambda} \partial_{\rho} \zeta_{\lambda\rho}^a)
\end{aligned} \tag{48}$$

and can be rewritten in the form which makes the desired invariance explicit:

$$\begin{aligned}
& (H_{\alpha\alpha}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot} + 43H_{\alpha\alpha}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}) \delta A_{\gamma\gamma}^a{}^{\cdot}{}^{\cdot} = +13\partial_{\alpha} \partial_{\alpha}{}^{\cdot} (\partial_{\rho} \zeta_{\rho\alpha}^a{}^{\cdot} - \partial_{\alpha}{}^{\cdot} \zeta_{\rho\rho}^a{}^{\cdot}) \\
& +13\partial_{\alpha} \partial_{\alpha}{}^{\cdot} (\partial_{\rho} \zeta_{\rho\alpha}^{\cdot} - \partial_{\alpha}{}^{\cdot} \zeta_{\rho\rho}^{\cdot}) - 43\partial_{\alpha}{}^{\cdot} \partial_{\alpha}{}^{\cdot} (\partial_{\rho} \zeta_{\rho\alpha}^a - \partial_{\alpha} \zeta_{\rho\rho}^a) \\
& -13\eta_{\alpha\alpha}{}^{\cdot} [\partial^2 (\partial_{\rho} \zeta_{\rho\alpha}^a{}^{\cdot} - \partial_{\alpha}{}^{\cdot} \zeta_{\rho\rho}^a{}^{\cdot}) + 2\partial_{\alpha}{}^{\cdot} \partial_{\lambda} (\partial_{\lambda} \zeta_{\rho\rho}^a - \partial_{\rho} \zeta_{\rho\lambda}^a)] \\
& -13\eta_{\alpha\alpha}{}^{\cdot} [\partial^2 (\partial_{\rho} \zeta_{\rho\alpha}^{\cdot} - \partial_{\alpha}{}^{\cdot} \zeta_{\rho\rho}^{\cdot}) + 2\partial_{\alpha}{}^{\cdot} \partial_{\lambda} (\partial_{\lambda} \zeta_{\rho\rho}^{\cdot} - \partial_{\rho} \zeta_{\rho\lambda}^{\cdot})] \\
& +13\eta_{\alpha\alpha}{}^{\cdot} [\partial^2 (\partial_{\rho} \zeta_{\rho\alpha}^a - \partial_{\alpha} \zeta_{\rho\rho}^a) + \partial_{\alpha} \partial_{\lambda} (\partial_{\lambda} \zeta_{\rho\rho}^a - \partial_{\rho} \zeta_{\rho\lambda}^a)].
\end{aligned} \tag{49}$$

From that we see that if the gauge parameter satisfies the conditions (44)

$$\partial_{\rho} \zeta_{\rho\lambda}^a - \partial_{\lambda} \zeta_{\rho\rho}^a = 0$$

the equation is indeed invariant with respect to a larger group of gauge transformations

$\tilde{\delta} A_{\mu\nu\lambda}^a = \partial_{\nu} \zeta_{\mu\lambda}^a + \partial_{\lambda} \zeta_{\mu\nu}^a$, because

$$(H_{\alpha\alpha}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot} + 43H_{\alpha\alpha}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{\cdot}) \tilde{\delta} A_{\gamma\gamma}^a{}^{\cdot}{}^{\cdot} = 0. \tag{50}$$

The final form of the equation is

$$\begin{aligned}
& \partial^2 (A_{\alpha\alpha}^a{}^{\cdot}{}^{\cdot} - 13A_{\alpha\alpha}^a{}^{\cdot}{}^{\cdot}{}^{\cdot} - 13A_{\alpha\alpha}^a{}^{\cdot}{}^{\cdot}{}^{\cdot}) - \partial_{\alpha} \partial_{\rho} (A_{\rho\alpha}^a{}^{\cdot}{}^{\cdot} - 13A_{\alpha\rho}^a{}^{\cdot}{}^{\cdot}{}^{\cdot} - 13A_{\alpha\rho}^a{}^{\cdot}{}^{\cdot}{}^{\cdot}) - \\
& -13\partial_{\alpha}{}^{\cdot} \partial_{\rho} (A_{\alpha\rho}^a{}^{\cdot}{}^{\cdot} + A_{\alpha\rho}^a{}^{\cdot}{}^{\cdot}{}^{\cdot} - A_{\rho\alpha}^a{}^{\cdot}{}^{\cdot}{}^{\cdot}) - 13\partial_{\alpha}{}^{\cdot} \partial_{\rho} (A_{\alpha\rho}^a{}^{\cdot}{}^{\cdot} + A_{\alpha\rho}^a{}^{\cdot}{}^{\cdot}{}^{\cdot} - A_{\rho\alpha}^a{}^{\cdot}{}^{\cdot}{}^{\cdot}) + \\
& +16\partial_{\alpha} \partial_{\alpha}{}^{\cdot} (A_{\alpha\rho\rho}^a{}^{\cdot} + A_{\rho\alpha\rho}^a{}^{\cdot} + A_{\rho\rho\alpha}^a{}^{\cdot}) + 16\partial_{\alpha} \partial_{\alpha}{}^{\cdot} (A_{\alpha\rho\rho}^a{}^{\cdot} + A_{\rho\alpha\rho}^a{}^{\cdot} + A_{\rho\rho\alpha}^a{}^{\cdot}) - 13\partial_{\alpha}{}^{\cdot} \partial_{\alpha}{}^{\cdot} A_{\alpha\rho\rho}^a{}^{\cdot} - \\
& -16\eta_{\alpha\alpha}{}^{\cdot} (\partial^2 A_{\alpha\rho\rho}^a{}^{\cdot} - \partial_{\alpha}{}^{\cdot} \partial_{\rho} A_{\rho\lambda\lambda}^a{}^{\cdot} + 2\partial^2 A_{\rho\rho\alpha}^a{}^{\cdot} - 2\partial_{\lambda} \partial_{\rho} A_{\rho\lambda\alpha}^a{}^{\cdot}) - \\
& -16\eta_{\alpha\alpha}{}^{\cdot} (\partial^2 A_{\alpha\rho\rho}^{\cdot} - \partial_{\alpha}{}^{\cdot} \partial_{\rho} A_{\rho\lambda\lambda}^{\cdot} + 2\partial^2 A_{\rho\rho\alpha}^{\cdot} - 2\partial_{\lambda} \partial_{\rho} A_{\rho\lambda\alpha}^{\cdot}) - \\
& +12\eta_{\alpha\alpha}{}^{\cdot} (\partial^2 A_{\alpha\rho\rho}^a - \partial_{\alpha} \partial_{\rho} A_{\rho\lambda\lambda}^a - 23\partial^2 A_{\rho\rho\alpha}^a + 23\partial_{\alpha} \partial_{\rho} A_{\lambda\lambda\rho}^a - 23\partial_{\lambda} \partial_{\rho} A_{\alpha\lambda\rho}^a + 23\partial_{\lambda} \partial_{\rho} A_{\lambda\rho\alpha}^a) = 0
\end{aligned} \tag{51}$$

and it is invariant with respect to the gauge group of transformations

$$\delta A_{\mu\nu\lambda}^a = \partial_{\mu} \zeta_{\nu\lambda}^a, \quad \tilde{\delta} A_{\mu\nu\lambda}^a = \partial_{\nu} \zeta_{\mu\lambda}^a + \partial_{\lambda} \zeta_{\mu\nu}^a, \quad \partial_{\rho} \zeta_{\rho\lambda}^a - \partial_{\lambda} \zeta_{\rho\rho}^a = 0. \tag{52}$$

One should stress that there are no restrictions on the gauge parameters $\zeta_{\mu\nu}^a$. The above invariance

of the equation now can be checked directly without referring to the previous analysis. In summary, we have the following Lagrangian for the third-rank gauge field $A_{\mu\nu\lambda}^a$:

$$\begin{aligned}
L_3 + 43L_3' &= -14G_{\mu\nu,\lambda\rho}^a G_{\mu\nu,\lambda\rho}^a - 18G_{\mu\nu,\lambda\lambda}^a G_{\mu\nu,\rho\rho}^a - 12G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda\rho\rho}^a - 18G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda\rho\rho}^a + \\
&+ 13G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu\rho}^a + 13G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho\lambda}^a + 13G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho\rho}^a + \\
&+ 13G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu\rho\rho}^a + 23G_{\mu\nu,\lambda}^a G_{\mu\rho,\nu\lambda\rho}^a + 13G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda\rho\rho}^a + 13G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda\rho\rho}^a.
\end{aligned} \tag{53}$$

We shall present the free equation of motion (51) also in terms of field strength tensors. The kinetic term of the above Lagrangian is

$$\begin{aligned}
L_3 + 43L_3' |_{free} &= -14F_{\mu\nu,\lambda\rho}^a F_{\mu\nu,\lambda\rho}^a - 18F_{\mu\nu,\lambda\lambda}^a F_{\mu\nu,\rho\rho}^a \\
&+ 13F_{\mu\nu,\lambda\rho}^a F_{\mu\lambda,\nu\rho}^a + 13F_{\mu\nu,\nu\lambda}^a F_{\mu\rho,\rho\lambda}^a + 13F_{\mu\nu,\nu\lambda}^a F_{\mu\lambda,\rho\rho}^a,
\end{aligned} \tag{54}$$

where

$$F_{\mu\nu,\lambda\rho}^a = \partial_\mu A_{\nu\lambda\rho}^a - \partial_\nu A_{\mu\lambda\rho}^a.$$

The variation of the above Lagrangian over the field $A_{\nu\lambda\rho}^a$ gives the free equation written in terms of field strength tensor $F_{\mu\nu,\lambda\rho}^a$ and it is identical to the equation (51)

$$\begin{aligned}
&\partial_\mu F_{\mu\nu,\lambda\rho}^a - 13\partial_\mu F_{\mu\lambda,\nu\rho}^a - 13\partial_\mu F_{\mu\rho,\nu\lambda}^a + 13\partial_\mu F_{\nu\lambda,\mu\rho}^a + 13\partial_\mu F_{\nu\rho,\mu\lambda}^a + \\
&+ 13\partial_\lambda F_{\nu\mu,\mu\rho}^a + 13\partial_\rho F_{\nu\mu,\mu\lambda}^a + 16\partial_\lambda F_{\nu\rho,\mu\mu}^a + 16\partial_\rho F_{\nu\lambda,\mu\mu}^a - \\
&- \eta_{\lambda\nu} (13\partial_\mu F_{\mu\sigma,\sigma\rho}^a + 16\partial_\mu F_{\mu\rho,\sigma\sigma}^a) - \eta_{\nu\rho} (13\partial_\mu F_{\mu\sigma,\sigma\lambda}^a + 16\partial_\mu F_{\mu\lambda,\sigma\sigma}^a) + \\
&+ \eta_{\lambda\rho} (12\partial_\mu F_{\mu\nu,\sigma\sigma}^a - 13\partial_\mu F_{\mu\sigma,\sigma\nu}^a + 13\partial_\mu F_{\nu\sigma,\sigma\mu}^a) = j_{\nu\lambda\rho}^a.
\end{aligned} \tag{55}$$

As we demonstrated, this equation is invariant with respect to the following gauge transformations:

$$\delta A_{\mu\nu\lambda}^a = \partial_\mu \xi_{\nu\lambda}^a, \quad \tilde{\delta} A_{\mu\nu\lambda}^a = \partial_\nu \zeta_{\mu\lambda}^a + \partial_\lambda \zeta_{\mu\nu}^a,$$

where the gauge parameters are totally symmetric tensors satisfying the condition (44). The initial invariance of the equation $\delta A_{\mu\nu\lambda}^a = \partial_\mu \xi_{\nu\lambda}^a$ imposes restriction on the current $j_{\nu\lambda\rho}^a$, in particular, on its conservation over the first index:

$$\partial_\nu j_{\nu\lambda\rho}^a = 0. \tag{56}$$

There are also additional constraints on the current which follow from the enhanced invariance $\tilde{\delta} A_{\mu\nu\lambda}^a = \partial_\nu \zeta_{\mu\lambda}^a + \partial_\lambda \zeta_{\mu\nu}^a$. In Fourier components the constraints on the group parameters are

$$\begin{aligned}
\omega\zeta_{03} + \kappa\zeta_{33} + \kappa(\zeta_{00} - \zeta_{11} - \zeta_{22} - \zeta_{33}) &= 0, \\
\omega\zeta_{01} + \kappa\zeta_{31} &= 0,
\end{aligned}$$

$$\begin{aligned}\omega\zeta_{02} + \kappa\zeta_{32} &= 0, \\ \omega\zeta_{00} + \kappa\zeta_{30} - \omega(\zeta_{00} - \zeta_{11} - \zeta_{22} - \zeta_{33}) &= 0,\end{aligned}$$

where $k^\mu = (\omega, 0, 0, \kappa)$, therefore

$$\begin{aligned}\zeta_{00} &= \zeta_{11} + \zeta_{22} - \omega\kappa\zeta_{03}, \\ \zeta_{31} &= -\omega\kappa\zeta_{01}, \\ \zeta_{32} &= -\omega\kappa\zeta_{02}, \\ \zeta_{33} &= -\zeta_{11} - \zeta_{22} - \kappa\omega\zeta_{03},\end{aligned}\tag{57}$$

and we have six independent gauge parameters

$$\zeta_{01}, \zeta_{02}, \zeta_{03}, \zeta_{11}, \zeta_{22}, \zeta_{12}.$$

From this it follows that the current components fulfill the following six relations:

$$\begin{aligned}k_\lambda(j_{1\lambda 0} + j_{0\lambda 1} + \omega\kappa j_{1\lambda 3} + \omega\kappa j_{3\lambda 1}) &= 0, \\ k_\lambda(j_{2\lambda 0} + j_{0\lambda 2} + \omega\kappa j_{2\lambda 3} + \omega\kappa j_{3\lambda 2}) &= 0, \\ k_\lambda(\omega\kappa j_{0\lambda 0} + \kappa\omega j_{3\lambda 3}) &= 0, \\ k_\lambda(j_{0\lambda 0} - j_{3\lambda 3} + j_{1\lambda 1}) &= 0, \\ k_\lambda(j_{0\lambda 0} - j_{3\lambda 3} + j_{2\lambda 2}) &= 0, \\ k_\lambda(j_{1\lambda 2} + j_{2\lambda 1}) &= 0,\end{aligned}\tag{58}$$

One can also use a different set of independent parameters, in particular: $\zeta_{11}, \zeta_{12}, \zeta_{13}, \zeta_{22}, \zeta_{23}, \zeta_{33}$.

6 Schwinger Equation for rank-3 Gauge Field

The Schwinger equation for symmetric massless rank-3 tensor gauge field has the form [12]

$$\begin{aligned}& + \partial^2 A_{\alpha\alpha\alpha} - \partial_\alpha \partial_\rho A_{\rho\alpha\alpha} - \partial_\alpha \partial_\rho A_{\alpha\rho\alpha} - \partial_\alpha \partial_\rho A_{\alpha\alpha\rho} + \\ & + \partial_\alpha \partial_\alpha A_{\alpha\rho\rho} + \partial_\alpha \partial_\alpha A_{\alpha\rho\rho} + \partial_\alpha \partial_\alpha A_{\alpha\rho\rho} - 3\partial_\alpha \partial_\alpha \partial_\alpha A - \\ & - \eta_{\alpha\alpha} (\partial^2 A_{\rho\rho\alpha} - \partial_\lambda \partial_\rho A_{\rho\lambda\alpha} + 12\partial_\alpha \partial_\rho A_{\rho\lambda\lambda}) - \\ & - \eta_{\alpha\alpha} (\partial^2 A_{\rho\rho\alpha} - \partial_\lambda \partial_\rho A_{\rho\lambda\alpha} + 12\partial_\alpha \partial_\rho A_{\rho\lambda\lambda}) - \\ & - \eta_{\alpha\alpha} (\partial^2 A_{\rho\rho\alpha} - \partial_\lambda \partial_\rho A_{\rho\lambda\alpha} + 12\partial_\alpha \partial_\rho A_{\lambda\lambda\rho}) = j_{\alpha\alpha\alpha}\end{aligned}\tag{59}$$

and contains the scalar field A which should satisfy the high-order differential equation

$$\partial^2 \partial^2 A - \partial^2 \partial_\lambda A_{\lambda\rho\rho} + 23 \partial_\alpha \partial_\lambda \partial_\rho A_{\alpha\lambda\rho} = 0. \quad (60)$$

Taking derivatives of the l.h.s. of the above equation one can get convinced that we have conservation of the totally symmetric current $j_{\alpha\alpha\alpha}$

$$\partial_\alpha j_{\alpha\alpha\alpha} = 0 \quad (61)$$

and the invariance of the equation with respect to the full gauge transformation

$$\delta A_{\mu\nu\lambda} = \partial_\mu \xi_{\nu\lambda} + \partial_\nu \xi_{\mu\lambda} + \partial_\lambda \xi_{\mu\nu} \quad (62)$$

without any restrictions on the symmetric gauge parameter $\xi_{\nu\lambda}$. The great advantage of this formulation is that we have conservation of current and full gauge symmetry (62). The disadvantage of this formulation is the appearance of the scalar field A and its high-order differential equation. The illuminating remark of Schwinger was to make a change of field variable of the form [12]

$$A_{\alpha\alpha\alpha} \rightarrow A_{\alpha\alpha\alpha} - 3 \partial^{-2} \partial_\alpha \partial_\alpha \partial_\alpha A,$$

which allows to eliminate the scalar field A from the field equation without changing its actual form! The equation will take a unique form [12]:

$$\begin{aligned} & + \partial^2 A_{\alpha\alpha\alpha} - \partial_\alpha \partial_\rho A_{\rho\alpha\alpha} - \partial_\alpha \partial_\rho A_{\alpha\rho\alpha} - \partial_\alpha \partial_\rho A_{\alpha\alpha\rho} + \\ & + \partial_\alpha \partial_\alpha A_{\alpha\rho\rho} + \partial_\alpha \partial_\alpha A_{\alpha\rho\rho} + \partial_\alpha \partial_\alpha A_{\alpha\rho\rho} - \\ & - \eta_{\alpha\alpha} (\partial^2 A_{\rho\rho\alpha} - \partial_\lambda \partial_\rho A_{\rho\lambda\alpha} + 12 \partial_\alpha \partial_\rho A_{\rho\lambda\lambda}) - \\ & - \eta_{\alpha\alpha} (\partial^2 A_{\rho\rho\alpha} - \partial_\lambda \partial_\rho A_{\rho\lambda\alpha} + 12 \partial_\alpha \partial_\rho A_{\rho\lambda\lambda}) - \\ & - \eta_{\alpha\alpha} (\partial^2 A_{\rho\rho\alpha} - \partial_\lambda \partial_\rho A_{\rho\lambda\alpha} + 12 \partial_\alpha \partial_\rho A_{\rho\lambda\lambda}) = j_{\alpha\alpha\alpha}, \end{aligned} \quad (63)$$

but it is not invariant any more with respect to the unrestricted gauge transformations (62). The gauge parameter should be traceless:

$$\xi_{\mu\mu} = 0. \quad (64)$$

This leads to the modification of the current conservation law:

$$\partial_\alpha j_{\alpha\alpha\alpha} - 14 \eta_{\alpha\alpha} j_{\alpha\rho\rho} = 0. \quad (65)$$

The conservation law for the current became more sophisticated because of the traceless restriction on the gauge parameters (64). One can see that the same phenomenon also happens in our case where the restriction on the gauge parameters $\zeta_{\nu\lambda}$ has the form (52) and the conservation law takes the form (58). Recent discussion of the Schwinger equation can be found in [30, 31].

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7 Appendix A

The invariance of the form L_3 can be demonstrated by explicit variation of each term in the sum (??). Indeed, the variation of the first term is

$$\delta_\xi G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu\rho}^a = 2gf^{abc} G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu\rho}^b \xi_\rho^c + 2gf^{abc} G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\rho\nu}^b \xi_\nu^c + 2gf^{abc} G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda}^b \xi_{\nu\rho}^c,$$

of the second term is

$$\delta_\xi G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho\lambda}^a = 2gf^{abc} G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho\lambda}^b \xi_\lambda^c + 2gf^{abc} G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\lambda\xi}^b \xi_\rho^c + 2gf^{abc} G_{\mu\nu,\nu\lambda}^a G_{\mu\rho}^b \xi_{\rho\lambda}^c,$$

of the third term is

$$\begin{aligned} \delta_\xi G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho\rho}^a &= 2gf^{abc} G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho\rho}^b \xi_\rho^c + gf^{abc} G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda}^b \xi_{\rho\rho}^c + gf^{abc} G_{\mu\lambda,\rho\rho}^a G_{\mu\nu,\nu\lambda}^b \xi_\lambda^c + \\ &+ gf^{abc} G_{\mu\lambda,\rho\rho}^a G_{\mu\nu,\lambda\xi}^b \xi_\nu^c + gf^{abc} G_{\mu\lambda,\rho\rho}^a G_{\mu\nu}^b \xi_{\nu\lambda}^c, \end{aligned}$$

of the fourth term is

$$\begin{aligned} \delta_\xi G_{\mu\nu,\lambda\xi}^a G_{\mu\lambda,\nu\rho\rho}^a &= gf^{abc} G_{\mu\lambda,\nu\rho\rho}^a G_{\mu\nu,\lambda\xi}^b \xi_\rho^c + 2gf^{abc} G_{\mu\lambda,\nu\rho}^a G_{\mu\nu,\lambda\xi}^b \xi_\rho^c + gf^{abc} G_{\mu\lambda,\rho\rho}^a G_{\mu\nu,\lambda\xi}^b \xi_\nu^c + \\ &+ gf^{abc} G_{\mu\nu,\lambda\xi}^a G_{\mu\lambda,\nu\xi}^b \xi_{\rho\rho}^c + 2gf^{abc} G_{\mu\nu,\lambda\xi}^a G_{\mu\lambda,\rho\xi}^b \xi_{\nu\rho}^c + gf^{abc} G_{\mu\nu,\lambda\xi}^a G_{\mu\lambda}^b \xi_{\nu\rho\rho}^c, \end{aligned}$$

of the fifth term is

$$\begin{aligned} \delta_\xi G_{\mu\nu,\lambda\xi}^a G_{\mu\rho,\nu\lambda\rho}^a &= \\ &gf^{abc} G_{\mu\rho,\nu\lambda\rho}^a G_{\mu\nu,\lambda\xi}^b \xi_\lambda^c + gf^{abc} G_{\mu\rho,\nu\lambda}^b G_{\mu\nu,\lambda\xi}^a \xi_\rho^c + gf^{abc} G_{\mu\rho,\nu\rho}^b G_{\mu\nu,\lambda\xi}^a \xi_\lambda^c + gf^{abc} G_{\mu\rho,\lambda\rho}^b G_{\mu\nu,\lambda\xi}^a \xi_\nu^c + \\ &+ gf^{abc} G_{\mu\nu,\lambda\xi}^a G_{\mu\rho,\nu\xi}^b \xi_{\lambda\rho}^c + gf^{abc} G_{\mu\nu,\lambda\xi}^a G_{\mu\rho,\lambda\xi}^b \xi_{\nu\rho}^c + gf^{abc} G_{\mu\nu,\lambda\xi}^a G_{\mu\rho,\rho\xi}^b \xi_{\nu\lambda}^c + gf^{abc} G_{\mu\nu,\lambda\xi}^a G_{\mu\rho}^b \xi_{\nu\lambda\rho}^c, \end{aligned}$$

of the sixth term is

$$\begin{aligned} \delta_\xi G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\lambda\rho\rho}^a &= \\ &gf^{abc} G_{\mu\lambda,\lambda\rho\rho}^a G_{\mu\nu,\nu\lambda}^b \xi_\nu^c + 2gf^{abc} G_{\mu\lambda,\lambda\rho}^b G_{\mu\nu,\nu\lambda}^a \xi_\rho^c + gf^{abc} G_{\mu\lambda,\rho\rho}^b G_{\mu\nu,\nu\lambda}^a \xi_\lambda^c + gf^{abc} G_{\mu\lambda,\lambda\xi}^b G_{\mu\nu,\nu\lambda}^a \xi_{\rho\rho}^c + \\ &+ 2gf^{abc} G_{\mu\lambda,\rho\xi}^b G_{\mu\nu,\nu\lambda}^a \xi_{\lambda\rho}^c + gf^{abc} G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda}^b \xi_{\lambda\rho\rho}^c \end{aligned}$$

and finally of the seventh term is

$$\begin{aligned} \delta_\xi G_{\mu,\nu\lambda}^a G_{\mu\lambda,\nu\lambda\rho\rho}^a &= \\ &2gf^{abc} G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda\rho\rho}^b \xi_\rho^c + gf^{abc} G_{\mu\nu}^a G_{\mu\lambda,\nu\rho\rho}^b \xi_\lambda^c + gf^{abc} G_{\mu\nu}^a G_{\mu\lambda,\lambda\rho\rho}^b \xi_\nu^c + gf^{abc} G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda}^b \xi_{\rho\rho}^c + \end{aligned}$$

$$\begin{aligned}
& 2gf^{abc}G_{\mu\nu}^aG_{\mu\lambda,\nu\rho}^b\xi_{\lambda\rho}^c + 2gf^{abc}G_{\mu\nu}^aG_{\mu\lambda,\lambda\rho}^b\xi_{\nu\rho}^c + gf^{abc}G_{\mu\nu}^aG_{\mu\lambda,\rho\rho}^b\xi_{\nu\lambda}^c + \\
& gf^{abc}G_{\mu\nu}^aG_{\mu\lambda,\nu}^b\xi_{\lambda\rho\rho}^c + gf^{abc}G_{\mu\nu}^aG_{\mu\lambda,\lambda}^b\xi_{\nu\rho\rho}^c + 2gf^{abc}G_{\mu\nu}^aG_{\mu\lambda,\rho}^b\xi_{\nu\lambda\rho}^c + gf^{abc}G_{\mu\nu}^aG_{\mu\lambda}^b\xi_{\nu\lambda\rho\rho}^c.
\end{aligned}$$

Some of the terms here are equal to zero, like: $gf^{abc}G_{\mu\nu,\lambda}^aG_{\mu\rho,\lambda}^b\xi_{\nu\rho}^c$, $gf^{abc}G_{\mu\lambda,\lambda}^aG_{\mu\nu,\nu}^b\xi_{\rho\rho}^c$ and $gf^{abc}G_{\mu\nu}^aG_{\mu\lambda}^b\xi_{\nu\lambda\rho\rho}^c$. Amazingly, all nonzero terms cancel each other.

8 Appendix B

The quadratic form $H'_{\alpha\alpha'\gamma\gamma'}$ can be extracted from (??) and should be symmetrized over the $\alpha' \leftrightarrow \alpha''$, $\gamma' \leftrightarrow \gamma''$ and over the exchange of two sets of indices $\alpha\alpha'\alpha'' \leftrightarrow \gamma\gamma'\gamma''$, so that in the momentum representation it has the form

$$\begin{aligned}
H'_{\alpha\alpha'\gamma\gamma'}(k) &= k^2 8 \{ +\eta_{\alpha\alpha'}(\eta_{\alpha\gamma'}\eta_{\gamma\gamma''} + \eta_{\alpha\gamma''}\eta_{\gamma\gamma'} + \eta_{\alpha\gamma'}\eta_{\gamma\gamma''}) \\
&+ \eta_{\alpha\alpha''}(\eta_{\alpha\gamma'}\eta_{\gamma\gamma''} + \eta_{\alpha\gamma''}\eta_{\gamma\gamma'} + \eta_{\alpha\gamma'}\eta_{\gamma\gamma''}) \\
&+ \eta_{\alpha\gamma'}(\eta_{\alpha\gamma''}\eta_{\alpha\gamma''} + \eta_{\alpha\gamma''}\eta_{\alpha\gamma''} + \eta_{\alpha\alpha''}\eta_{\gamma\gamma''}) \\
&+ \eta_{\alpha\gamma''}(\eta_{\alpha\gamma'}\eta_{\alpha\gamma''} + \eta_{\alpha\gamma''}\eta_{\alpha\gamma''} + \eta_{\alpha\alpha''}\eta_{\gamma\gamma''}) \} \\
&- 18 \{ +k_{\alpha\alpha'}k_{\alpha\alpha''}(\eta_{\alpha\gamma'}\eta_{\gamma\gamma''} + \eta_{\alpha\gamma''}\eta_{\gamma\gamma'} + \eta_{\alpha\gamma'}\eta_{\gamma\gamma''}) \\
&+ k_{\alpha\alpha'}k_{\alpha\alpha''}(\eta_{\alpha\gamma'}\eta_{\gamma\gamma''} + \eta_{\alpha\gamma''}\eta_{\gamma\gamma'} + \eta_{\alpha\gamma'}\eta_{\gamma\gamma''}) \\
&+ k_{\alpha\alpha'}k_{\alpha\gamma'}(\eta_{\alpha\gamma''}\eta_{\alpha\gamma''} + \eta_{\alpha\gamma''}\eta_{\alpha\gamma''} + \eta_{\alpha\alpha''}\eta_{\gamma\gamma''}) \\
&+ k_{\alpha\alpha'}k_{\alpha\gamma''}(\eta_{\alpha\gamma'}\eta_{\alpha\gamma''} + \eta_{\alpha\gamma''}\eta_{\alpha\gamma''} + \eta_{\alpha\alpha''}\eta_{\gamma\gamma''}) \\
&+ k_{\gamma'}k_{\alpha\alpha''}(\eta_{\alpha\gamma'}\eta_{\alpha\gamma''} + \eta_{\alpha\gamma''}\eta_{\alpha\gamma''} + \eta_{\alpha\alpha''}\eta_{\gamma\gamma''}) \\
&+ k_{\gamma''}k_{\alpha\alpha''}(\eta_{\alpha\gamma'}\eta_{\alpha\gamma''} + \eta_{\alpha\gamma''}\eta_{\alpha\gamma''} + \eta_{\alpha\alpha''}\eta_{\gamma\gamma''}) \\
&+ k_{\gamma'}k_{\gamma''}(\eta_{\alpha\alpha''}\eta_{\alpha\gamma''} + \eta_{\alpha\alpha''}\eta_{\alpha\gamma''} + \eta_{\alpha\gamma'}\eta_{\alpha\alpha''}) \\
&+ k_{\gamma''}k_{\gamma'}(\eta_{\alpha\alpha''}\eta_{\alpha\gamma''} + \eta_{\alpha\alpha''}\eta_{\alpha\gamma''} + \eta_{\alpha\gamma'}\eta_{\alpha\alpha''}) \} \\
&+ 14 \{ +\eta_{\alpha\gamma'}(k_{\alpha\gamma''}k_{\alpha\gamma''}\eta_{\alpha\gamma''} + k_{\alpha\gamma''}k_{\alpha\gamma''}\eta_{\alpha\gamma''} + k_{\alpha\gamma''}k_{\alpha\gamma''}\eta_{\alpha\gamma''}) \\
&+ k_{\alpha\gamma''}k_{\alpha\gamma''}\eta_{\alpha\gamma''} + k_{\alpha\gamma''}k_{\alpha\gamma''}\eta_{\alpha\gamma''} + k_{\alpha\gamma''}k_{\alpha\gamma''}\eta_{\alpha\gamma''} \}.
\end{aligned} \tag{66}$$

or combining some of the terms together we shall get an equivalent form

$$\begin{aligned}
H'_{\alpha\alpha\alpha\gamma\gamma\gamma}(k) &= 18\{+(k^2\eta_{\alpha\alpha} - k_\alpha k_\alpha)(\eta_{\alpha\gamma}\eta_{\gamma\gamma} + \eta_{\alpha\gamma}\eta_{\gamma\gamma} + \eta_{\alpha\gamma}\eta_{\gamma\gamma}) \\
&+ (k^2\eta_{\alpha\alpha} - k_\alpha k_\alpha)(\eta_{\alpha\gamma}\eta_{\gamma\gamma} + \eta_{\alpha\gamma}\eta_{\gamma\gamma} + \eta_{\alpha\gamma}\eta_{\gamma\gamma}) \\
&+ (k^2\eta_{\alpha\gamma} - k_\alpha k_\gamma)(\eta_{\alpha\gamma}\eta_{\alpha\gamma} + \eta_{\alpha\gamma}\eta_{\alpha\gamma} + \eta_{\alpha\alpha}\eta_{\gamma\gamma}) \\
&+ (k^2\eta_{\alpha\gamma} - k_\alpha k_\gamma)(\eta_{\alpha\gamma}\eta_{\alpha\gamma} + \eta_{\alpha\gamma}\eta_{\alpha\gamma} + \eta_{\alpha\alpha}\eta_{\gamma\gamma})\} \\
&- 18\{+k_\gamma k_\alpha(\eta_{\alpha\gamma}\eta_{\alpha\gamma} + \eta_{\alpha\gamma}\eta_{\alpha\gamma} + \eta_{\alpha\alpha}\eta_{\gamma\gamma}) \\
&+ k_\gamma k_\alpha(\eta_{\alpha\gamma}\eta_{\alpha\gamma} + \eta_{\alpha\gamma}\eta_{\alpha\gamma} + \eta_{\alpha\alpha}\eta_{\gamma\gamma}) \\
&+ k_\gamma k_\gamma(\eta_{\alpha\alpha}\eta_{\alpha\gamma} + \eta_{\alpha\alpha}\eta_{\alpha\gamma} + \eta_{\alpha\gamma}\eta_{\alpha\alpha}) \\
&+ k_\gamma k_\gamma(\eta_{\alpha\alpha}\eta_{\alpha\gamma} + \eta_{\alpha\alpha}\eta_{\alpha\gamma} + \eta_{\alpha\gamma}\eta_{\alpha\alpha})\} \\
&+ 14\{+\eta_{\alpha\gamma}(k_\alpha k_\gamma \eta_{\alpha\gamma} + k_\alpha k_\gamma \eta_{\alpha\gamma} + k_\alpha k_\gamma \eta_{\alpha\gamma}) \\
&+ k_\alpha k_\gamma \eta_{\alpha\gamma} + k_\alpha k_\alpha \eta_{\gamma\gamma} + k_\gamma k_\gamma \eta_{\alpha\alpha}\}. \tag{67}
\end{aligned}$$

This expression can be used to calculate divergences. Indeed,

$$\begin{aligned}
k_\alpha H'_{\alpha\alpha\alpha\gamma\gamma\gamma}(k) &= 18\{+(k^2\eta_{\alpha\alpha} - k_\alpha k_\alpha)(k_\gamma \eta_{\gamma\gamma} + k_\gamma \eta_{\gamma\gamma} + k_\gamma \eta_{\gamma\gamma}) \\
&+ (k^2\eta_{\alpha\gamma} - k_\alpha k_\gamma)(k_\gamma \eta_{\alpha\gamma} + k_\gamma \eta_{\alpha\gamma} + k_\alpha \eta_{\gamma\gamma}) \\
&+ (k^2\eta_{\alpha\gamma} - k_\alpha k_\gamma)(k_\gamma \eta_{\alpha\gamma} + k_\gamma \eta_{\alpha\gamma} + k_\alpha \eta_{\gamma\gamma})\} \\
&- 18\{+k^2 k_\gamma(\eta_{\alpha\gamma}\eta_{\alpha\gamma} + \eta_{\alpha\gamma}\eta_{\alpha\gamma} + \eta_{\alpha\alpha}\eta_{\gamma\gamma}) \\
&+ k_\gamma k_\alpha(2k_\gamma \eta_{\alpha\gamma} + 2k_\gamma \eta_{\alpha\gamma} + k_\alpha \eta_{\gamma\gamma}) \\
&+ k_\gamma k_\gamma(k_\alpha \eta_{\alpha\gamma} + 2k_\gamma \eta_{\alpha\alpha}) + k_\gamma k_\gamma k_\alpha \eta_{\alpha\gamma}\} \\
&+ 14\{+\eta_{\alpha\gamma}(k^2 k_\gamma \eta_{\alpha\gamma} + k^2 k_\gamma \eta_{\alpha\gamma} + k^2 k_\alpha \eta_{\gamma\gamma} + 3k_\alpha k_\gamma k_\gamma)\}
\end{aligned}$$

or using the operator $H_{\alpha\gamma} = k^2\eta_{\alpha\gamma} - k_\alpha k_\gamma$ one can get

$$\begin{aligned}
k_\alpha H'_{\alpha\alpha\alpha\gamma\gamma\gamma}(k) &= 18\{+H_{\alpha\alpha}(k_\gamma \eta_{\gamma\gamma} + k_\gamma \eta_{\gamma\gamma} + k_\gamma \eta_{\gamma\gamma}) \\
&+ H_{\alpha\gamma}(k_\gamma \eta_{\alpha\gamma} + k_\gamma \eta_{\alpha\gamma} + k_\alpha \eta_{\gamma\gamma}) \\
&+ H_{\alpha\gamma}(k_\gamma \eta_{\alpha\gamma} + k_\gamma \eta_{\alpha\gamma} + k_\alpha \eta_{\gamma\gamma})\} \\
&- 18\{+H_{\alpha\alpha} k_\gamma \eta_{\gamma\gamma} + H_{\alpha\gamma} k_\gamma \eta_{\alpha\gamma} + H_{\alpha\gamma} k_\gamma \eta_{\alpha\gamma}\}
\end{aligned}$$

$$\begin{aligned}
& +k_\gamma k_\alpha (2k_\gamma \eta_{\alpha\gamma} + 2k_\gamma \eta_{\alpha\gamma} + 2k_\alpha \eta_{\gamma\gamma}) \\
& +k_\gamma k_\gamma (2k_\alpha \eta_{\alpha\gamma} + 2k_\gamma \eta_{\alpha\alpha}) + 2k_\gamma k_\gamma k_\alpha \eta_{\alpha\gamma} \} \\
& +14\{+\eta_{\alpha\gamma}(k^2 k_\gamma \eta_{\alpha\gamma} + k^2 k_\gamma \eta_{\alpha\gamma} + k^2 k_\alpha \eta_{\gamma\gamma} + 3k_\alpha k_\gamma k_\gamma) \}
\end{aligned}$$

and canceling the identical terms we shall get

$$\begin{aligned}
k_\alpha H'_{\alpha\alpha\alpha\gamma\gamma\gamma}(k) & = 18\{+H_{\alpha\alpha}(k_\gamma \eta_{\gamma\gamma} + k_\gamma \eta_{\gamma\gamma}) \\
& + H_{\alpha\gamma}(k_\gamma \eta_{\alpha\gamma} + k_\alpha \eta_{\gamma\gamma}) \\
& + H_{\alpha\gamma}(k_\gamma \eta_{\alpha\gamma} + k_\alpha \eta_{\gamma\gamma}) \} \\
& -14\{+k_\gamma k_\alpha (k_\gamma \eta_{\alpha\gamma} + k_\gamma \eta_{\alpha\gamma}) + k_\gamma k_\gamma k_\gamma \eta_{\alpha\alpha} \} \\
& +14\{+H_{\alpha\gamma} k_\gamma \eta_{\alpha\gamma} + H_{\alpha\gamma} k_\gamma \eta_{\alpha\gamma} + H_{\alpha\gamma} k_\alpha \eta_{\gamma\gamma} \\
& + 3\eta_{\alpha\gamma} k_\alpha k_\gamma k_\gamma \}.
\end{aligned}$$

Again collecting terms we shall get the final expression:

$$\begin{aligned}
k_\alpha H'_{\alpha\alpha\alpha\gamma\gamma\gamma}(k) & = 18\{+H_{\alpha\alpha}(k_\gamma \eta_{\gamma\gamma} + k_\gamma \eta_{\gamma\gamma}) \\
& + H_{\alpha\gamma}(k_\gamma \eta_{\alpha\gamma} + k_\alpha \eta_{\gamma\gamma}) \\
& + H_{\alpha\gamma}(k_\gamma \eta_{\alpha\gamma} + k_\alpha \eta_{\gamma\gamma}) \} \tag{68} \\
& -14\{+k_\gamma k_\alpha (k_\gamma \eta_{\alpha\gamma} + k_\gamma \eta_{\alpha\gamma}) + k_\gamma k_\gamma k_\gamma \eta_{\alpha\alpha} - 3\eta_{\alpha\gamma} k_\alpha k_\gamma k_\gamma \} \\
& +14\{+H_{\alpha\gamma}(k_\gamma \eta_{\alpha\gamma} + k_\gamma \eta_{\alpha\gamma} + k_\alpha \eta_{\gamma\gamma}) \},
\end{aligned}$$

which has been used in the main text.

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