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TOEPLITZ OPERATORS ON WEIGHTED BESOV SPACES OF HOLOMORPHIC FUNCTIONS ON THE POLYDISK

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Abstract. We characterize symbols h for which the corresponding Toeplitz operators are bounded on weighted Besov spaces $B_p(\omega)$ of holomorphic functions on the polydisc for some $1 \le p < \infty$[1].

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1. Introduction and auxiliary constructions

Let $U^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| < 1, 1 \le j \le n\}$ be the unit polydisk in the n-dimensional complex plane $\mathbb{C}^n$ and let $T^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| = 1, 1 \le j \le n\}$ be its torus. We denote by $H(U^n)$ the set of holomorphic functions on U^n, by $L^\infty(U^n)$ the set of bounded measurable functions on U^n and by $H^\infty(U^n)$ the subspace of $L^\infty(U^n)$ consisting of holomorphic functions.

Let S be the class of all non-negative measurable functions ω on $(0,1)$ for which there exist positive numbers M_ω, q_ω, m_ω ($m_\omega, q_\omega \in (0,1)$) such that

$$m_\omega \le \frac{\omega(\lambda r)}{\omega(r)} \le M_\omega$$

for all $r \in (0,1)$ and $\lambda \in [q_\omega, 1]$. Some properties of functions of the class S can be found in [12]. We put

$$\alpha_\omega = \frac{\log m_\omega}{\log q_\omega^{-1}} \quad \text{and} \quad \beta_\omega = \frac{\log M_\omega}{\log q_\omega^{-1}}.$$

For example, $\omega \in S$ if $\omega(t) = t^\alpha$ with $-1 < \alpha < \infty$. Using the results of [12] one can prove that

$$\omega(t) = \exp\left\{\eta(t) + \int_t^1 \frac{\zeta(u)}{u} du\right\},$$

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where $\eta(u)$ and $\zeta(u)$ are bounded measurable functions and $-\alpha_\omega < \zeta(u) < \beta_\omega$. Without loss of generality we assume that $\eta(t) = 0$. Then we have

$$t^{\alpha_\omega} \le \omega(t) \le t^{-\beta_\omega}, \ 0 < t < 1.$$

Below, for convenience of notations, for $\xi = (\xi_1, \dots, \xi_n)$ and $z = (z_1, \dots, z_n)$ we put

$$\omega(1-|z|) = \prod_{j=1}^n \omega_j(1-|z_j|), \quad 1-|z| = \prod_{j=1}^n (1-|z_j|), \quad 1-\bar{\xi} z = \prod_{j=1}^n (1-\bar{\xi}_j z_j).$$

Furthermore, for $m = (m_1, \dots, m_n)$ we put $m+1 = \prod_{j=1}^n (m_j+1)$.

The notation $|f| \asymp |g|$ will mean that $C_1|f(z)| \le |g(z)| \le C_2|f(z)|$ for some positive constants C_1, C_2 that are independent of z.

Throughout the paper the capital letters $C(...)$ and C_k will stand for different positive constants depending only on the indicated parameters.

Let $\omega_j \in S$, $1 \le j \le n$. It is not difficult to show that

$$\omega(1-|z'|) \asymp \omega(1-|z''|), \ if \ |z'| \asymp |z''|, \ z', z'' \in U^n. \tag{1.1}$$

Definition 1.1. Let $1 \le p < \infty$. We denote by $L_p(\omega)$ the set of all measurable functions on U^n for which

$$||f||^p_{L_p(\omega)} := \int_{U^n} |f(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^2} dm_{2n}(z) < \infty,$$

where $dm_{2n}(z)$ is the $2n$−dimensional Lebesgue measure on U^n.

Now define the notion of the fractional derivative.

Definition 1.2. (1) For a holomorphic function $f(z) = \sum_{(k)=(0)}^{(\infty)} a_k z^k$, $z \in U^n$ and for $\beta = (\beta_1, \dots, \beta_n)$, $\beta_j > -1$, $1 \le j \le n$, we define the fractional derivative D^β as follows:

$$D^\beta f(z) = \sum_{(k)=(0)}^{(\infty)} \prod_{j=1}^n \frac{\Gamma(\beta_j+1+k_j)}{\Gamma(\beta_j+1)\Gamma(k_j+1)} a_k z^k, \quad k = (k_1, \dots, k_n), \ z \in U^n,$$

where $\Gamma(\cdot)$ is the Gamma function and $\sum_{(k)=(0)}^{(\infty)} = \sum_{k_1=0}^\infty \cdots \sum_{k_n=0}^\infty$.

(2) The inverse operator $D^{-\beta}$ is defined to satisfy $D^{-\beta} D^\beta f(z) = f(z)$ for $z \in U^n$.

We put $Df(z) = D^\beta f(z)$ if $\beta = (1, \dots, 1)$. It is not difficult to show, that

$$f(z) = \int_0^1 Df(rz) dr. \tag{1.2}$$

Next, we define the holomorphic Besov spaces on the polydisk (see [8]).

Definition 1.3. Let $1 \le p < \infty$. The holomorphic Besov space $B_p(\omega)$ is defined to be the class of functions $f \in H(U^n)$ satisfying

$$||f||^p_{B_p(\omega)} = \int_{U^n} |Df(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} dm_{2n}(z) < \infty.$$

Numerous authors have contributed to holomorphic Besov spaces in the unit disk of C and in the unit ball of C^n (see, e.g., [1], [2], [4], [9], [13]). The study of holomorphic Besov space on the polydisk thus is of special interest. In Theorem 2.1 below we show that $B_p(\omega)$ is a Banach space with respect to the norm $||\cdot||_{B_p(\omega)}$. We first define the Toeplitz operator on the spaces $H(U^n)$. Let $L^1(T^n)$ be the class of all integrable functions on T^n.

Definition 1.4. A Toeplitz operator with symbol $h \in L^1(T^n)$ is defined by formula:

$$T_h(f)(z) := \frac{1}{(2\pi i)^n} \int_{T^n} \frac{f(\xi)h(\xi)}{\xi - z} d\xi =$$

$$= \frac{1}{(2\pi i)^n} \int_{T^n} \frac{f(\xi_1, \dots, \xi_n)h(\xi_1, \dots, \xi_n)}{(\xi_1 - z_1)\dots(\xi_n - z_n)} d\xi_1 \dots d\xi_n, \quad f \in H(U^n).$$

Remark 1.1. *The above defined Toeplitz operator T_h can be extended to functions $f \in B_p(\omega)$ as follows. First, one can consider T_h on some everywhere dense subset of $B_p(\omega)$, for instance, on the set of all polynomials, where T_h is obviously well-defined. Then one can show that if the operator T_h is bounded with respect to the norm $||\cdot||_{B_p(\omega)}$ on the set of polynomials, then it has a unique extension to $B_p(\omega)$, again denoted by T_h.*

The present paper extends the result obtained in [6]. Our main aim is to describe the symbols h, for which T_h defines a bounded operator $B_p(\omega) \to B_p(\omega)$. To this end, in Section 2 we show that the set of polynomials is dense in $B_p(\omega)$. In Section 3, we give a description of a special class of bounded Toeplitz operators on $B_p(\omega)$ in the case where $1 \le p < \infty$.

2. Auxiliary results

In this section we show that $B_p(\omega)$ is a Banach space for $1 \le p < \infty$, and the set of polynomials is dense in $B_p(\omega)$. The proofs of our main results are based on the following lemmas.

Lemma 2.1. *The following statements hold.*

(a) If $f \in H(U^n)$, $\gamma_j \geq \beta_j + \alpha_{\omega_j} - 2$ $(1 \leq j \leq n)$ and $(1-|z|^2)^\beta D^\beta f \in L^1(\omega)$, then for some $C > 0$

$$|Df(z)| \leq C \int_{U^n} \frac{(1-|\xi|^2)^\gamma |D^\beta f(\xi)|}{|1-\bar{\xi}z|^{\gamma+3-\beta}} dm_{2n}(\xi). \tag{2.1}$$

(b) If $f \in B_1(\omega)$, $\gamma_j > \alpha_{\omega_j} - 1$, $(1 \leq j \leq n)$, $\gamma = (\gamma_1, \ldots, \gamma_n)$, then for some $C > 0$

$$|D^\beta f(z)| \leq C \int_{U^n} \frac{(1-|\xi|^2)^\gamma |Df(\xi)|}{|1-\bar{\xi}z|^{\gamma+1+\beta}} dm_{2n}(\xi). \tag{2.2}$$

For the proof we refer to [8], Lemma 1.

Lemma 2.2. *Let $f \in B_p(\omega)$. Then*

$$|Df(z)| \leq \frac{C||f||_{B_p(\omega)}}{\omega^{1/p}(1-|z|)(1-|z|)}.$$

Proof. The function $|Df(z)|^p$ is subharmonic in U^n, where $0 < p < \infty$. Therefore

$$|Df(z)|^p \leq \frac{1}{|\tilde{U}^n(z)|} \int_{\tilde{U}^n(z)} |Df(\xi)|^p dm_{2n}(\xi), \quad \xi \in U^n,$$

where $\tilde{U}^n(z) = \{\xi; |\xi_j - z_j| < (1-|z_j|)/2, 1 \leq j \leq n\}$. It is clear that $\tilde{U}^n(z) \subset U^n$ and $1 - |\xi_j| \asymp 1 - |z_j|$ $(1 \leq j \leq n)$, $|\tilde{U}^n(z)| = 2^{-n}\pi^n(1-|z|)^2$. Then using (1.1), we get

$$\begin{aligned} |Df(z)|^p(1-|z|^2)^p\omega(1-|z|) &\leq C \int_{\tilde{U}^n(z)} |Df(\xi)|^p(1-|\xi|^2)^{p-2}\omega(1-|\xi|) dm_{2n}(\xi) \\ &\leq C \int_{U^n} |Df(\xi)|^p \frac{\omega(1-|\xi|)}{(1-|\xi|)^{2-p}} dm_{2n}(\xi) = C||f||^p_{B_p(\omega)}, \end{aligned}$$

and the result follows. ☐

Lemma 2.3. *For any $K \subset\subset U^n$ and any number $s \in \{0, 1\}$ there exists a constant $C_s = C_s(s, p, \omega, K)$ such that*

$$\max_{z \in K} |D^s f(z)| \leq C_s ||f||_{B_p(\omega)} \quad \textit{for all } f \in B_p(\omega).$$

Proof. We have $\omega(t) \geq t^{\alpha_\omega}$. Then, using Lemma 2.2, we get

$$|Df(z)| \leq \frac{C||f||_{B_p(\omega)}}{(1-|z|)^{\alpha_\omega/p+1}} \leq \frac{C||f||_{B_p(\omega)}}{(1-|z|)^{[\alpha_\omega/p]+2}},$$

implying that $\max_{z\in K} |Df(z)| \leq C_1 ||f||_{B_p(\omega)}$.

Let $s = 0$. Using (1.2) we get

$$\begin{aligned} |f(z)| &\leq C \int_0^1 \frac{||f||_{B_p(\omega)} dz}{(1-r|z|)^{[\alpha_\omega/p]+2}} = \frac{C(p,\alpha_\omega)}{|z|} \left(\frac{1}{(1-|z|)^{[\alpha_\omega/p]+1}} - 1 \right) ||f||_{B_p(\omega)} \\ &= \frac{C(p,\alpha_\omega)}{|z|} \left(\frac{1-(1-|z|)^{[\alpha_\omega/p]+1}}{(1-|z|)^{[\alpha_\omega/p]+1}} \right) ||f||_{B_p(\omega)}. \end{aligned}$$

Setting

$$g(|z|) \equiv \frac{C(p,\alpha_\omega)}{|z|}(1-(1-|z|)^{[\alpha_\omega/p]+1}),$$

we get $|f(z)| \leq g(|z|)(1-|z|)^{-[\alpha_\omega/p]-1}||f||_{B_p(\omega)}$, implying that $\max_{z\in K}|f(z)| \leq C_0||f||_{B_p(\omega)}$. □

Lemma 2.4. *Let $n=1$, $\omega\in S$, $a+1-\beta_\omega>0$, $b>1$ and $b-a-2>\alpha_\omega$. Then*

$$\int_U \frac{(1-|\xi|^2)^a\omega(1-|\xi|)}{|1-\bar{\xi}z|^b}dm_2(\xi) \leq C\frac{\omega(1-|z|)}{(1-|z|^2)^{b-a-2}}.$$

For the proof see [5], Lemma 1.6.

Definition 2.1. Let $p\geq 1$, $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_j>-1$ $(1\leq j\leq n)$. We define the class $A^p(\alpha)$ to be the set of all functions $f\in H(U^n)$ satisfying

$$||f|^p_{A^p(\alpha)} := \int_{U^n}|f(z)|^p(1-|z|)^\alpha dm_{2n}(z)<\infty.$$

The integral representation formula for functions of the class $A^p(\alpha)$ that follows is a trivial consequence of the well-known Djrbashian representation formula of the one-dimensional case (for details we refer to [3, 11]):

$$f(z)=\frac{\alpha+1}{\pi^n}\int_{U^n}\frac{(1-|\xi|^2)^\alpha}{(1-\bar{\xi}z)^{\alpha+2}}f(\xi)dm_{2n}(\xi). \tag{2.3}$$

Note that the generalization of $A^p(\alpha)$ spaces in terms of $\omega-$weighted spaces was first studied by F.A.Shamoyan, who greatly contributed to the theory of weighted classes of functions on the polydisk (see, e.g., [11]).

Lemma 2.5. *Let $f\in B_p(\omega)$ and $\alpha_j>\alpha_{\omega_j}-2+p$ for all j. Then*

$$f(z)=C(\alpha,\pi)\int_{U^n}\frac{(1-|\xi|^2)^\alpha Df(\xi)P(\bar{\xi},z)}{(1-\bar{\xi}z)^{\alpha+1}}dm_{2n}(\xi),$$

where

$$P(\bar{\xi},z)=(1-(1-\bar{\xi}z)^{\alpha+1})/z, \qquad \alpha=(\alpha_1,...,\alpha_n),\ \alpha_i\in\mathbb{N}.$$

Proof. Note that $Df\in A^1(\alpha)$ if $\alpha_j>\alpha_{\omega_j}-2+p$ $(1\leq j\leq n)$. Then using (2.3), we get

$$Df(z)=\frac{\alpha+1}{\pi^n}\int_{U^n}\frac{(1-|\xi|^2)^\alpha}{(1-\bar{\xi}z)^{\alpha+2}}Df(\xi)dm_{2n}(\xi).$$

On the other hand by (1.2) we have

$$\begin{aligned} f(z) &= C(\alpha,\pi)\int_{U^n}(1-|\xi|^2)^\alpha Df(\xi)\int_0^1\frac{dz}{(1-r\bar{\xi}z)^{\alpha+2}}dm_{2n}(\xi)= \\ &= C(\alpha,\pi)\int_{U^n}\frac{(1-|\xi|^2)^\alpha Df(\xi)}{(1-\bar{\xi}z)^{\alpha+1}z}(1-(1-\bar{\xi}z)^{\alpha+1})dm_{2n}(\xi). \end{aligned}$$

Setting

$$P(\bar{\xi}, z) = \frac{1-(1-\bar{\xi}z)^{\alpha+1}}{z}, \qquad \alpha = (\alpha_1, ..., \alpha_n),\ \alpha_i \in \mathbb{N},$$

we get the desired result. □

Theorem 2.1. *$B_p(\omega)$ is a Banach space with respect to the norm $||\cdot||_{B_p(\omega)}$ for any $1 \le p < \infty$.*

Proof. We use the standard arguments. First, it is not difficult to show that $||\cdot||_{B_p(\omega)}$ is a norm in $B_p(\omega)$ for any $1 \le p < \infty$. So, it remains to prove that $B_p(\omega)$ is closed. Let $\{f_n\} \subset B_p(\omega)$ be a Cauchy sequence. It is clear that $\{Df_n\}$ is a Cauchy sequence in $L^p(\tilde{\omega})$, where $\tilde{\omega}(t) = \omega(t)t^p$. We assume that $g \in L^p(\tilde{\omega})$ is the limit function of $\{Df_n\}$ in $L^p(\tilde{\omega})$, and show that $g(z) = Df(z)$ for some $f \in B_p(\omega)$. Let $K \subset\subset U^n$. Using Lemma 2.3, we conclude that there exists a constant $C = C(p, \omega, K)$ such that

$$\max_{z\in K} |Df_n(z) - Df_m(z)| \le C||f_n - f_m||_{B_p(\omega)}.$$

Therefore $\{Df_n(z)\}_{n=1}^{\infty}$ is a uniformly convergent sequence of holomorphic functions in K which converges to some h on K.

Next, we show that $h(z) = g(z)$, $z \in K$. To this end observe that

$$\left(\int_K |g(z)-h(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} dm_{2n}(z)\right)^{1/p} \le$$

$$\le \left(\int_K |Df_n(z)-g(z)|^p \frac{\omega(1-|z|)dm_{2n}(z)}{(1-|z|^2)^{2-p}}\right)^{1/p} +$$

$$+ \left(\int_K |Df_n(z)-h(z)|^p \frac{\omega(1-|z|)dm_{2n}(z)}{(1-|z|^2)^{2-p}}\right)^{1/p}. \tag{2.4}$$

Taking into account that Df_n converges uniformly to $h(z)$ on K, for the second integral on the right-hand side of (2.4) we obtain

$$\int_K |Df_n(z)-h(z)|^p \frac{\omega(1-|z|)dm_{2n}(z)}{(1-|z|^2)^{2-p}} \to 0 \ \text{ as } \ n \to \infty.$$

For the first integral on the right-hand side of (2.4) we have

$$\int_K |Df_n(z)-g(z)|^p \frac{\omega(1-|z|)dm_{2n}(z)}{(1-|z|^2)^{2-p}}$$

$$\le \int_{U^n} |Df_n(z)-g(z)|^p \frac{\tilde{\omega}(1-|z|)dm_{2n}(z)}{(1-|z|^2)^2} = ||Df_n - g||_{L^p(\tilde{\omega})} \to 0,$$

as $n \to \infty$. Therefore

$$\int_K |g(z)-h(z)|^p \frac{\omega(1-|z|)dm_{2n}(z)}{(1-|z|^2)^{2-p}} = 0.$$

In particular, for any $\delta \in (0,1)$ we have

$$I_\delta = \int_{\delta U^n} |g(z) - h(z)|^p \frac{\omega(1-|z|)dm_{2n}(z)}{(1-|z|^2)^{2-p}} = 0.$$

Hence

$$\lim_{\delta\to 1} I_\delta = \int_K |g(z) - h(z)|^p \frac{\omega(1-|z|)dm_{2n}(z)}{(1-|z|^2)^{2-p}} = 0,$$

implying that $g(z) = h(z)$ in U^n. Taking $f(z) = D^{-1}g(z)$ we conclude the proof of the theorem. Theorem 2.1 is proved. □

Theorem 2.2. *Let $f \in B_p(\omega)$ and $f_\tau(z) = f(\tau z)$. Then $||f_\tau - f||_{B_p(\omega)} \to 0$ as $\tau \to 1-0$.*

Proof. We have

$$||f_\tau - f||^p_{B_p(\omega)} = \int_{U^n} |Df_\tau(z) - Df(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} dm_{2n}(z)$$
$$\le \int_{\delta U^n} |Df_\tau(z) - Df(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} dm_{2n}(z)$$
$$+ \int_{U^n\setminus\delta U^n} |Df_\tau(z) - Df(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} dm_{2n}(z),\ 1 > \delta > 0.$$

Using the fact that $|Df(z)|^p$ is a subharmonic function, and hence

$$\int_{U^n\setminus\delta U^n} |Df_\tau(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} dm_{2n}(z) \le \int_{U^n\setminus\delta U^n} |Df(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} dm_{2n}(z),$$

we can write

$$||f_\tau - f||^p_{B_p(\omega)} \le C \int_{\delta U^n} |Df_\tau(z) - Df(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} dm_{2n}(z) +$$
$$+2^{p+1} \int_{U^n\setminus\delta U^n} |Df(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} dm_{2n}(z).$$

Letting $\delta, \tau \to 1-0$, we complete the proof of the theorem. □

Theorem 2.3. *The set of polynomials is dense in $B_p(\omega)$.*

Proof. Let $f \in B_p(\omega)$. Then by Theorem 2.2 we have $||f_\tau - f||_{B_p(\omega)} \to 0$ as $\tau \to 1-0$. Taking into account that the Taylor polynomials of the function $f_\tau(z)$ converge uniformly to $f_\tau(z)$, we complete the proof of theorem 2.3. □

To prove the main theorem we also will need the following lemma.

Lemma 2.6. *If $f \in B_p(\omega)$, then $z_j f \in B_p(\omega)$ $(1 \le j \le n)$.*

Proof. We assume that $j = 1$. Then

$$D(f(z)z_1) = z_1 Df(z) + \frac{\partial^{n-1}(f(z)z)}{\partial z_2 \ldots \partial z_n}.$$

Using Lemma 2.5 and the inequality $(1 - |\xi_1|^2) \leq 1$ we can write

$$\left|\frac{\partial^{n-1}(f(z)z)}{\partial z_2 \ldots \partial z_n}\right| = C(m,\pi)\left|\int_{U^n} \frac{(1-|z|^2)^m Df(\xi) P(\bar{\xi_1}, z_1) dm_{2n}(\xi)}{(1-\bar{\xi_1}z_1)^{m_1+1}\prod_{j=2}^n (1-\bar{\xi_j}z_j)^{m_j+2}}\right| \leq$$

$$\leq C(m,\pi)\int_{U^n} \frac{(1-|\xi|)^k |Df(\xi)|}{|1-\bar{\xi}z|^{k+2}} dm_{2n}(\xi),$$

where $P(\bar{\xi_1}, z_1) = (1 - (1-\bar{\xi_1}z_1)^{m_1+1})/z_1$, $m_j \in N$, $1 \leq j \leq n$, and $k_1 = m_1 - 1$, $k_j = m_j$, $2 \leq j \leq n$.

In the case $p > 1$ with some $\delta > 0$ we get

$$\left|\frac{\partial^{n-1}(f(z)z)}{\partial z_2 \ldots \partial z_n}\right|^p \leq \frac{C(m,p)}{(1-|z|^2)^{\delta p/q}} \int_{U^n} \frac{(1-|\xi|^{k-\delta+\delta p}|Df(\xi)|^p}{|1-\bar{\xi}z|^{k+2}} dm_{2n}(\xi).$$

Then, using Lemma 2.4, we obtain

$$\int_{U^n} \left|\frac{\partial^{n-1}(f(z)z)}{\partial z_2 \ldots \partial z_n}\right|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} dm_{2n}(z) \leq$$

$$\leq C(m,p)\int_{U^n} (1-|\xi|^2)^{k-\delta+\delta p}|Df(\xi)|^p \frac{\omega(1-|\xi|)dm_{2n}(\xi)}{(1-|\xi|)^{k+\delta p/q+2-p}} =$$

$$= C(m,p)\int_{U^n} (1-|\xi|^2)^p \frac{|Df(\xi)|^p \omega(1-|\xi|)}{(1-|\xi|^2)^2} dm_{2n}(\xi) = ||f||^p_{B_p(\omega)} < \infty.$$

Let now $p = 1$. Then we can write

$$\int_{U^n} \left|\frac{\partial^{n-1}(f(z)z)}{\partial z_2 \ldots \partial z_n}\right| \frac{\omega(1-|z|)}{1-|z|^2} dm_{2n}(z) \leq$$

$$\leq \int_{U^n} (1-|\xi|^2)^k |Df(\xi)| \int_{U^n} \frac{\omega(1-|z|)dm_{2n}(\xi)dm_{2n}(z)}{|1-\bar{\xi}z|^{k+2}(1-|z|^2)} \leq$$

$$\leq \int_{U^n} \frac{(1-|\xi|^2)^k|Df(\xi)|\omega(1-|\xi|)dm_{2n}(\xi)}{(1-|\xi|^2)^k(1-|\xi|^2)} = ||f||_{B_1(\omega)}.$$

Therefore $z_1 f \in B_1(\omega)$, and the result follows. □

3. Toeplitz operators on $B_p(\omega)$

In this section we give a description of those symbols h for which the corresponding Toeplitz operator is a bounded mapping $B_p(\omega) \to B_p(\omega)$.

Definition 3.1. A function $g \in H(U^n)$ is called a factor of the space $B_p(\alpha)$ if $fg \in B_p(\alpha)$ for any $f \in B_p(\alpha)$.

Definition 3.2. We say that $h \in L^1(T^n)$ is of the class LR if the Fourier coefficients of h vanish outside $\mathbb{R}^n_+ \cup \mathbb{R}^n_-$.

Theorem 3.1. *Let $p > 1$ and $\beta_\omega < 0$. Then the following assertions are equivalent:*

1) $T_{\bar{h}}$ is a bounded operator $B_p(\omega) \to B_p(\omega)$ $(p > \alpha_{\omega_j},\ 1 \leq j \leq n)$.

2) h has the form $h = h_1 + \bar{h}_2$, where h_1 is a factor of $B_p(\omega)$ and $h_2 \in B_q(\omega^*)$, where $\omega^*(t) = \omega^{-q/p}(t)t^q$ and $1/p + 1/q = 1$.

Proof. 1) $\Rightarrow$ 2). Let $T_{\bar{h}} : B_p(\omega) \to B_p(\omega)$ be a bounded operator on $B_p(\omega)$. Then we have $\|T_h(f)\|_{B_p(\omega)} < \infty$. It is known that the operator T_h is bounded on $B_p(\omega)$ if and only if

$$\|T_h\| = \sup_{\|f\|_{B_p(\omega)} \leq 1} \|T_h(f)\|_{B_p(\omega)} < \infty.$$

Now we prove that for every $z \in U^n$ the functional $T_h(f)(z)$ is bounded on $B_p(\omega)$.

To this end, let $z = (r_1 e^{i\theta_1}, \dots r_1 e^{i\theta_1})$ and V^n be the polydisk centered at $(r_1, ..., r_n)$ with radius of $V^n \asymp (1-r)$ and $V^n \subset U^n$. Then using the fact that $|T_h(f)(z)|^p$ is n–subharmonic we get

$$|T_h(f)(z)|^p \leq \frac{1}{(1-r)^2} \int_{V^n} |T_h(f)(w)|^p dm_{2n}(w)$$

$$\leq \frac{\omega^{-1}(1-r)}{(1-r)^2} \int_{V^n} |T_h(f)(w)|^p \omega(1-|w|) dm_{2n}(w)$$

$$\leq C(|z|) \int_{U^n} |T_h(f)(w)|^p \omega(1-|w|) dm_{2n}(w) = C(|z|) \|T_h(f)(w)\|^p_{B_p(\omega)},$$

which proves our assertion. Consequently $T_{\bar{h}}(f)(0)$ is a linear bounded functional on $B_p(\omega)$ and can be written in the form

$$T_{\bar{h}}(f)(0) = \int_{U^n} (1-|\xi|^2)^\beta Df(\xi)\overline{Dg(\xi)} dm_{2n}(\xi),$$

where $\beta \in \mathbb{R}^n$, $g \in B_q(\tilde{\omega})$ with $\tilde{\omega}_j(t) = t^{\beta_j q} \omega_j^{-q/p}(t)$, $\beta_j > \alpha_{\omega_j} + p - 2$ $(1 \leq j \leq n)$ (for details see Theorem 1.2 in [7]).

Using the representation

$$Df(\xi) = \frac{1}{(2\pi i)^n} \int_{T^n} \frac{f(t) t dt}{(t-\xi)^2},$$

we can write

$$\begin{aligned} T_{\bar h}(f)(0) &= \int_{U^n}(1-|\xi|^2)^\beta\overline{Dg(\xi)}\frac{1}{(2\pi i)^n}\int_{T^n}\frac{tf(t)dt}{(t-\xi)^2}dm_{2n}(\xi) \\ &= \frac{1}{(2\pi i)^n}\int_{T^n}tf(t)\int_{U^n}\frac{(1-|\xi|^2)^\beta\overline{Dg(\xi)}}{(t-\xi)^2}dm_{2n}(\xi)dt \\ &= \frac{1}{(2\pi i)^n}\int_{T^n}f(t)\bar t\int_{U^n}\frac{(1-|\xi|^2)^\beta\overline{Dg(\xi)}}{(1-\bar t\xi)^2}dm_{2n}(\xi)dt. \end{aligned}$$

Next, we set

$$\bar F(t)=\bar t\int_{U^n}\frac{(1-|\xi|^2)^\beta\overline{Dg(\xi)}}{(1-t\xi)^2}dm_{2n}(\xi),$$

and show that $F\in B_q(\omega^*)$, where $\omega^*(t)=\omega^{-q/p}(t)\,t^q$. To this end we use the Hölder inequality to obtain

$$|DF(t)|^q\le\frac{C}{(1-|t|)^{\delta q/p}}\int_{U^n}\frac{(1-|\xi|^2)^{1-\delta+\delta q+(\beta-1)q}|Dg(\xi)|^q}{|1-\xi\bar t|^3}dm_{2n}(\xi),\ \delta>0.$$

Then we have

$$\begin{aligned} &\int_{U^n}|DF(t)|^q\frac{\omega^*(1-|t|)}{(1-|t|)^{2-q}}dm_{2n}(t) \\ &\le C\int_{U^n}(1-|\xi|^2)^{1-\delta+\delta q+(\beta-1)q}|Dg(\xi)|^q\int_{U^n}\frac{\omega^*(1-|t|)dm_{2n}(t)dm_{2n}(\xi)}{(1-|t|^2)^{2-q+\delta q/p}|1-\bar\xi t|^3} \\ &\le C\int_{U^n}(1-|\xi|^2)^{\beta q}|Dg(\xi)|^q\frac{\omega^{-q/p}(1-|\xi|)}{(1-|\xi|)^{2-q}}dm_{2n}(\xi)=C||g||^q_{B_q(\bar\omega)}. \end{aligned}$$

So, we have proved that $F\in B_q(\omega^*)$. Then $T_{\bar h}$ has the form

$$T_{\bar h}(f)(0)=(2\pi i)^{-n}\int_{T^n}f(t)\overline{F(t)}dt.$$

On the other hand, we have

$$T_{\bar h}(f)(0)=(2\pi i)^{-n}\int_{T^n}f(\xi)\overline{h(\xi)}d\xi.$$

Setting $f(t)=t^k$ we get $\overline{h(\xi)}-\overline{F(\xi)}=h_1(\xi)\in H(U^n)$. So, $h=\bar h_1+h_2$, where $h_1\in H(U^n)$ and $h_2\in B_q(\omega^*)$. Thus $T_{\bar h}(f)=T_{\bar h_2}(f)+fh_1$.

Next, we show that if $h_2\in B_q(\omega^*)$, then $T_{\bar h_2}(f)\in B_p(\omega)$. Using Lemma 2.5, we get

$$\begin{aligned} T_{\bar h_2}(f)(z)&=\frac{1}{(2\pi i)^n}\int_{U^n}(1-|\xi|^2)^mDf(\xi)\int_{T^n}\frac{\overline{h_2(t)}P(\bar t,\xi)dtdm_{2n}(\xi)}{(1-\bar\xi t)^{m+1}(t-z)} \\ &=\frac{1}{(2\pi i)^n}\int_{U^n}(1-|\xi|^2)^mDf(\xi)\overline{\int_{T^n}\frac{h_2(t)P(t,\bar\xi)t^mdt}{(1-t\bar z)(t-\xi)^{m+1}}}dm_{2n}(\xi). \end{aligned}$$

Without loss of generality, we assume that $P(\bar{t},\xi) = \bar{t}^l \xi^l$. Then we get

$$T_{\bar{h}_2}(f)(z) = \frac{1}{(2\pi i)^n} \int_{U^n} (1-|\xi|^2)^m Df(\xi) \int_{T^n} \frac{\overline{h_2(t)} t^l \xi^l \bar{t}^m dt}{(t-\xi)^{m+1}(1-\bar{t}z)} dm_{2n}(\xi)$$
$$= \frac{1}{(2\pi i)^n m!} \int_{U^n} (1-|\xi|^2)^m Df(\xi) \xi^l \frac{\partial^m}{\partial \xi^m} \overline{\left(\frac{h_2(\xi)\xi^{m-l}}{1-\xi\bar{z}} \right)} dm_{2n}(\xi).$$

We set $h_2(\xi)\xi^{m-l} = \tilde{h}_2(\xi)$. It follows from Lemma 2.6 that $\tilde{h}_2 \in B_p(\omega^*)$ (we can take $m \in \mathbb{N}^n$.) Therefore

$$T_{\bar{h}}(f)(z) = \frac{1}{(2\pi i)^n m!} \int_{U^n} (1-|\xi|^2)^m Df(\xi)\xi^l \sum_{(k)=(0)}^{(m)} z^{m+k-1} C_m^k \frac{\partial^{|k|}\tilde{h}_2(\xi)}{\partial \xi_1^{k_1} \dots \partial \xi_n^{k_n}} \frac{dm_{2n}(\xi)}{(1-\bar{\xi}z)^{m-k+1}}.$$

We set

$$\Phi_k(z) = \int_{U^n} \frac{(1-|\xi|^2)^m Df(\xi)\xi^l}{(1-\bar{\xi}z)^{m-k+1}} \frac{\partial^{|k|}\tilde{h}_2(\xi)}{\partial \xi_1^{k_1} \dots \partial \xi_n^{k_n}} dm_{2n}(\xi),$$

and show that $\Phi_k(z) \in B_p(\omega)$. One can prove that

$$I = \int_{U^n} \left| \frac{\partial^{|k|}\tilde{h}_2(z)}{\partial z_1^{k_1} \dots \partial z_n^{k_n}} \right|^q \frac{\omega^*(1-|z|)}{(1-|z|^2)^{2-kq}} dm_{2n}(\xi) \le C\|h\|_{B_q(\omega^*} < \infty$$

for all $0 \le k_j \le m_j$ $(1 \le j \le n)$. To this end observe that

$$\tilde{h}_2(z) = \int_{U^n} \frac{(1-\xi|^2)^s}{(1-\bar{\xi}z)^{s+1}} P(\bar{\xi},z) D\tilde{h}_2(\xi) dm_{2n}(\xi).$$

By Holder's inequality we get

$$\left| \frac{\partial^{|k|}\tilde{h}_2(z)}{\partial z_1^{k_1} \dots \partial z_n^{k_n}} \right|^q \le \frac{C}{(1-|z|^2)^{(\delta+k-1)q/p}} \int_{U^n} \frac{(1-|\xi|^2)^{s-\delta+\delta q}}{|1-\bar{\xi}z|^{s+k+1}} |D\tilde{h}_2(\xi)|^q dm_{2n}(\xi).$$

Then we can write

$$\begin{aligned} I &\le \int_{U^n} (1-|\xi|^2)^{s-\delta+\delta q} |D\tilde{h}_2(\xi)|^q \int_{U^n} \frac{\omega^*(1-|z|) dm_{2n}(z) dm_{2n}(\xi)}{|1-\bar{\xi}z|^{s+k+1}(1-|z|^2)^{(\delta+k-1)q/p+2-kq}} \\ &= \int_{U^n} \frac{(1-|\xi|^2)^{s-\delta+\delta q} |D\tilde{h}_2(\xi)|^q \omega^*(1-|z|)}{(1-|\xi|^2)^{s+k-1+(\delta+k-1)q/p+2-kq}} dm_{2n}(\xi) \\ &= \int_{U^n} \frac{|D\tilde{h}_2(\xi)|^q \omega^*(1-|z|)}{(1-|\xi|^2)^{2-q}} dm_{2n}(\xi) < +\infty. \end{aligned}$$

Hence we have

$$|D\Phi_k(z)| \le \int_{U^n} \frac{(1-|\xi|^2)^m}{|1-\bar{\xi}z|^{m-k+2}} \left| \frac{\partial^{|k|}\tilde{h}_2(\xi,\bar{\xi})}{\partial \xi_1^{k_1} \dots \partial \xi_n^{k_n}} \right| |Df(\xi)| dm_{2n}(\xi)$$
$$= \int_{U^n} \frac{(1-|\xi|^2)^{m-k+2/q}(1-|\xi|^2)^k}{|1-\bar{\xi}z|^{m-k+2}(\omega^*(1-|\xi|))^{1/q}} \left| \frac{\partial^{|k|}\tilde{h}_2(\xi,\bar{\xi})}{\partial \xi_1^{k_1} \dots \partial \xi_n^{k_n}} \right| Df(\xi)| \frac{(\omega^*(1-|\xi|))^{1/q}}{(1-|\xi|^2)^{2/q}} dm_{2n}(\xi)$$
$$\le C\|\tilde{h}_2\|_{B_q(\omega^*)} \left(\int_{U^n} \frac{(1-|\xi|^2)^{(m-k+2/q)p} |Df(\xi)|^p dm_{2n}(\xi)}{|1-\bar{\xi}z|^{(m-k+2)p}(\omega^*(1-|\xi|))^{p/q}} \right)^{1/p}.$$

Therefore

$$\int_{U^n}(1-|z|^2)^p\frac{|D\Phi_k(z)|^p\omega(1-|z|)}{(1-|z|^2)^2}dm_{2n}(z)\le ||\tilde{h}_2||^p_{B_q(\omega^*}\times$$

$$\times\int_{U^n}(1-|\xi|^2)^{(m-k+2/q)p}|Df(\xi)|^p\omega(1-|\xi|)\int_{U^n}\frac{(1-|z|^2)^p\omega(1-|z|)dm_{2n}(\xi)}{|1-\bar{\xi}z|^{(m-k+2)p}(1-|z|^2)^2}.$$

Now we estimate the following integral in the one dimensional case

$$I(\xi)=\int_{U^n}\frac{(1-|\xi|^2)^{p-2}\omega(1-|\xi|)}{|1-\bar{\xi}z|^{(m-k+2)p}}dm_2(z).$$

(i) If $(m-k+2)p-1-p+2+\beta_\omega<1$, and hence $(m-k+1)p<-\beta_\omega$, then $I(\xi)\le C$ with some constant C.

(ii) If $(m-k+1)p>-\beta_\omega$, then $I(\xi)\le C/(1-|\xi|)^{(m-k+1)p+\beta_\omega}$ with some constant C.

(iii) If $(m-k+1)p=-\beta_\omega$, then $I(\xi)\le\log(1-|\xi|)^{-1}$.

For the case (i) we have

$$\int_U(1-|\xi|^2)^{(m-k+2/q)p}|Df(\xi)|^p\frac{\omega(1-|\xi|)}{(1-|\xi|^2)^p}dm_2(\xi)$$
$$\le\int_U|Df(\xi)|^p\frac{\omega(1-|\xi|)}{(1-|\xi|^2)^{2-p}}dm_2(\xi)=\|f\|^p_{B_p(\omega)},$$

where $p(m-k)+2p/q+2-2p=p(m-k)\ge 0$.

For the case (ii) we have

$$\int_U(1-|\xi|^2)^{(m-k+2/q)p}|Df(\xi)|^p\omega(1-|\xi|)\frac{dm_2(\xi)}{(1-|\xi|)^{(m-k+1)p+\beta_\omega}}$$
$$=\int_U|Df(\xi)|^p\frac{\omega(1-|\xi|)}{(1-|\xi|^2)^{2-p}}\frac{dm_2(\xi)}{(1-|\xi|^2)^{\beta_\omega+2p-2p/q-2}}\le C||f||^p_{B_p(\omega)},$$

provided that $\beta_\omega+2p-2p/q-2=\beta_\omega<0$.

Let $(m-k+1)p=-\beta_\omega$. We have

$$\int_U(1-|\xi|^2)^{(m-k+2/q)p}|Df(\xi)|^p\omega(1-|\xi|)\log\left(\frac{1}{1-|\xi|}\right)dm_2(\xi)$$
$$=\int_U|Df(\xi)|^p\frac{\omega(1-|\xi|)}{(1-|\xi|^2)^{2-p}}(1-|\xi|^2)^{-\beta_\omega-p+2p/q-p+2}\log\left(\frac{1}{1-|\xi|}\right)dm_2(\xi)$$
$$\le C||f||^p_{B_p(\omega)}.$$

Then it follows that $\Phi_k\in B_p(\omega)$, and therefore $T_{\tilde{h}_2}(f)\in B_p(\omega)$. So, we have $T_h(f)(z)=T_{h_1}(f)(z)+T_{\tilde{h}_2}(f)(z)$, where T_h and $T_{\tilde{h}_2}$ are bounded operators $B_p(\omega)\to B_p(\omega)$. Then T_{h_1} is a bounded operator $B_p(\omega)\to B_p(\omega)$. On the other hand, we have $T_{h_1}(f)(z)=h_1(z)f(z)$, showing that h_1 is a factor of $B_p(\omega)$. This completes the proof of implication 1) $\Rightarrow$ 2).

To prove the implication 2) $\Rightarrow$ 1), let $h = h_1 + \bar{h}_2$. Then we have $T_h(f)(z) = T_{h_1}(f)(z) + T_{\bar{h}_2}(f)(z)$. We have proved that, if $h_2 \in B_q(\omega^*)$, then T_{h_2} is a bounded operator. The boundedness of T_{h_1} is evident. Thus, T_h is a bounded operator $B_p(\omega) \to B_p(\omega)$. □

Next, we consider the case $p = 1$. We need the definition of holomorphic Bloch spaces:

Definition 3.3. The holomorphic Bloch space B_ω is defined to be the class of functions $f \in H(U^n)$ satisfying

$$||f||_{B_\omega} = \sup_{z \in U^n} \left\{ \frac{(1-|z|^2)}{\omega(1-|z|^2)} |Df(z)| \right\} < +\infty.$$

Definition 3.1 can be regarded as the definition of Besov space $B_p(\omega)$ in the case $p = \infty$, that is, $B_\infty(\omega) = B_\omega$. The following theorem holds.

Theorem 3.2. *Let $\beta_\omega < 1$. Then T_h is a bounded operator $B_1(\omega) \to B_1(\omega)$ if and only if $h = h_1 + \bar{h}_2$, where h_1 is a factor of $B_1(\omega)$ and $h_2 \in B_{\tilde{\omega}}$, where $\tilde{\omega}_j = \omega_j(t) t^{1-\beta_j}$, $\beta_j > \alpha_{\omega_j} - 1$, $1 \le j \le n$.*

Proof. Assume that $T_h : B_1(\omega) \to B_1(\omega)$ is a bounded operator. Then as in the case of $p > 1$, $T_h(f)(0)$ is a bounded linear functional on $B_1(\omega)$, and by Theorem 3 of [7], has a representation of the form

$$T_h(f)(0) = \int_{U^n} Df(\xi)\overline{Dh(\xi)}(1-|\xi|^2)^\beta dm_{2n}(\xi),$$

where $\beta_j > \alpha_{\omega_j} - 1$, $1 \le j \le n$, and $h \in B_{\tilde{\omega}}$, $\tilde{\omega}_j(t) = \omega_j(t) t^{1-\beta_j}$, $1 \le j \le n$. As in the case $p > 1$ we obtain

$$T_h(f)(0) = \frac{1}{(2\pi i)^n} \int_{T^n} f(t) t \int_{U^n} \frac{(1-|\xi|^2)^\beta \overline{Dh(\xi)}}{(t-\xi)^2} dm_{2n}(\xi) dt.$$

We set

$$\overline{F(t)} = \bar{t} \int_{U^n} \frac{(1-|\xi|^2)^\beta \overline{Dh(\xi)}}{(1-t\bar{\xi})^2} dm_{2n}(\xi),$$

and show that $F \in B_\omega$. Indeed, we have

$$\begin{aligned} |DF(t)| &\le C \int_{U^n} \frac{(1-|\xi|^2)^\beta |Dh(\xi)|}{|1-t\bar{\xi}|^3} dm_{2n}(\xi) \\ &\le C||h||_{B_{\tilde{\omega}}} \int_{U^n} \frac{\omega(1-|\xi|^2) dm_{2n}(\xi)}{|1-t\bar{\xi}|^3} \le C||h||_{B_{\tilde{\omega}}} \frac{\omega(1-|t|)}{1-|t|}. \end{aligned}$$

Hence $F \in B_\omega$. Therefore we have

$$T_h(f)(0) = \frac{1}{(2\pi i)^n} \int_{T^n} f(t)\overline{F(t)} dm_{2n}(t)$$

and

$$T_h(f)(0) = (2\pi i)^{-n} \int_{T^n} f(t)h(t)dm_{2n}(t).$$

As in the case $p > 1$ we have $h(z) = h_1(z) + \overline{h_2(z)}$, where $h_1 \in H(U^n)$ and $h_2 \in B_\omega$. So, we obtain $T_h(f) = T_{h_1}(f) + T_{\bar{h}_2}(f)$. Next, we show that if $h_2 \in B_\omega$, then $T_{\bar{h}_2}(f) \in B_1(\omega)$ for all $f \in B_1(\omega)$. Indeed, we have

$$\begin{aligned} T_{\bar{h}_2}(f)(z) &= \frac{1}{(2\pi i)^n} \int_{T^n} \frac{\overline{h_2(\xi)}}{(\xi - z)} \int_{U^n} \frac{(1-|\xi|^2)^m Df(t)}{(1-\bar{t}\xi)^{m+1}} dm_{2n}(\xi)dt \\ &= \int_{U^n} (1-|t|^2)^m Df(t) \sum_{(k)=(0)}^{(m)} C_m^k \frac{\partial^{|k|}\tilde{h}_2(t)}{\partial t^k} \frac{dm_{2n}(t)}{(1-\bar{t}z)^{m-k}}. \end{aligned}$$

Hence we can write

$$\begin{aligned} &||T_{\bar{h}_2}(f)|| \\ &\le \sum_{|k|=0}^{|m|} C_m^\alpha \int_{U^n} \frac{\omega(1-|z|)}{1-|z|^2} \int_{U^n} \frac{(1-|t|^2)^m |Df(t)|}{|1-\bar{t}z|^{m-k+1}} \left| \frac{\partial^{|k|}\tilde{h}_2(t)}{\partial t^k} \right| dm_{2n}(t)dm_{2n}(z) \\ &\le \int_{U^n} (1-|t|^2)^m |Df(t)| \cdot \left| \frac{\partial^{|k|}\tilde{h}_2(t)}{\partial t^k} \right| \int_{U^n} \frac{\omega(1-|z|)dm_{2n}(z)dm_{2n}(t)}{|1-\bar{t}z|^{m-k+1}(1-|z|)} \\ &\le \int_{U^n} (1-|t|^2)^m |Df(t)| \cdot \left| \frac{\partial^{|k|}\tilde{h}_2(t)}{\partial t^k} \right| \frac{\omega(1-|t|}{(1-|t|)^{m-k}} dm_{2n}(t) \\ &\le ||\tilde{h}_2||_{B_\omega} \int_{U^n} |Df(z)| \frac{\omega(1-|z|)}{1-|z|} (1-|z|)\omega(1-|z|) dm_{2n}(z), \end{aligned}$$

and $(1-|z|)\omega(1-|z|) \le C$ with some constant C provided that $\beta_\omega < 1$. So, we have that $T_{\bar{h}_2}P : B_1(\omega) \to B_1(\omega)$ is a bounded operator. It follows that $T_h : B_1(\omega) \to B_1(\omega)$ is a bounded operator. On the other hand, we have $T_{h_1}(f)(z) = f(z)h_1(z)$. Hence $f(z)h_1(z) \in B_1(\omega)$ for all $f \in B_1(\omega)$. Therefore h_1 is a factor of $B_1(\omega)$.

Conversely, let $h = h_1 + \bar{h}_2$, where $h_2 \in B_\omega$ and h_1 is a factor of $B_1(\omega)$. In the first part of the proof we have shown that $T_{\bar{h}_2} : B_1(\omega) \to B_1(\omega)$ is bounded if $h_2 \in B_\omega$. It follows from the definition of a factor that $T_{h_1}(f) \in B_1(\omega)$ for all $f \in B_1(\omega)$ provided that $h_2 \in B_\omega$. □

Now we provide an application of our results to division theorems in the spaces $B_p(\omega)$, $(p \ge 1)$. To this end, we need the following well-known definitions (see [10]).

Definition 3.4. A function $g \in H^\infty(U^n)$ is called an inner function, if its radial boundary values satisfy $|g^*(w)| = 1$ almost everywhere on T^n.

Definition 3.5. An inner function $g \in H^\infty(U^n)$ is said to be good if $u[g] = 0$, where $u[g]$ is the least n-harmonic majorant of $\log |g|$ in U^n.

Theorem 3.3. *Let $\beta_\omega < 1$ if $p = 1$ and $\beta_\omega < 0$ if $p > 1$. Let $f \in B_p(\omega)$ and J be a good inner function and $F = f/J \in H(U^n)$. Then $F \in B_p(\omega)$.*

Proof. We have

$$T_{\overline{J}}(f)(z) = \frac{1}{(2\pi i)^n} \int_{T^n} \frac{f(\zeta)\overline{J}(\zeta)}{\zeta - z} d\zeta =$$

$$= \frac{1}{(2\pi i)^n} \int_{T^n} \frac{f(\zeta)/\overline{J}(\zeta)}{\zeta - z} d\zeta = \frac{1}{(2\pi i)^n} \int_{T^n} \frac{F(\zeta)}{\zeta - z} d\zeta = F(z).$$

By Theorems 3.2 and 3.3 we obtain $F \in B_p(\omega)$. □

Список литературы

[1] J. Arazy, S. Fisher, J. Peetre, "Mobius invariant function spaces", J. Reine Angew. Math., **363**, 110 – 145 (1985).

[2] O. Blasco, "Multipliers on weighted Besov spaces of analytic functions", Contemporary Mathematics, **144**, 23 – 33 (1993).

[3] M. M. Djrbashian, "On the representation problem of analytic functions", Soobsh. Inst. Matem. Mekh. Akad. Nauk Arm. SSR, **2**, 3 – 40 (1948).

[4] V.S. Guliyev, W. Zhijian, "Weighted holomorphic Besov spaces and their boundary values", Anal. Theory App., **21**, no. 2, 143 – 156 (2005).

[5] A. V. Harutyunyan, "Bloch spaces of holomorphic functions in the polydisk", J. Funct. Spaces Appl. **5**, no. 3, 213 – 230 (2007).

[6] A. V. Harutyunyan, G. V. Harutyunyan, "Holomorphic Besov spaces in the polydisc and bounded Toeplitz operators", Analysis: **30**, no. 4, 365 – 381 (2010).

[7] A. V. Harutyunyan, W. Lusky, "Duals of holomorphic Besov spaces on the polydisk and diagonal mappings", Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences) **45**, no. 3, 128 – 135 (2010).

[8] A. V. Harutyunyan, W. Lusky, "Weighted holomorphic Besov spaces on the polydisks", J. Funct. Spaces and Appl. **9**, 1 – 16 (2011).

[9] M. Jevtic, Holomorphic Besov Spaces $B^{\cdot}$ $0 < p < 1$ on Bounded Symetric Domains", Filomat, no 12, part 1 (1988).

[10] W. Rudin, Function Theory in Polydiscs, New York (1969).

[11] F. Shamoyan, "Diagonal mappincs and questions of presentation in anisotropic spaces in Polydisk" [in Russian], Sib. mat. Journ., **3**, no. 2, 197 – 215 (1990).

[12] E. Seneta, Functions of Regular Variation [in Russian], Nauka, Moscow (1985).

[13] K. Zhu, Operator Theory in Function Spaces, Marcel Dekker, New York (1990).

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