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SOLVING DECONVOLUTION TYPE PROBLEMS BY WAVELET DECOMPOSITION METHODS

E. P. SERRANO, M. I. TROPAREVSKY AND M. A. FABIO

Universidad de San Martín, Argentina,
Universidad de Buenos Aires, Argentina,
E-mails: *eduardo.eduser@gmail.com; mariainestro@gmail.com; mfabio@unsam.edu.ar*

Abstract. In this paper we consider an inverse problem associated with equations of the form $\mathcal{K}f = g$, where $\mathcal{K}$ is a convolution-type operator. The aim is to find a solution f for given function g. We construct approximate solutions by applying a wavelet basis that is well adapted to this problem. For this basis we calculate the elementary solutions that are the approximate preimages of the wavelets. The solution for the inverse problem is then constructed as an appropriate finite linear combination of the elementary solutions. Under certain assumptions we estimate the approximation error and discuss the advantages of the proposed scheme.

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1. Introduction

We consider convolution type operators of the form

$$\mathcal{K}f(x) = 2\pi \int_{\mathbb{R}} \kappa(w) f(w) e^{iwx}\, dx, \tag{1.1}$$

where $\kappa \in \mathcal{C}^1(\mathbb{R})$ and f is a measurable function. Defining

$$\mathcal{D}_\kappa = \left\{\text{measurable } f : (\kappa f) \in L^2(\mathbb{R})\right\} \tag{1.2}$$

it results $\mathcal{K} : \mathcal{D}_\kappa \to L^2(\mathbb{R})$. If $g = \mathcal{K}f$, then its Fourier transform is $\widehat{g}(\omega) = (\kappa f)(\omega)$.

Although, in the general case, the integral operators $\mathcal{K}$ defined by (1.1) do not represent a convolution, by analogy, we call them *convolution-type operators*. The inverse problem (IP), associated with these operators consist in finding $f \in \mathcal{D}_\kappa$ such that

$$\mathcal{K}f = g \tag{1.3}$$

for a given $g \in L^2(\mathbb{R})$.

Note that in general we cannot assure that a solution f of the IP is the Fourier transform of a distribution $\check{f}$. In the ideal case we could construct f as : $f(\omega) = \widehat{g}(\omega)(\kappa(\omega))^{-1}$, but it is generally non viable because of its numerical instability if κ approaches to zero on a set of positive measure, or due to irregularities on $\widehat{g}$.

In order to find an approximate solution for the IP, we construct a sequence $f_J \in \mathcal{D}_\kappa \cap L^2(\mathbb{R})$ such that

$$\lim_{J\to+\infty} \|\mathcal{K}f_J - g\| = 0, \tag{1.4}$$

and the approximate solution will be the limiting function f: $f_J \to f$.

If we choose band-limited functions $f_J(\omega)$, $|\omega| \le \omega_J$, we have $f_J \in \mathcal{D}_\kappa \cap L^2(\mathbb{R})$. In that case if we set $\kappa_J(\omega)$, the restriction of $\kappa(\omega)$ to the band of frequencies, then $\mathcal{K}f_J$ is actually a convolution and we can write

$$\lim_{J\to+\infty} \|\check{\kappa}_J * \check{f}_J - g\| = 0. \tag{1.5}$$

Based on this idea, we construct f_J, the solution of the IP restricted to a compact set of frequencies, and consequently, the approximate solution f as the limit of f_J.

In this paper we choose an orthonormal wavelet basis $\psi_{jk}(x)$ associated with a hierarchical structure of the space, the multiresolution analysis (MRA) (see [15]). The scale function and the wavelets belong to the Schwartz class $\mathcal{S}$. They are smooth and infinitely oscillating functions with fast decay and compact support in two-sided bands Ω_j. They are well localized in both, time and frequency domains, and are well adapted to this problem (see [20]). Under suitable hypothesis on κ, it is possible to construct the *elementary solutions*, smooth functions μ_{jk} that are *nearly* the preimages of the wavelet basis $\psi_{jk}(x)$:

$$\begin{aligned} \mathcal{K}\mu_{jk}(x) &= (\check{\kappa} * \check{\mu}_{jk})(x) = 2\pi \int_{\Omega_j} \kappa(\omega)\mu_{jk}(\omega)e^{i\omega x}dw \\ &\cong \psi_{jk}(x). \end{aligned}$$

In this way, from the coefficients of the decomposition of g in the wavelet basis, we can estimate the components of the solution f on the subspaces generated by the elementary solutions (band limited) and the f_J are obtained. We also estimate the error of the approximation and discuss the advantages of the proposed scheme.

Integral operators, and in particular deconvolution problems, have been extensively studied, since they appear in various applications (electromagnetic measurements, design of digital filters , etc.). For instance, in [18] an efficient method for solving one dimensional deconvolution problems with noisy discrete data is presented and a regularizer is introduced, provided that the underlying operator can be decomposed into a sum of a compact and an invertible operators. Numerical aspects of this problem were analyzed in [19] and an optimal regularizer is constructed. The case of sampled independent identically distributed variables with random measurement error with deconvolution kernel density estimator is revisited in [7]. The relationship between deconvolution and correct sampling is explored in [13].

The Galerkin method for the case of integral operators with Hilbert kernels was proposed in [1]. In [11], [12] and [24] solvability and properties of the solutions of some integral and integro-differential equations were studied.
Techniques based on Wavelet Galerkin Methods and Wavelet Vaguelet Decomposition were studied in [5], [8] and [9] to find approximate solutions to IP associated to Pseudodifferential Operators (see [23]). Inverse problems associated with this kind of operators have been studied in [5], [8] and [9], where techniques based on wavelet Galerkin methods and wavelet Vaguelet decompositions were developed. In [22], using deconvolution techniques, we obtained approximate solutions for the equation $\mathcal{K}f = \check{\kappa} * \check{f} = g$ with $\kappa(\omega) = (1+\omega^2)^{-\alpha}$, $\alpha > 1$ for $n = 1$. In [22] a wavelet Vaguelet decomposition (WVD) was used to approximate f for a more general case. In [21] the inverse problem associated with a pseudodifferential operator whose symbol has separated variables is analyzed.

The present paper is organized as follows. In Section 2 we introduce the wavelet basis. In Section 3 we construct an approximate solution for IP, and analyze the approximation error. Two examples are discussed in Section 4. Finally, we state our conclusions in Section 5.

2. The Wavelet Basis

We choose a mother wavelet ψ, which is well localized in both time and frequency domains and possesses the following properties (see [14]):

- the family $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbb{Z}$, forms an orthonormal basis in $L^2(\mathbb{R})$ associated to MRA;
- $\psi \in \mathcal{S}$, where $\mathcal{S}$ is the Schwartz space, is a smooth, infinitely oscillating mother wavelet with fast decay;
- the spectrum $|\widehat{\psi}(2^{-j}\omega)|$ is supported on the two-sided band
  $\Omega_j = \{\omega : 2^j(\pi - \alpha) \le |\omega| \le 2^{j+1}(\pi + \alpha)\}$, for some $0 < \alpha \le \pi/3$.
- the numerical implementation of the associated scheme is efficient.

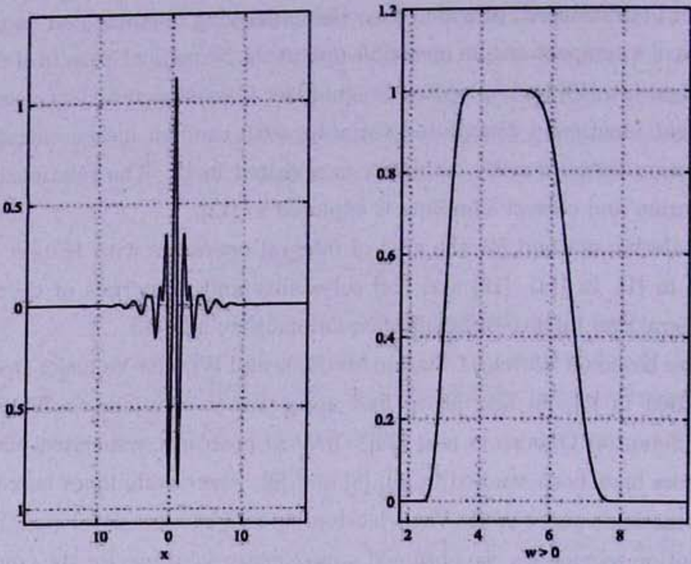

FIG. 1. (a) Mother wavelet with $\alpha = \pi/4$, (b) $|\widehat{\psi}|$ for $\omega \geq 0$

There also exists $\phi \in V_0$ such that $\{\phi(x-k),\ k \in Z\}$ is an orthonormal basis for V_0. The design of this basis and the implementation algorithm based on the Fourier fast transform (FFT) have been developed by authors in [22].

Let $W_j = \text{span}\{\psi_{jk}, k \in \mathbb{Z}\}$ and $V_J = \bigoplus_{j<J} W_j$ be the wavelet and the scales subspaces, respectively. Observe that each family $\phi_{jn}(x) = 2^{j/2}\phi(2^j x - n)$, $n \in \mathbb{Z}$, is an orthonormal basis of V_J.

For any signal $s \in L^2(\mathbb{R})$ we denote by $\mathcal{Q}_j(s)$ and $_j(s)$ the orthogonal projections of s onto the subspaces W_j and V_j, respectively. Then for any index J, the following representation holds:

$$\begin{aligned} s(x) &= \sum_{j\in\mathbb{Z}} \mathcal{Q}_j s(x) = \mathcal{P}_J s(x) + \sum_{j\geq J} \mathcal{Q}_j s(x) = \\ &= \sum_{n\in\mathbb{Z}} \langle s, \phi_{Jn}\rangle\, \phi_{Jn}(x) + \sum_{j\geq J}\sum_{k\in\mathbb{Z}} \langle f, \psi_{jk}\rangle\, \psi_{jk}(x). \end{aligned} \tag{2.1}$$

We remark that $\widehat{\mathcal{Q}}_J s(\omega)$ is supported on Ω_j, while $\widehat{\mathcal{P}}_J s(\omega)$ is supported on $\cup_{j\leq J}\Omega_j$.

Notice also that the properties of ψ ensure uniform convergence on each W_j. In addition, since ψ is infinitely oscillating, it has vanishing moments: $\int_{\mathbb{R}} x^n \psi(x)\,dx = 0$ for all $n \in \mathbb{N}_0$, and the same occurs to its polynomials' components.

3. An approximate solution to the IP

3.1. The alternative problem. Without loss of generality we can assume that g is a real function and that the real and imaginary parts of the kernel $\kappa(w)$ are even and odd functions, respectively. If it is not the case, we can split the corresponding problems into two subproblems associated to the kernels $\kappa_1(w) = \frac{\kappa(w)+\kappa^*(-w)}{2}$ and $\kappa_2(w) = \frac{\kappa(w)-\kappa^*(-w)}{2}$, respectively.

Regarding the given function g, called also the data function, we suppose that there exist an integer J such that an appropriate approximation $g_J \in V_J$ of g is available, given by

$$g_J(x) = \mathcal{P}_J g(x) = 2\pi \int_{|w|\leq w_{max}} \kappa(\omega) f(\omega) e^{ix\omega}\, d\omega,$$

where $\omega_{max} = 2^{J+1}(\pi+\alpha)$. Then we consider the following alternative problem:

$$\mathcal{K}f = g_J. \tag{3.1}$$

Taking into account that

$$g_J(x) = \sum_{j<J} \mathcal{Q}_j g(x), \tag{3.2}$$

it seems natural to solve the IP on each subspace W_j:

$$g_j(x) = \mathcal{Q}_j g(x) = 2\pi \int_{\Omega_j} \kappa(\omega) f(\omega) e^{ix\omega}\, d\omega,$$

$$g_j(x) \triangleq \mathcal{K}_j f(x). \tag{3.3}$$

Observe that, in general, the subspaces W_j are not in the rank of the operator and we cannot assure that the problem (3.3) has a solution in the strict sense. In order to solve it, we propose to construct functions $f_j \in \mathcal{D}_\kappa \cap L^2(\mathbb{R})$ to satisfy

$$\mathcal{K}_j f_j(x) = g_j(x) + e_j$$

with $\|e_j\| \leq \epsilon \parallel g_j \parallel$ for some small ϵ.

In that case we will have

$$g_J = \sum_{j<J} \mathcal{K}_j f_j + e_J = \mathcal{K}\left(\sum_{j<J} f_j\right) + e_J$$

with $\|e_J\| \leq \epsilon \parallel g_J \parallel$. We remark that each f_j is a band limited function supported on Ω_j, and hence the approximate solution $f_J = \sum_{j<J} f_j$ is a band limited function in $\mathcal{D}_\kappa \cap L^2(\mathbb{R})$. We hope that under suitable assumptions, the sequence f_J will converge in some sense, to be explained later, to an appropriate approximate solution of

(1.3). In order to construct such solutions f_J, in the next subsection we consider the decomposition (3.2) and define the *preimages* of the wavelet basis.

3.2. The elementary solutions. In this subsection we construct the *elementary solutions* μ_{jk} to be the preimages of the wavelet basis ψ_{jk}. To this end, we will distinguish the following two cases:

- the functions μ_{jk} are the preimages of the wavelet basis ψ_{jk}: $\mathcal{K}_j\mu_{jk} = \psi_{jk}$.
- the wavelet basis ψ_{jk} is not in the rank of $\mathcal{K}_j$.

In the first case, it is worth noting that if $|\kappa(w)| \geq c > 0$ on Ω_j, then we can compute $\mu_{jk} = \frac{\widehat{\psi}_{jk}}{\kappa}$.

In the second case, where the wavelets ψ_{jk} are not in the rank of the operator $\mathcal{K}_j$, we can construct approximate preimages μ_{jk} that satisfy $\mathcal{K}_j\mu_{jk} = \widetilde{\psi}_{jk} \cong \psi_{jk}$.

We observe that if $|\kappa(\omega)| \equiv 0$ on some interval $\omega_1 \leq |\omega| \leq \omega_2$ in Ω_j, then the same occurs to $|kf_j(\omega)|$. In this case the problem $\mathcal{K}_j f_j = g_j$ either has no unique solution or it is incompatible. For these reasons, in what follows we assume:

Hypothesis 3.1. *The set $\emptyset_j = \{\omega \in \Omega_j : |\kappa(\omega)| = 0\}$ has null measure for all $j \in \mathbb{Z}$.* Then the eventual roots of κ must be isolated.

Definition 3.1. Under the Hypothesis 3.1 the *elementary solutions* μ_{jk}, $j, k \in \mathbb{Z}$, for the operator $\mathcal{K}_j$ are defined by

$$\mu_{jk}(\omega) = \frac{\widehat{\psi}_{jk}(\omega)\, \kappa^*(\omega)}{|\kappa(\omega)|^2 + \rho_j^2(\omega)}\,,\ \omega \in \Omega_j, \tag{3.4}$$

where ρ_j are even smooth functions satisfying $\rho_j(\omega) \geq \rho > 0$ in some neighborhood of the roots of $\kappa(\omega)$ in Ω_j.

Definition 3.2. For $j, k \in \mathbb{Z}$ we define $\widetilde{\psi}_{jk} \triangleq \mathcal{K}_j\mu_{jk}$. For the following proposition we refer to [21].

Proposition 3.1. *Under the Hypothesis 3.1 the following properties hold:*

- $\mu_{jk}(\omega) = \mu_{j0}(\omega)\, e^{-i\omega k/2^j}$, *that is,* $\check{\mu}_{jk}(x) = \check{\mu}_{j0}(x - 2^{-j}k)$.
- *If μ_{j0} is a band limited function on Ω_j, then $\check{\mu}_{j0}$ inherits the smoothness properties of ψ.*
- *If $|k(w)| \geq c > 0$, then $\rho_j \equiv 0$ on Ω_j. On the other hand, ρ_j can be chosen based on $k(\omega)$ for each level j, such that $|support(\rho_j)| \leq \epsilon$.*
- *In such cases,* $\| \psi_{jk} - \widetilde{\psi}_{jk} \| \leq \epsilon$.

- *For each j there exist constants m_j and M_j such that for all $w \in \Omega_j$*

$$\frac{|k(w)|^2}{(|k(w)|^2+|\rho_j(w)|^2)^2} \le m_j, \ \ \|\tilde{\mu}_{jk}\|^2 \le M_j, \ k \in K_j.$$

- *The functions ρ_j can be chosen such that μ_{j0} preserves the smoothness properties of $\widehat{\psi}$.*
- *The family $\{\tilde{\mu}_{jk},\ k \in \mathbb{Z}\}$ consists of linearly independent elements. Moreover, it is a basis of the subspace $\mathcal{U}_j = span\{\tilde{\mu}_{jk},\ k \in \mathbb{Z}\} \subset L^2(\mathbb{R})$ and it is a Bessel sequence (see [4]).*

3.3. The proposed solution. Observe first that by Proposition 3.1, the families $\{\widetilde{\psi}_{jk}\}$ and $\{\psi_{jk}\}$ consist of exponentially decaying functions and are nearly biorthogonal, that is, $\left\langle \widetilde{\psi}_{jk}, \psi_{jk} \right\rangle \simeq 1$ and $\left\langle \widetilde{\psi}_{jn}, \psi_{jk} \right\rangle \simeq 0$ for $n \neq k$. Also, the family $\{\widetilde{\psi}_{jk}\}$ consist of linearly independent functions and, in addition, it is a Bessel sequence.

In order to calculate a solution for the alternative problem (3.1), for each $j < J$ we choose appropriate finite subsets $\mathbb{K}_j$, and construct an approximate solution on $\widetilde{W}_j = \text{span}\{\widetilde{\psi}_{jk}, k \in \mathbb{Z}\} \subset L_2(\mathbb{R})$ as follows: $\widetilde{\mathcal{Q}}_j g = \sum_{k\in\mathbb{K}_j} d_{jk}\widetilde{\psi}_{jk}$. We define $\widetilde{\mathcal{Q}}_j f = \sum_{k\in\mathbb{K}_j} d_{jk}\mu_{jk}$, and observe that

$$\mathcal{K}\widetilde{\mathcal{Q}}_j f = \widetilde{\mathcal{Q}}_j g = \mathcal{Q}_j g + \Delta_j g.$$

We propose to select the coefficients d_{jk} to satisfy the linear system:

$$\sum_{k\in\mathbb{K}_j} \langle g, \psi_{jk}\rangle\, \psi_{jk} = \sum_{k\in\mathbb{K}_j} d_{jk}\, \widetilde{\psi}_{jk},$$

that is,

$$\langle g, \psi_{jn}\rangle = \sum_{k\in\mathbb{K}_j} d_{jk} \left\langle \psi_{jn}, \widetilde{\psi}_{jk} \right\rangle, \ \ n \in \mathbb{K}_j,$$

yielding

$$\Delta_j g = \sum_{k\notin\mathbb{K}_j} \langle g, \psi_{jk}\rangle\,.$$

We observe that $\Delta_j g$ is a projection that is nearly orthogonal to W_j on $\widetilde{W}_j$.

Since the error of the approximation will depend on the wavelet coefficients of the data function g, it can be controlled by choosing for each $J_0 \le j < J$, the finite sets $\mathbb{K}_j$ to satisfy

$$\|\Delta_j g\|^2 = \sum_{k\notin\mathbb{K}_j} |\,\langle g, \psi_{jk}\rangle\,|^2 < \frac{\epsilon}{2}\, \|\, \mathcal{Q}_j g\,\|^2 \tag{3.5}$$

for some small $\epsilon > 0$.

To carry out the numerical implementation of the scheme, we choose a minimum level J_0, such that

$$\| g_J - \sum_{j<J_0} \mathfrak{Q}_j g \|^2 < \frac{\epsilon}{2} \| g_J \|^2 . \tag{3.6}$$

Finally,

$$\widetilde{f}_J = \sum_{J_0 \le j < J} \widetilde{\mathfrak{Q}}_j f = \sum_{J_0 \le j < J} \sum_{k \in \mathbb{K}_j} d_{jk} \mu_{jk}$$

is an approximate solution for the IP associated with the alternative problem (3.1).

We have

$$\mathcal{K} \widetilde{f}_J = \sum_{J_0 \le j < J} \widetilde{\mathfrak{Q}}_j g = \sum_{J_0 \le j < J} \sum_{k \in \mathbb{K}_j} \langle g, \psi_{jk} \rangle \, \psi_{jk} = g_J + \Delta_J g.$$

3.4. **The Error.** Under the hypothesis and definitions described above, from (3.5) - (3.6) we have

$$\begin{aligned} \| \Delta_J g \|^2 &= \| g_J - \mathcal{K} \widetilde{f}_J \|^2 \\ &= \| \sum_{j<J_0} \mathfrak{Q}_j g \|^2 + \sum_{J_0 \le j < J} \sum_{k \notin \mathbb{K}_j} | \langle g, \psi_{jk} \rangle |^2 \\ &< \frac{\epsilon}{2} \| g_J \|^2 + \frac{\epsilon}{2} \sum_{J_0 \le j < J} \| \mathfrak{Q}_j g \|^2 = \epsilon \| g_J \|^2 \end{aligned}$$

Assuming that the initial approximate data function g_J satisfies

$$\| g - g_J \|^2 = \| \sum_{j \ge J} \langle g, \psi_{jk} \rangle \, \psi_{jk} \|^2 \| g \|^2 < \epsilon,$$

we obtain $\| g - \mathcal{K} \widetilde{f}_J \|^2 < 2\epsilon$.

Remark 3.1. We note that it is not always convenient or even possible to disregard the low frequency components around $\omega = 0$. In particular, this is the case if g is not an oscillating function or if it is a distribution. In such cases it is important to include the component in the scale space V_{J_0}, and consequently the low frequencies of elementary solutions must be defined and included in the approximation scheme.

4. Examples

Example 4.1. We consider integro-differential operators with kernels:

$$\kappa(\omega) = (1 + |\omega|^2)^{\alpha},$$

where $0 < |\alpha| < 1$. In this case the kernel $\kappa(\omega)$ is real and even.

Note that if $\alpha \le -\frac{1}{2}$ we have $\kappa \in L^2(\mathbb{R})$ and $\mathcal{D}_\kappa \supseteq L^2(\mathbb{R})$. On the other hand, if $\alpha > -\frac{1}{2}$, then $\mathcal{D}_\kappa \subset L^2(\mathbb{R})$. Since $\kappa(\omega) > 0$, we have $\widetilde{\psi}_{jk}(\omega) = \psi_{jk}$ for all j and k.

Example 4.2. We consider transfer functions of the form:

$$H(s) = \frac{P(s)}{Q(s)},$$

where P and Q are polynomials of degree m and n $(m < n)$, respectively. Observe that H describes the relationship between the input U and the output Y of a linear time-invariant system:

$$Y(s) = H(s)U(s) = \frac{P(s)}{Q(s)}U(s).$$

When the poles and zeros of H lie in the half-plane $\mathbb{C}^-$, the system is stable and we have $\kappa(\omega) = H(\imath\omega) \in L^2(\mathbb{R})$. In this case, for a given output Y it is possible to identify the input U based on the relationship between the Fourier transform and the convolution operator. In this case the assertion $\widetilde{\psi}_{jk}(\omega) = \psi_{jk}$ remains true.

When the output Y is a distribution, it is necessary to consider the low frequency elementary solutions as explained in the Remark 3.1.

In this frame, we could also propose an identification scheme, that is, a scheme that identifies H, considering the kernels $\kappa_n = U_n$ as test-inputs.

5. Conclusions

In this paper approximate solutions for inverse problems, associated with the equation $\mathcal{K}f = g$, where $\mathcal{K}$ is a convolution-type operator, are constructed, and the corresponding approximation error is analyzed. A perturbed data function g is considered. The data function g is decomposed in a suitable orthonormal wavelet basis ψ_{jk} and the elementary solutions $\tilde{\mu}_{jk}$ are calculated, which are the preimages of the wavelet basis trough $\mathcal{K}$. The case when ψ_{jk} are not in the rank of $\mathcal{K}$ is also studied. In both cases the data function g can be expressed as a linear combination of the images of the elementary solutions via the operator $\mathcal{K}$. The approximate solution of the IP is constructed as an appropriate finite linear combination of the elementary solutions. In this way the proposed scheme takes into account both, the characteristics of the data we want to invert and the properties of the underlying operator.

The proposed scheme can be adapted to make it applicable for the cases of noisy data and for convolution-type operators with more general kernels.

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