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# ON THE GENERALIZATIONS OF POLYNOMIAL INEQUALITIES IN THE COMPLEX DOMAIN

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Abstract. In this paper we establish some generalizations of Bernstein-type inequalities for polynomials having zeros in the closed interior or closed exterior of a circle of radius $|z| = K$.

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## 1. Introduction and statement of results

The geometrical relation between the maximum modulus of a complex polynomial on a circle and the location of zeros of this polynomial within or outside this circle is one of the attractive and fertile subjects in geometry of polynomials. Bernstein-type inequalities play a fundamental role for many propositions in the area of polynomial inequalities. There are many results on Bernstein's theorems and their generalizations in different forms. Before proceeding towards specific results concerning the zeros of polynomials, we find it useful to consider certain fundamental theorems, which will be used throughout this paper. We begin by stating a classical result due to Bernstein [5]. Let $P(z)$ be a polynomial of degree $n$. Then

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)|. \tag{1.1}$$

We also state an inequality which is a simple consequence of maximum principle (see [9, 20]).

$$\max_{|z|=R}|P(z)| \leq R^n \max_{|z|=1}|P(z)|, \quad R \geq 1. \tag{1.2}$$

The above results are best possible and the equalities hold for polynomials having zeros at the origin.

Observe that (1.2) can also be obtained from (1.1) by using Gauss-Lucas theorem (see [12]). Refinements of inequalities (1.1) and (1.2) can be found in a number of important papers (see [2, 4, 6, 8, 10, 11, 16, 18, 19], and references therein).

If $P(z)$ is a polynomial of degree $n$, having all its zeros in $|z| < 1$, then Aziz and Dawood [3] proved that for $|z| = 1$,

$$\min_{|z|=R} |P(z)| \geq R^n \min_{|z|=1} |P(z)|, \quad R \geq 1. \tag{1.3}$$

This inequality is sharp for the polynomial $P(z) = me^{i\beta} z^n, \quad m > 0$.

Jain [14], generalized inequality (1.2) by proving that if $P(z)$ is a polynomial of degree $n$, then for $|z| = 1$ and $|\alpha| \leq 1$,

$$|P(Rz) + \alpha(\frac{R+1}{2})^n P(z)| \leq |R^n + \alpha(\frac{R+1}{2})^n| \max_{|z|=1} |P(z)|, \quad R \geq 1. \tag{1.4}$$

This result is best possible for $P(z) = \beta + \gamma z^n$, where $|\beta| = |\gamma|$.

It was shown by Ankeny and Rivlin [1] that if $P(z) \neq 0$ in $|z| < 1$, then (1.2) can also be replaced by

$$\max_{|z|=R} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|, \quad R \geq 1. \tag{1.5}$$

Inequality (1.5) is sharp for $P(z) = \beta + \gamma z^n$, where $|\beta| = |\gamma| = 1/2$.

Aziz and Dawood [3], improved the above inequality by introducing the minimum value of $|P(z)|$ on $|z| = 1$ as follows:

$$\max_{|z|=R} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)| - \frac{R^n - 1}{2} \min_{|z|=1} |P(z)|, \quad R \geq 1. \tag{1.6}$$

This result is best possible for $P(z) = \beta + \gamma z^n$, where $|\beta| \geq |\gamma|$.

Jain [15], improved (1.4) for polynomials having no zeros in $|z| < 1$ with $|\alpha| \leq 1, \ R \geq 1$ and $|z| = 1$ as given below

$$|P(Rz) + \alpha(\frac{R+1}{2})^n P(z)|$$
$$\leq \frac{1}{2}\{|R^n + \alpha(\frac{R+1}{2})^n| + |1 + \alpha(\frac{R+1}{2})^n|\} \max_{|z|=1} |P(z)|. \tag{1.7}$$

Equality in (1.7) holds for $P(z) = \beta + \gamma z^n$, where $|\beta| = |\gamma| = 1/2$.

Recently, Dewan and Hans [7], proved a result concerning minimum modulus of polynomials $P(z)$, which is an analog of inequality (1.3).

**Theorem A.** *If $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| < 1$, then for every real or complex number $\beta$ with $|\beta| \leq 1$ and $R \geq 1$,*

$$\min_{|z|=1} \left| P(Rz) + \beta(\frac{R+1}{2})^n P(z) \right| \geq \left| R^n + \beta(\frac{R+1}{2})^n \right| \min_{|z|=1} |P(z)|. \tag{1.8}$$

*The inequality (1.8) is best possible and equality holds for $P(z) = me^{i\beta} z^n, \quad m > 0$.*

Dewan and Hans [7] also improved inequality (1.7) by proving the following theorem.

**Theorem B.** *If $P(z)$ is a polynomial of degree $n$ having no zeros in $|z| < 1$, then for every real or complex number $\beta$ with $|\beta| \leq 1,\ R \geq 1$, and $|z| = 1$,*

$$|P(Rz) + \beta(\frac{R+1}{2})^n P(z)| \leq \frac{1}{2}\{|R^n + \beta(\frac{R+1}{2})^n| + |1 + \beta(\frac{R+1}{2})^n|\} \max_{|z|=1} |P(z)|$$

$$-\frac{1}{2}\{|R^n + \beta(\frac{R+1}{2})^n| - |1 + \beta(\frac{R+1}{2})^n|\} \min_{|z|=1} |P(z)|. \tag{1.9}$$

*Equality in (1.9) holds for $P(z) = \alpha + \gamma z^n$, where $|\alpha| = |\gamma| = 1/2$.*

Recently, Mezerji et al. [17], generalized Theorems A and B to a class of polynomials having zeros in the closed interior and closed exterior of a circle $|z| = K$, given by the following two theorems.

**Theorem C.** *If $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq K,\ K \leq 1$, then for every real or complex number $\beta$ with $|\beta| \leq 1$, and $R \geq 1$,*

$$\min_{|z|=1} \left| P(Rz) + \beta \left(\frac{R+K}{1+K}\right)^n P(z) \right| \geq \frac{1}{K^n} \left| R^n + \beta \left(\frac{R+K}{1+K}\right)^n \right| \min_{|z|=K} |P(z)|. \tag{1.10}$$

*The result is the best possible and equality holds for $P(z) = \alpha z^n$.*

**Theorem D.** *If $P(z)$ is a polynomial of degree $n$ having no zeros in $|z| < K,\ K \geq 1$, then for every real or complex number $\beta$ with $|\beta| \leq 1,\ R \geq 1$, and $|z| = 1$,*

$$|P(RK^2z) + \beta\left(\frac{RK+1}{1+K}\right)^n P(K^2z)|$$

$$\leq \frac{1}{2}\{K^n|R^n + \beta\left(\frac{RK+1}{1+K}\right)^n| + |1 + \beta\left(\frac{RK+1}{1+K}\right)^n|\} \max_{|z|=K} |P(z)|$$

$$-\frac{1}{2}\{K^n|R^n + \beta\left(\frac{RK+1}{1+K}\right)^n| - |1 + \beta\left(\frac{RK+1}{1+K}\right)^n|\} \min_{|z|=K} |P(z)|. \tag{1.11}$$

*The result is best possible and equality in (1.11) holds for $P(z) = z^n - K^n$.*

While making an attempt towards the generalization of the above inequalities, the author found that there is a room for the generalization of the condition $R \geq 1$ in the above theorems to $R \geq r > 0$, which induces inequalities towards more generalized form. The essence in the papers by Govil et al. [13] and Mezerji et al. [17] is the origin of thought for the new inequalities presented in this paper.

Now we are in position to state our main results. Our first result, Theorem 1.1, is a further generalization of Theorem C. It involves an inequality on a class of polynomials having all its zeros in $|z| \leq K,\ \ K > 0$.

**Theorem 1.1.** *If $P(z)$ is a polynomial of degree $n$, having all its zeros in $|z| \leq K,\ K > 0$, then for every real or complex number $\beta$ with $|\beta| \leq 1,\ \ |z| \geq 1$, and*

$R \geq r,\ Rr \geq K^2$,

$$\min_{|z|=1} |P(Rz) + \beta(\frac{R+K}{r+K})^n P(rz)| \geq \frac{1}{K^n} |R^n + \beta r^n (\frac{R+K}{r+K})^n| \min_{|z|=K} |P(z)|. \tag{1.12}$$

*The result is best possible and equality in (1.12) holds for $P(z) = me^{i\beta} z^n,\ m > 0$.*

**Remark 1.1.** *If $r = 1$ and $K \leq 1$, then Theorem 1.1 reduces to Theorem C; if $K = 1$, then it further reduces to Theorem A, and if, in addition, $\beta = 0$, then inequality (1.12) becomes inequality (1.3).*

**Remark 1.2.** *If $r = K$, then inequality (1.12) takes the following simple form:*

$$|P(Rz) + \beta\left(\frac{R+K}{2K}\right)^n P(Kz)| \geq \left|\frac{R^n}{K^n} + \beta\left(\frac{R+K}{2K}\right)^n\right| \min_{|z|=K} |P(z)|.$$

Our second theorem extends Theorem D to the class of polynomials having no zeros in $|z| < K,\ K > 0$.

**Theorem 1.2.** *If $P(z)$ is a polynomial of degree $n$, having no zeros in $|z| < K$, $K > 0$, then for every real or complex number $\beta$ with $|\beta| \leq 1$ and $R \geq r,\ rR \geq \frac{1}{K^2}$, $|z| = 1$,*

$$|P(RK^2z) + \beta\left(\frac{RK+1}{rK+1}\right)^n P(rK^2z)|$$

$$\leq \frac{1}{2}\{K^n|R^n + \beta\left(\frac{RK+1}{rK+1}\right)^n| + |1 + \beta\left(\frac{RK+1}{rK+1}\right)^n|\} \max_{|z|=K} |P(z)|$$

$$-\frac{1}{2}\{K^n|R^n + \beta r^n\left(\frac{RK+1}{rK+1}\right)^n| - |1 + \beta\left(\frac{RK+1}{rK+1}\right)^n|\} \min_{|z|=K} |P(z)|. \tag{1.13}$$

*The result is best possible and equality in (1.13) holds for $P(z) = az^n + bK^n$, $|b| \geq |a|$.*

**Remark 1.3.** *If $r = 1$ and $K \geq 1$, then Theorem 1.4 reduces to the Theorem D, and if $\beta = 0$ and $K = 1$, then inequality (1.13) becomes inequality (1.6).*

**Remark 1.4.** *If $\beta = 0$, then inequality (1.13) becomes*

$$|P(RK^2z)| \leq \frac{1}{2}(K^nR^n + 1) \max_{|z|=K} |P(z)| - (K^nR^n - 1) \min_{|z|=K} |P(z)|.$$

## 2. Lemmas

We begin with a lemma due to Govil et al. [13].

**Lemma 2.1.** *If $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq K$, $K > 0$, then for every $R \geq r$ and $Rr \geq K^2$, we have*

$$|P(Rz)| \geq \left(\frac{R+K}{r+K}\right)^n |P(rz)| \quad for \quad |z| = 1. \tag{2.1}$$

**Lemma 2.2.** *If $P(z)$ is a polynomial of degree $n$ having no zeros in $|z| < K$, $K > 0$, then for any $\beta$ with $|\beta| \leq 1$, $R \geq r$, $rR \geq \frac{1}{K^2}$, and $|z| = 1$, we have*

$$(2.2)\quad |P(RK^2z) + \beta\left(\frac{RK+1}{rK+1}\right)^n P(rK^2z)| \leq K^n|Q(Rz) + \beta\left(\frac{RK+1}{rK+1}\right)^n Q(rz)|,$$

*where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.*

**Proof.** Since $P(z) \neq 0$ in $|z| < K$, the polynomial $Q(z)$ has all its zeros in $|z| \leq \frac{1}{K}$. Note that $|Q(z)| = \frac{1}{K^n}|P(K^2z)|$ for $|z| = \frac{1}{K}$. Therefore by Rouche's theorem, the polynomial $S(z) = K^nQ(z) - \alpha P(K^2z)$ of degree $n$ has all its zeros in $|z| \leq \frac{1}{K}$ for $|\alpha| < 1$. Hence using Lemma 2.1, for $R \geq r$, $Rr \geq \frac{1}{K^2}$ and $|z| = 1$, we have

$$|S(Rz)| \geq \left(\frac{RK+1}{rK+1}\right)^n |S(rz)|,$$

implying

$$|K^nQ(Rz) - \alpha P(RK^2z)| \geq \left(\frac{RK+1}{rK+1}\right)^n |K^nQ(rz) - \alpha P(rK^2z)|.$$

Denote

$$T(z) := K^n\{Q(Rz)+\beta\left(\frac{RK+1}{rK+1}\right)^n Q(rz)\} - \alpha\{P(RK^2z)+\beta\left(\frac{RK+1}{rK+1}\right)^n P(rK^2z)\},$$

and note that $T(z) \neq 0$ for $|\beta| < 1$ and $|z| = 1$. This implies that (2.2) is true. Indeed, if it is not true, then there exists a point $z = z_0$ with $|z_0| = 1$ such that

$$|P(RK^2z_0) + \beta\left(\frac{RK+1}{rK+1}\right)^n P(rK^2z_0)| > K^n|Q(Rz_0) + \beta\left(\frac{RK+1}{rK+1}\right)^n Q(rz_0)|.$$

We take

$$\alpha = \frac{K^n\{Q(Rz_0) + \beta(\frac{RK+1}{rK+1})^n Q(rz_0)\}}{P(RK^2z_0) + \beta(\frac{RK+1}{rK+1})^n P(rK^2z_0)},$$

and observe that $|\alpha| < 1$. Clearly with this choice of $\alpha$, we have $T(z_0) = 0$ for $|z_0| = 1$, yielding a contradiction to the fact that $T(z)$ is nonzero on the circle $|z| = 1$. If $|\beta| = 1$, inequality (2.2) follows by continuity. Hence the proof is complete.

**Lemma 2.3.** *If $P(z)$ is a polynomial of degree $n$, then for every real or complex number $\beta$ with $|\beta| \leq 1$ and $K > 0$, and for every $R \geq r$, $rR \geq K^2$ and $|z| \geq 1$, we have*

$$(2.3)\qquad |P(Rz) + \beta(\frac{R+K}{r+K})^n P(rz)| \leq \frac{|z|^n}{K^n}|R^n + \beta r^n(\frac{R+K}{r+K})^n)| \max_{|z|=K} |P(z)|.$$

**Proof.** Consider the reciprocal polynomial of $P(z)$ given by $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$. Take $M = \max_{|z|=\frac{1}{K}} |Q(z)| = \frac{1}{K^n}\max_{|z|=K}|P(z)|$. By Rouche's theorem, for every $\alpha$ with $|\alpha| > 1$, the polynomial $I(z) = Q(z) - \alpha M$ does not vanish in $|z| < \frac{1}{K}$. Hence the polynomial

$$Y(z) = z^n\overline{I(\frac{1}{\bar{z}})} = P(z) - \overline{\alpha}Mz^n$$

has all its zeros in $|z| \le K$, $K > 0$. Applying Lemma 2.1 to the polynomial $Y(z)$, for $|z| = 1$ and every $R \ge r$ and $rR \ge K^2$, we obtain

$$|Y(Rz)| \ge (\frac{R+K}{r+K})^n|Y(rz)|.$$

Since $Y(Rz)$ has all its zeros in $|z| \le \frac{K}{R} \le 1$, again applying Rouche's theorem, we conclude that for every $\beta$ with $|\beta| < 1$, all the zeros of the polynomial

$$Y(Rz) + \beta(\frac{R+K}{r+K})^nY(rz)$$

lie in $|z| < 1$. In other words, all the zeros of the polynomial

$$T(z) = P(Rz) + \beta(\frac{R+K}{r+K})^nP(rz) - \overline{\alpha}((MR^nz^n) + \beta(\frac{R+K}{r+K})^nMr^nz^n) \tag{2.4}$$

lie in $|z| < 1$. We claim that this implies inequality (2.3). We prove the claim by contradiction. If the claim is not true, then there exists a point $z = z_0$ with $|z_0| \ge 1$, such that

$$|P(Rz_0) + \beta\left(\frac{R+K}{r+K}\right)^nP(rz_0)| > \frac{|z_0|^n}{K^n}|R^n + \beta r^n\left(\frac{R+K}{r+K}\right)^n|\max_{|z|=K}|P(z)|,$$

or equivalently,

$$|P(Rz_0) + \beta\left(\frac{R+K}{r+K}\right)^nP(rz_0)| > M|z_0|^n|R^n + \beta r^n\left(\frac{R+K}{r+K}\right)^n|.$$

We take

$$\overline{\alpha} = \frac{P(Rz_0) + \beta(\frac{R+K}{r+K})^nP(rz_0)}{Mz_0^n\{R^n + \beta r^n(\frac{R+K}{r+K})^n\}},$$

and observe that $|\alpha| > 1$. In view of (2.4), it is easy to see that with this choice of $\alpha$, we have $T(z_0) = 0$ for $|z_0| \ge 1$, yielding a contradiction to the fact that $T(z)$ is nonzero in the closed exterior of the circle $|z| = 1$. If $|\beta| = 1$, inequality (2.3) follows by continuity. Hence the proof is complete.

**Lemma 2.4.** *If $P(z)$ is a polynomial of degree $n$, then for any $\beta$ with $|\beta| \le 1$, $K > 0$ and $R \ge r$, $Rr \ge \frac{1}{K^2}$, and $|z| = 1$, we have*

$$|P(RK^2z)+\beta\Big(\frac{RK+1}{rK+1}\Big)^nP(rK^2z)|+K^n|Q(Rz)+\beta\Big(\frac{RK+1}{rK+1}\Big)^nQ(rz)|$$

$$\leq\{K^n|R^n+\beta r^n\left(\frac{RK+1}{rK+1}\right)^n|+|1+\beta\left(\frac{RK+1}{rK+1}\right)^n|\}\max_{|z|=K}|P(z)|, \tag{2.5}$$

*where* $Q(z)=z^n\overline{P(\frac{1}{\bar z})}$.

**Proof.** Let $M=\max\limits_{|z|=K}|P(z)|$. Then $|P(z)|\leq M$ for $|z|=K$. Therefore, for a given real or complex number $\lambda$ with $|\lambda|>1$, it follows from Rouche's theorem that the polynomial $T(z)=P(z)+\lambda M$ does not vanish in $|z|<K$. Hence applying Lemma 2.2 to the polynomial $T(z)$ for $\beta$ with $|\beta|\leq 1$, $|z|=1$ and $R\geq r$, $Rr\geq\frac{1}{K^2}$, we get

$$|P(RK^2z)+\lambda M+\beta\Big(\frac{RK+1}{rK+1}\Big)^n(P(rK^2z)+\lambda M)|$$

$$\leq K^n|Q(Rz)+\overline{\lambda}MR^nz^n+\beta\Big(\frac{RK+1}{rK+1}\Big)^n(Q(rz)+\overline{\lambda}Mr^nz^n)|,$$

implying

$$|P(RK^2z)+\beta\Big(\frac{RK+1}{rK+1}\Big)^nP(rK^2z)+\lambda M(1+\beta\Big(\frac{RK+1}{rK+1}\Big)^n|$$

$$\leq K^n|Q(Rz)+\beta\Big(\frac{RK+1}{rK+1}\Big)^nQ(rz)+\overline{\lambda}Mz^n(R^n+\beta r^n\Big(\frac{RK+1}{rK+1}\Big)^n|.$$

Now choosing the argument of $\lambda$ appropriately, we get

$$|P(RK^2z)+\beta\Big(\frac{RK+1}{rK+1}\Big)^nP(rK^2z)|-|\lambda|M|1+\beta\Big(\frac{RK+1}{rK+1}\Big)^n|$$

$$\leq||\lambda|MK^n|z|^n|R^n+\beta r^n\Big(\frac{RK+1}{rK+1}\Big)^n|-K^n|Q(Rz)+\beta\Big(\frac{RK+1}{rK+1}\Big)^nQ(rz)||.$$

Next, applying Lemma 2.3 to the polynomial $Q(z)$, we can write

$$|\lambda|MK^n|z|^n|R^n+\beta r^n\Big(\frac{RK+1}{rK+1}\Big)^n|\geq K^n|Q(Rz)+\beta\Big(\frac{RK+1}{rK+1}\Big)^nQ(rz)|.$$

Therefore

$$|P(RK^2z)+\beta\Big(\frac{RK+1}{rK+1}\Big)^nP(rK^2z)|-|\lambda|M|1+\beta\Big(\frac{RK+1}{rK+1}\Big)^n|$$

$$\leq|\lambda|MK^n|z|^n|R^n+\beta\Big(\frac{RK+1}{rK+1}\Big)^n|-K^n|Q(Rz)+\beta\Big(\frac{RK+1}{rK+1}\Big)^nQ(rz)|,$$

implying

$$|P(RK^2z)+\beta\big(\tfrac{RK+1}{rK+1}\big)^nP(rK^2z)|+K^n|Q(Rz)+\beta\big(\tfrac{RK+1}{rK+1}\big)^nQ(rz)|$$

$$\leq|\lambda|MK^n|R^n+\beta r^n\left(\frac{RK+1}{rK+1}\right)^n)|+|\lambda|M|1+\beta\Big(\frac{RK+1}{rK+1}\Big)^n|$$

on $|z|=1$.

Letting $|\lambda| \to 1$ in the last inequality, we get the desired inequality (2.5), and thus the proof is complete.

## 3. Proofs of the theorems

**Proof of Theorem 1.1.** Let $m = \min_{|z|=K} |P(z)|$, then $0 < m \leq |P(z)|$ for $|z| = K$. Therefore if $\lambda$ is a complex number such that $|\lambda| < 1$, then by Rouche's theorem, it follows that the polynomial

$$G(z) = P(z) - \frac{\lambda}{K^n} m z^n \tag{3.1}$$

of degree $n$ has all its zeros in $|z| < K$. Applying Lemma 2.1 to the polynomial $G(z)$ with $K > 0$, $R \geq r$ and $rR \geq K^2$, for $|z| = 1$ we get

$$|G(Rz)| \geq \left(\frac{R+K}{r+K}\right)^n |G(rz)|.$$

Since $G(Rz)$ has all its zeros in $|z| < \frac{K}{R} \leq 1$, then applying Rouche's theorem, for real or complex number $\beta$ with $|\beta| \leq 1$, one can show that the polynomial

$$T(z) = G(Rz) + \beta \left(\frac{R+K}{r+K}\right)^n G(rz) \tag{3.2}$$

has all its zeros in $|z| < 1$. Substituting $G(z)$ from (3.1) into (3.2), we conclude that for every $\lambda$ with $|\lambda| < 1$, $|\beta| \leq 1$ and $|z| \geq 1$ the polynomial

$$T(z) = \{|P(Rz) + \beta \left(\frac{R+K}{r+K}\right)^n P(rz)\} - \frac{\lambda}{K^n} m\{R^n z^n + \beta r^n \left(\frac{R+K}{r+K}\right)^n z^n\} \tag{3.3}$$

is nonzero. In view of the above facts, we can conclude that for every $\beta$ with $|\beta| \leq 1$ and $|z| \geq 1$

$$|P(Rz) + \beta \left(\frac{R+K}{r+K}\right)^n P(rz)| \geq \frac{1}{K^n} |R^n + \beta r^n \left(\frac{R+K}{r+K}\right)^n |m|z|^n. \tag{3.4}$$

We prove our conclusion by contradiction. If the inequality (3.4) is not true, then there exists a point $z = z_0$ with $|z_0| \geq 1$, such that

$$K^n |P(Rz_0) + \beta \left(\frac{R+K}{r+K}\right)^n P(rz_0)| < |R^n + \beta r^n \left(\frac{R+K}{r+K}\right)^n |m|z_0|^n.$$

We take

$$\lambda = \frac{K^n (P(Rz_0) + \beta \left(\frac{R+K}{r+K}\right)^n P(rz_0))}{(R^n + \beta r^n \left(\frac{R+K}{r+K}\right)^n) m z_0^n},$$

and observe that $|\lambda| < 1$. In view of (3.3), it is easy to see that with this choice of $\lambda$, we have $T(z_0) = 0$ for $|z_0| \geq 1$, yielding a contradiction to the fact that $T(z) \neq 0$ for $|z| \geq 1$. This completes the proof.

**Proof of Theorem 1.4.** By the assumption the polynomial $P(z)$ has all its zeros in $|z| \geq K$. Let $m = \min_{|z|=K} |P(z)|$. Then $|P(z)| \geq m$ for $|z| = K$. If $\alpha$ is a complex number such that $|\alpha| < 1$, then it follows from Rouche's theorem that the polynomial $H(z) = P(z) - \alpha m$ has no zeros in $|z| < K$. Hence by Lemma 2.2, we get for $|z| = 1$,

$$\left|\{P(RK^2z) + \beta\left(\frac{RK+1}{rK+1}\right)^n P(rK^2z)\} - \alpha m\{1 + \beta\left(\frac{RK+1}{rK+1}\right)^n\}\right|$$

$$\leq K^n|\{Q(Rz) + \beta\left(\frac{RK+1}{rK+1}\right)^n Q(rz)\} - m\bar{\alpha}z^n\{R^n + \beta r^n\left(\frac{RK+1}{rK+1}\right)^n\}|.$$

By a proper choice of argument of $\alpha$, we get

$$|P(RK^2z) + \beta\left(\frac{RK+1}{rK+1}\right)^n P(rK^2z)| - |\alpha|m\left|1 + \beta\left(\frac{RK+1}{rK+1}\right)^n\right|$$

$$\leq |K^n|Q(Rz) + \beta(\frac{RK+1}{rK+1})^n Q(rz)| - mK^n|\alpha||z|^n\left|R^n + \beta r^n\left(\frac{RK+1}{rK+1}\right)^n\right|. \tag{3.5}$$

An application of Theorem 1.1 to the polynomial $Q(z)$ yields

$$|Q(Rz) + \beta\left(\frac{RK+1}{rK+1}\right)^n Q(z)| \geq |\alpha|m|R^n + \beta r^n\left(\frac{RK+1}{rK+1}\right)^n|, \quad |z| = 1,$$

and hence (3.5) can be rewritten as

$$|P(RK^2z) + \beta\left(\frac{RK+1}{rK+1}\right)^n P(rK^2z)| - |\alpha|m|1 + \beta(\frac{RK+1}{rK+1})^n|$$

$$\leq K^n|Q(Rz) + \beta\left(\frac{RK+1}{rK+1}\right)^n Q(z)| - |\alpha|mK^n|R^n + \beta r^n\left(\frac{RK+1}{rK+1}\right)^n|.$$

Therefore

$$|P(RK^2z) + \beta\left(\frac{RK+1}{rK+1}\right)^n P(rK^2z)| - K^n|Q(Rz) + \beta\left(\frac{RK+1}{rK+1}\right)^n Q(z)|$$

$$\leq m|\alpha|\left|1 + \beta\left(\frac{RK+1}{rK+1}\right)^n\right| - |\alpha|mK^n\left|R^n + \beta r^n\left(\frac{RK+1}{rK+1}\right)^n\right|, \quad |z| = 1.$$

Letting $|\alpha| \to 1$, we obtain for $|z| = 1$,

$$|P(RK^2z) + \beta(\frac{RK+1}{rK+1})^n P(rK^2z)| - K^n|Q(Rz) + \beta(\frac{RK+1}{rK+1})^n Q(z)|$$

$$\leq -\{|K^n|R^n + \beta r^n\left(\frac{RK+1}{rK+1}\right)^n| - |1 + \beta\left(\frac{RK+1}{rK+1}\right)^n|\}. \tag{3.6}$$

Next, by Lemma 2.4, we have

$$|P(RK^2z) + \beta\left(\frac{RK+1}{rK+1}\right)^n P(rK^2z)| + K^n|Q(Rz) + \beta\left(\frac{RK+1}{rK+1}\right)^n Q(rz)|$$

$$\leq \{K^n|R^n + \beta r^n\left(\frac{RK+1}{rK+1}\right)^n| + \left|1 + \beta\left(\frac{RK+1}{rK+1}\right)^n\right|\} \max_{|z|=K} |p(z)|, \quad |z| = 1. \tag{3.7}$$

Finally, adding (3.6) and (3.7) and rearranging, we get the desired result. Hence the proof is complete. We conclude the paper by the following remark.

**Remark 3.1.** *It would be of interest to find the analogues of the above theorems for polynomials all of whose critical points lie within a unit distance away from each root.*

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