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On the Socles of Commutator Invariant Submodules of $QTAG$-Modules

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Abstract. In [9, 8, 10], respectively, socles have been studied with the aid of fully invariant, characteristic and projection invariant submodules of $QTAG$-modules. Here we focus our attention on the socles of commutator invariant submodules and introduce a new class of modules, which we term commutator socle-regular $QTAG$-modules. After establishing some crucial properties of commutator socle-regularity, we show that the addition of separable summand to a module does not influence commutator socle-regularity.

Key Words: $QTAG$-modules, commutator invariant submodules, socles, commutator socle-regular $QTAG$-modules
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Introduction and background material

Following [11], a unital module M_R is called a $QTAG$-module if it satisfies the following condition: Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

Through a number of papers it has been seen that the structure theory of these modules is similar to that of torsion abelian groups and that these modules occur over any ring. Here the rings are almost restriction-free and the $QTAG$-modules satisfy a simple condition. Several authors have worked extensively on these modules. Many interesting results have surfaced, but there is yet much to explore.

All rings examined in the current paper contain unity ($1 \neq 0$) and modules are unital $QTAG$-modules. A module in which the lattice of its submodules is totally ordered is called a serial module; in addition, if it has finite composition length, it is called a uniserial module. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module, and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and

$H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. M is called separable if $M^1 = 0$.

A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. A submodule $B \subseteq M$ is a basic submodule [6] of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible. A fully invariant submodule $L \subseteq M$ is large [6], if $L + B = M$, for every basic submodule B in M. A submodule $N \subset M$ is nice [4] in M, if $H_\sigma(M/N) = (H_\sigma(M) + N)/N$ for all ordinals σ, i.e. every coset of M modulo N may be represented by an element of the same height.

A family $\mathcal{N}$ of nice submodules of M is called a *nice system* in M if

(i) $0 \in \mathcal{N}$;

(ii) if $\{N_i\}_{i\in I}$ is any subset of $\mathcal{N}$, then $\sum_{i\in I} N_i \in \mathcal{N}$;

(iii) given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated [5].

An h-reduced $QTAG$-module M is called totally projective if it has a nice system.

Mehran et al. [7] proved that almost all the results which hold for TAG-modules also hold good for $QTAG$-modules. Our notations and terminology are standard and follow essentially those from [1, 2]. As usual, End(M) denotes the endomorphism ring of a module M.

1 The class of commutator socle-regular $QTAG$-modules

The classes of transitive and fully transitive $QTAG$-modules were generalized in [8, 9] by focusing on the possible socles of characteristic and fully invariant submodules. In [10] full invariance was replaced by projection invariance and the current work continues this theme by replacing full invariance with commutator invariance. We begin with the following useful concept.

Definition 1 *A submodule N of a $QTAG$-module M is said to be commutator invariant if $\pi(N) \subseteq N$ for every $\pi \in End(M)$ that is of the form*

$\pi = [\phi, \psi] = \phi\psi - \psi\phi$, *where* $\phi, \psi \in End(M)$.

Clearly each fully invariant submodule is commutator invariant, whereas the converse fails. Nevertheless, in some concrete situations, commutator invariant submodules are fully invariant. Specifically, the following result holds:

Proposition 1 *Suppose* M *is a QTAG-module such that* $M = \bigoplus_{i\in I} M'$ *for some module* M'*, where* $|I| > 1$*. Then any commutator invariant submodule of* M *is fully invariant.*

Proof. Let N be an arbitrary commutator invariant submodule of M. If $|I|$ is infinite, then every element of $\mathrm{End}(M)$ is a sum of commutators, and so if N is commutator invariant, it is then certainly fully invariant.

Suppose that $M = \bigoplus_{i=1}^{n} M_i'$, $n > 1$, where each $M_i' \cong M'$, say. Let $A_{ij}(s)$ be the $n \times n$ matrix over the ring $R = \mathrm{End}(M')$ with ij^{th}-entry equal to s and all other entries zero. Recall that an arbitrary endomorphism of M can be represented as an $n \times n$ matrix Δ over R,

$$\Delta = \begin{pmatrix} a_{11} \cdots a_{1n} \\ \ddots \\ a_{n1} \cdots a_{nn} \end{pmatrix}.$$

Now $A_{ij}(a_{ij})A_{jj}(1) = A_{ij}(a_{ij})$ while $A_{jj}(1)A_{ij}(a_{ij}) = 0$ provided $i \neq j$. So, for $i \neq j$, $A_{ij}(a_{ij})$ is a commutator. Hence $\Delta = \mathrm{diag}\{a_{11}, \ldots, a_{nn}\} + \Delta'$, where Δ' is a sum of commutators. Thus, to establish that N is fully invariant, it suffices to show that N is invariant under the diagonal matrix $\mathrm{diag}\{a_{11}, \ldots, a_{nn}\}$; in fact, it follows easily that it will suffice to show that N is invariant under the diagonal matrix $\mathrm{diag}\{a, 0, \ldots, 0\}$, where $a = a_{11}$.

Now $A_{n1}(a)$ is a commutator, so if $(b_1, \ldots, b_n)^t \in N$ $-$ we are writing elements of M as column vectors and using $(\cdot)^t$ to denote transposes $-$ then it follows that the matrix product $A_{n1}(a) \cdot (b_1, \ldots, b_n)^t = (0, \ldots, 0, ab_1)^t$ is also an element of N. However, the matrix obtained by interchanging the first and last columns of the identity matrix and 0 elsewhere is also a commutator:

$$A_{1n}(1) + A_{n1}(1) = [(A_{1n}(1) + A_{n1}(-1)), A_{nn}(1)].$$

It follows immediately that

$$\mathrm{diag}\{a, 0, \ldots, 0\} \cdot (b_1, \ldots, b_n)^t = (ab_1, 0, \ldots, 0)^t \in N$$

and so N has the required invariance property. $\square$

Motivated by similar definitions used in [9, 8, 10], we define the following:

Definition 2 *A $QTAG$-module M is said to be commutator socle-regular if, for each commutator invariant submodule N of M, there exists an ordinal σ (depending on N) such that $Soc(N) = Soc(H_\sigma(M))$.*

These notions have a great degree of similarity since they may be defined in a common way as follows: A $QTAG$-module M is said to be $*$-socle-regular if every $*$-submodule N of M has the property that $Soc(N) = Soc(H_\sigma(M))$ for some ordinal σ.

When $*$-submodule corresponds to fully invariant (characteristic) submodule, we get the notions that were called socle-regular (strongly socle-regular) modules in [9, 8]; when $*$-submodule corresponds to projection invariant (commutator invariant) submodule, we get the notion of projectively socle-regular (commutator socle-regular) modules introduced in [10] and the present work, respectively.

It is easy to see that the class of socle-regular $QTAG$-modules contains each of the other three classes. It is found in [8, 10] that the strongly socle-regular and projectively socle-regular classes are properly contained in the class of socle-regular $QTAG$-modules. It was also established in [9] that fully transitive $QTAG$-modules are socle-regular, while in [8] that transitive $QTAG$-modules are strongly socle-regular.

Now we investigate some of the fundamental properties of commutator socle-regular $QTAG$-modules. Our first observation is that the property of a $QTAG$-module M being commutator socle-regular is inherited by certain submodules.

Proposition 2 *If M is a commutator socle-regular $QTAG$-module, then so is $H_\beta(M)$ for all ordinals β.*

Proof. Let N be a commutator invariant submodule of $H_\beta(M)$. Since the latter is fully invariant in M, it follows that N is commutator invariant in M. Consequently, there is an ordinal α such that $Soc(N) = Soc(H_\alpha(M))$. Intersecting both sides of the last equality with $H_\beta(M)$, we obtain that $Soc(N) = Soc(H_\gamma(M))$ where $\gamma = \max(\alpha, \beta)$. But we have $\gamma = \beta + \delta$ for some $\delta \geq 0$, so that we can write $Soc(N) = Soc(H_\delta(H_\beta(M)))$, as desired. □

The next result allows us to restrict our attention hereafter to h-reduced $QTAG$-modules.

Theorem 1 *Let D be an h-divisible $QTAG$-module and T an h-reduced $QTAG$-module. If $D \oplus T$ is commutator socle-regular, then both D and T are commutator socle-regular. Moreover, if T is commutator socle-regular, then $D \oplus T$ is also commutator socle-regular.*

Proof. If N is a commutator invariant submodule of D, then it follows from Proposition 1 that N is fully invariant in D. Then N has the form $N = D$ or $N = Soc^n(D)$ for some non-negative integer n. Hence, in both situations, we have $Soc(N) = Soc(Soc^n(D)) = Soc(D)$, as desired.

Now suppose that L is an arbitrary commutator invariant submodule of T. We claim that $D \oplus L$ is then a commutator invariant submodule of $D \oplus T$. Assuming we have established this, it follows that

$$\begin{aligned} Soc(D \oplus L) = Soc(D) \oplus Soc(L) &= Soc(H_\alpha(D \oplus T)) \\ &= Soc(H_\alpha(D)) \oplus Soc(H_\alpha(T)) \end{aligned}$$

for some ordinals α. Thus it readily follows that $Soc(L) = Soc(H_\alpha(T))$. Hence it remains only to establish the claim.

Since endomorphisms of $D \oplus T$ have matrix representations as upper triangular matrices, an easy calculation shows that any commutator homomorphism in $\mathrm{End}(D \oplus T)$ must have the form

$$\Delta = \begin{pmatrix} [\alpha, \alpha'] & \gamma \\ 0 & [\beta, \beta'] \end{pmatrix}.$$

for endomorphisms α, α' of D, β, β' of T and a homomorphism $\gamma : T \to D$. Since L is commutator invariant in T, it follows easily that $\Delta(D \oplus L) \subseteq D \oplus L$, as desired.

Conversely, suppose that K is an arbitrary commutator invariant submodule of $D \oplus T$, then K has one of the forms $K = D \oplus L$ or $K = Soc^t(D) \oplus L$ for some t, where in both cases L is a commutator invariant submodule of T. In the first case,

$$\begin{aligned} Soc(K) &= Soc(D) \oplus Soc(L), \\ &= Soc(D) \oplus Soc(H_\sigma(T)), \\ &= Soc(D \oplus H_\sigma(T)), \\ &= Soc(H_\sigma(D) \oplus H_\sigma(T)) \\ &= Soc(H_\sigma(D \oplus T)), \end{aligned}$$

as required. For the second case we have

$$\begin{aligned} Soc(K) &= Soc(Soc^t(D)) \oplus Soc(L), \\ &= Soc(D) \oplus Soc(L), \\ &= Soc(H_\sigma(D \oplus T)), \end{aligned}$$

as desired. $\square$

Let us recall the terminology used in [9]: For a submodule N of M, put $\sigma = \min\{H_M(x| \; x \in Soc(N)\}$ and denote $\sigma = \inf(Soc(N))$. Here $Soc(N) \subseteq Soc(H_\sigma(M))$.

Our next result illustrates some elementary but useful properties of the function inf.

Proposition 3 *If N is a commutator invariant submodule of the QTAG-module M and inf$(Soc(N)) = n$, a natural number, then*

$$Soc(N) = Soc(H_n(M)).$$

Proof. Suppose that N is a commutator invariant submodule of M and $\inf(Soc(N)) = n$, a finite integer. Therefore, there is an element $x \in Soc(N)$ such that $H_M(x) = n$ and so $d\left(\dfrac{yR}{xR}\right) = n$ for some $y \in M$. Since every element of exponent one and finite height can be embedded in a direct summand, by [3] yR is a summand of M containing x. Therefore $M = yR \oplus M'$ for some submodule M' of M. If z is an arbitrary element of $Soc(H_n(M)) \setminus Soc(H_{n+1}(M))$, then there exists $w \in H^{n+1}(M)$ such that $d\left(\dfrac{wR}{zR}\right) = n$, and hence $M = wR \oplus M''$ for some M'' of M. Now $d(wR) = d(yR) = n + 1$, implying that $wR \cong yR$. Then there is a commutator endomorphism ϕ of M such that $\phi(y) = w$ or $\phi(y) = w - uy$. Thus we have $\phi(x) = vz$ or $\phi(x) = z - ux$ for some u. Since $x \in N$ and N is commutator invariant in M, either $z \in N$ or $z - ux \in N$; in either case we can conclude that $z \in N$.

If now s is an arbitrary element of $Soc(H_{n+1}(M))$, then

$$z + s \in Soc(H_n(M)) \setminus Soc(H_{n+1}(M))$$

and so $z+s \in N$, whence $s \in N$. Hence $Soc(H_n(M)) \subseteq N$. As $\inf(Soc(N)) = n$, we certainly have $Soc(N) \subseteq Soc(H_n(M))$ and so we obtain the desired equality

$$Soc(N) = Soc(H_n(M)).$$

□

Our next result shows that commutator socle-regularity is inherited by large submodules.

Proposition 4 *Let M be an h-reduced commutator socle-regular QTAG-module and L a large submodule of M. Then L is also commutator socle-regular.*

Proof. Let N be a commutator invariant submodule of a large submodule L of M. If we suppose that $\inf(Soc(N))$ is finite, n, then it follows from Proposition 3 that $Soc(N) = Soc(H_n(L))$. If $\inf(Soc(N))$ is infinite, then so is $\inf(Soc(L))$, and thus $Soc(N) \subseteq Soc(H_\alpha(M))$ for some infinite ordinal α. Since N is commutator invariant in M as well, $Soc(N) = Soc(H_\beta(M))$ for some ordinal β, and immediately $\alpha \leq \beta$ is infinite. It follows that $H_\beta(M) = H_\beta(L)$, whence $Soc(N) = Soc(H_\beta(L))$. Thus L is commutator socle-regular, as claimed. □

Remark 1 *For any fully invariant submodule F of M, $H_\omega(F) = H_\omega(M)$, therefore fully invariant submodules are commutator socle-regular.*

Our next proposition is somewhat technical but will enable us to deduce some interesting consequences.

Proposition 5 *Let N is a submodule of the $QTAG$-module M such that $H_\omega(M) = N$ and for each $\phi \in End(N)$ there is an endomorphism $\phi' \in End(M)$ with $\phi'|N = \phi$, then M is commutator socle-regular if, and only if, N is commutator socle-regular.*

Proof. The necessity follows from Proposition 2 above.

Conversely, suppose that N is commutator socle-regular and let K be an arbitrary commutator invariant submodule of M. If $\inf(Soc(K))$ is finite, then it follows from Proposition 3 that $Soc(K) = Soc(H_n(M))$ for some finite n. If $\inf(Soc(K))$ is infinite, then $Soc(K) \subseteq N$. We claim that $Soc(K)$ is actually a commutator invariant submodule of N. Assuming this for the moment, we conclude, as N is commutator socle-regular, that $Soc(K) = Soc(H_\beta(N))$ for some ordinal β, and hence

$$Soc(K) = Soc(H_\beta(H_\omega(M))) = Soc(H_{\omega+\beta}(M)),$$

as required.

It remains then to establish the claim. If $\pi = \phi\psi - \psi\phi$ is any commutator in $\mathrm{End}(N)$, then $\pi' = \phi'\psi' - \psi'\phi'$ is commutator in $\mathrm{End}(M)$. But if $x \in N$, then

$$(\phi'\psi')(x) = \phi'(\psi(x))$$

since $\psi'|N = \psi$; note that $y = \psi(x) \in N$ because $\psi \in \mathrm{End}(N)$. Thus

$$(\phi'\psi')(x) = \phi'(y) = \phi(y) = \phi(\psi(x)) = (\phi\psi)(x)$$

and we have $(\phi'\psi')|N = \phi\psi$; similarly $(\psi'\phi')|N = \psi\phi$. In particular, if $x \in Soc(K)$, then $\pi(x) = \pi'(x) \in Soc(K)$ since K is a commutator invariant submodule of M which in turn makes $Soc(K)$ commutator invariant in M. Since π was an arbitrary commutator in $\mathrm{End}(N)$, we conclude that $Soc(K)$ is a commutator invariant submodule of N, as claimed. □

In the proof of our next theorem we shall need an easy extension of a well-known result on extending automorphisms from the submodule $H_n(M)$, n an integer, to automorphisms of the module M.

Lemma 1 *Let M be a QTAG-module and ϕ an arbitrary endomorphism of the submodule $H_n(M)$ of M, for some finite n, then ϕ extends to an endomorphism ϕ' of M.*

Proof. Consider the module $M' = M \oplus M$ and note that $H_n(M') = H_n(M) \oplus H_n(M)$. Regard endomorphisms of M' as 2×2 matrices over $\mathrm{End}(M)$ and endomorphisms of $H_n(M')$ as 2×2 matrices over $\mathrm{End}(H_n(M))$. Let $\phi \in \mathrm{End}(H_n(M))$ be arbitrary. Then

$$\Delta = \begin{pmatrix} \phi & 1_{H_n(M)} \\ 1_{H_n(M)} & 0 \end{pmatrix}$$

is an endomorphism of $H_n(M)$ which is easily seen to actually be an automorphism. Thus, Δ extends to an automorphism

$$\Delta' = \begin{pmatrix} \eta & \theta \\ \lambda & \mu \end{pmatrix}$$

of M', where $\eta, \theta, \lambda, \mu \in \mathrm{End}(M)$. Thus $\Delta \begin{pmatrix} x \\ 0 \end{pmatrix} = \Delta' \begin{pmatrix} x \\ 0 \end{pmatrix}$ for all $x \in H_n(M)$, i.e.,

$$\begin{pmatrix} \phi(x) \\ x \end{pmatrix} = \begin{pmatrix} \eta(x) \\ \lambda(x) \end{pmatrix}.$$

Set $\phi' = \eta$, an endomorphism of M, and note that $\phi'|H_n(M) = \eta|H_n(M)$, as desired. $\square$

Our next result demonstrates that the class of commutator socle-regular $QTAG$-modules is quite large.

Theorem 2 *The following statements hold.*

(i) If M is a QTAG-module such that either $H_\omega(M) = 0$ or $d(H_\omega(M)) = n$ for some finite n, then M is commutator socle-regular.

(ii) A QTAG-module M is commutator socle-regular if, and only if, $H_n(M)$ is commutator socle-regular for some n.

(iii) If M is a QTAG-module such that $M/H_\sigma(M)$ is totally projective for some ordinal $\sigma < \omega^2$, then M is commutator socle-regular if, and only if, $H_\sigma(M)$ is commutator socle-regular.

(iv) Totally projective modules of length $< \omega^2$ are commutator socle-regular.

Proof. Statement (i) follows immediately from Proposition 5 and the observation that in both cases the endomorphisms of $H_\omega(M)$ are scalars and hence give rise in a natural way to the desired homomorphism.

The necessity in statement (ii) follows directly from Proposition 2. The proof of sufficiency is similar to the proof of Proposition 5; let N be a commutator invariant submodule of M, and if we suppose that $\inf(Soc(N))$ is finite, k, then with the aid of Proposition 3 we may write $Soc(N) = Soc(H_k(M))$, as desired. Otherwise, if we have $\inf(Soc(N)) \geq \omega$, then clearly $Soc(N) \subseteq H_\omega(M) \subseteq H_n(M)$. We assert that $Soc(N)$ is a commutator invariant submodule of $H_n(M)$. This follows as Proposition 5 uses Lemma 1 to deduce that endomorphisms of $H_n(M)$ extend to endomorphisms of M. Since $H_n(M)$ is commutator socle-regular, we have $Soc(N) = Soc(H_\sigma(H_n(M)))$ for some ordinal σ. Consequently, $Soc(N) = Soc(H_{n+\sigma}(M))$ and M is commutator socle-regular, as required.

We will establish (iii) by first considering the case $\sigma = \omega$. In this special case the proof follows from Proposition 5 and the observation that as $M/H_\omega(M)$ is totally projective, it follows that every endomorphism of $H_\omega(M)$ extends to an endomorphism of M, thereby giving the extension property required to apply Proposition 5.

Suppose now the ordinal σ has the form $\sigma = \omega \cdot k$ for some $1 < k < \omega$. Since the *QTAG*-module $H_\sigma(M) = H_{\omega\cdot k}(M) = H_\omega(H_{\omega\cdot(k-1)}(M))$ is commutator socle-regular and the quotient $M/H_\sigma(M) = M/H_{\omega\cdot k}(M)$ is totally projective, whence so is the quotient

$$H_{\omega\cdot(k-1)}(M/H_\sigma(M)) = H_{\omega\cdot(k-1)}(M)/H_{\omega\cdot k}(M),$$

we apply the preceding case $\sigma = \omega$ for $K = H_{\omega\cdot(k-1)}(M)$ to derive that $H_{\omega\cdot(k-1)}(M)$ is commutator socle-regular. Moreover, as $M/H_\sigma(M)$ is totally projective so is $M/H_\tau(M)$ for any $\tau < \sigma$. Thus, after $k-1$ steps, we deduce that $H_\omega(M)$ is commutator socle-regular and $M/H_\omega(M)$ is a direct sum of uniserial modules. Again by what we have shown in the previous paragraph, M will be commutator socle-regular, finishing this case.

Finally, consider the case where $\sigma = \omega \cdot k + n$ with $k, n < \omega$. Since the *QTAG*-module $H_\sigma(M) = H_{\omega\cdot k+n}(M) = H_n(H_{\omega\cdot k}(M))$ is commutator socle-regular, we can conclude from (ii) above that the same holds for $H_{\omega\cdot k}(M)$. As already observed, if $M/H_\sigma(M)$ is totally projective, then so is $M/H_{\omega.k}(M)$. We therefore may employ the previous step to conclude that M is commutator socle-regular, indeed.

Part (iv) follows immediately from (iii) by choosing σ to be the length of M. $\square$

Nevertheless, in certain specific cases, the following direct summand property holds:

Theorem 3 *Suppose that $M = P \oplus Q$ and Q is separable. Then M is commutator socle-regular if, and only if, P is commutator socle-regular.*

Proof. Suppose that P is commutator socle-regular and L is a commutator invariant submodule of M. If $\inf(Soc(L))$ is finite, then by Proposition 3, we have $Soc(L) = Soc(H_n(M))$ for some finite n. So, supposing $\inf(Soc(L))$ is infinite, then $Soc(L) \subseteq Soc(H_\omega(M)) = Soc(H_\omega(P))$, as Q is separable. However, L is a commutator invariant submodule of M and so $Soc(L)$ is a commutator invariant submodule of M which is actually contained in P. Since endomorphisms of P extend trivially to endomorphisms of M, it is easy to see that $Soc(L)$ is actually a commutator invariant submodule of M, and so $Soc(L) = Soc(H_\sigma(P))$ for some ordinal σ. Thus $Soc(H_\sigma(P)) \subseteq Soc(H_\omega(P))$ and so $\sigma \geq \omega$. It follows immediately that $Soc(L) = Soc(H_\sigma(P)) = Soc(H_\sigma(M))$, since $H_\sigma(Q) = 0$.

Conversely, suppose that M is commutator socle-regular and let N be an arbitrary commutator invariant submodule of P. As before, if $\inf(Soc(N))$ is finite, then Proposition 3 assures that $Soc(N) = Soc(H_n(P))$ for some positive integer n. Suppose then that $\inf(Soc(N))$ is infinite, so that $Soc(N) \subseteq Soc(H_\omega(P)) = Soc(H_\omega(M))$. We claim that $Soc(N)$ is a commutator invariant submodule of M. Assuming for the moment that we have established this claim, it follows that $Soc(N) = Soc(H_\sigma(M))$ for some ordinal σ. Hence $Soc(N) = Soc(H_\sigma(M)) \subseteq Soc(H_\omega(M))$, yielding $\sigma \geq \omega$. Since we have $H_\sigma(M) = H_\sigma(P)$ for $\sigma \geq \omega$, we get the required result that $Soc(N) = Soc(H_\sigma(P))$ for some σ. It remains then only to establish the claim.

Observe firstly that if $\phi = \begin{pmatrix} \eta & \theta \\ \lambda & \mu \end{pmatrix}$ and $\psi = \begin{pmatrix} \eta' & \theta' \\ \lambda' & \mu' \end{pmatrix}$ are arbitrary endomorphisms of M (in the standard matrix representation), then the commutator $[\phi, \psi]$ can be represented as a matrix

$$\Delta = \begin{pmatrix} [\eta, \eta'] & \pi \\ \pi' & [\mu, \mu'] \end{pmatrix}$$

where $\pi : Q \to P$, $\pi' : P \to Q$ are homomorphisms. Note, however, that as Q is separable and $Soc(N) \subseteq Soc(H_\omega(P))$, the image under π' of each element of $Soc(N)$ is necessarily zero. Identifying $Soc(N)$ with $Soc(N) \oplus 0$, a straightforward calculation shows that $\Delta(Soc(N)) = [\eta, \eta'](Soc(N)$ and this is clearly contained in $Soc(N)$ since N is, by assumption, a commutator invariant submodule of P. $\square$

We end the paper with a question as follows:

Question. Does there exist a commutator socle-regular $QTAG$-module of length $\geq \omega^2$; in particular, is the restriction on the ordinal σ in the Theorem 2.2(iii) necessary?

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