Известия НАН Армении. Математика, том 47, н. 4, 2012, стр. 73-80.[1]

THE RIEMANN–HILBERT BOUNDARY VALUE PROBLEM FOR MATRICES ON NON-SMOOTH ARC

B. A. KATS, S. R. MIRONOVA AND A. YU. POGODINA

Kazan Federal University, Russia
Kazan State Technical University, Russia
Saratov State Technical University, Russia
E-mail: *srmironova@yandex.ru*

Abstract. We consider the Riemann–Hilbert boundary value problem for holomorphic matrices (the Fokas–Its–Kitaev version) on certain class of non-smooth arcs. The main result is sufficient condition for its solvability.

MSC2010 number: 30E25.

Keywords: Riemann–Hilbert boundary value problem; countable set of curves; non-smooth arc.

1. Introduction

Let Γ be a Jordan arc in complex plane $\mathbb{C}$. We consider boundary value problem on evaluation of holomorphic in $\mathbb{C} \setminus \Gamma$ matrix

$$Y(z) = \begin{pmatrix} Y_{11}(z) & Y_{12}(z) \\ Y_{21}(z) & Y_{22}(z) \end{pmatrix}$$

such that

$$Y^+(t) = Y^-(t)G(t), t \in \Gamma \setminus \{a_1, a_2\}, \tag{1.1}$$

where $Y^+(t)$ and $Y^-(t)$ stand for boundary values of matrix Y at a point $t \in \Gamma \setminus \{a_1, a_2\}$ from the left and from the right correspondingly, a_1 and a_2 are starting and end points of Γ, and $G(t)$ is defined on Γ triangular matrix

$$G(t) = \begin{pmatrix} 1 & w(t) \\ 0 & 1 \end{pmatrix}.$$

In addition, the matrix Y must satisfy certain restrictions

$$Y(z) = O(|z - a_j|^{-\gamma}), \gamma = \gamma(Y) < 1, z \to a_j, j = 1, 2, \tag{1.2}$$

[1]The research is partially supported by Russian Foundation for Basic Researches, grants no. 12-01-00636 and 12-01-97015-r_povolzhie_a.

on its growth near points a_1, a_2, and condition

$$Y(z) = \left(I + O(z^{-1})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, z \to \infty, \tag{1.3}$$

where I stands for unit matrix and n is fixed positive integer.

This version of the matrix Riemann–Hilbert boundary value problem was studied first by A. Fokas, A. Its and A. Kitaev [1]. It has numerous applications in theory of orthogonal polynomials, theory of rational approximations and other branches of analysis (see, for instance, [2], [3]). But all known results on this problem concern the cases of very smooth arc Γ. In the present paper we solve it on certain class of non-smooth arcs.

In the next two sections we prove auxiliary results concerning the jump problem. Then in the last section we establish a sufficient condition for solvability of the problem (1.1) on non-smooth arcs.

2. The jump problem on countable set of disjoint closed curves

Let D be finite measurable domain of the complex plane with boundary Γ.

Definition 2.1. *The value*

$$S_\alpha(D) = \iint_D \frac{dxdy}{(dist(z,\Gamma))^\alpha}, \quad z = x + iy,$$

is called α-size of the domain D.

Obviously, $S_0(D)$ is area of D.

Lemma 2.1. *If boundary Γ of domain D is rectifiable Jordan curve, then for $0 \leq \alpha < 1$ the $\alpha-$size of D is finite, and $S_\alpha(D) \leqslant C\lambda(\Gamma)\omega^{1-\alpha}(\Gamma)$, where $\lambda(\Gamma)$ is length of Γ, $\omega(\Gamma)$ is diameter of the most disk lying inside D, and the constant C depends only on α.*

Proof. We consider the Whitney decomposition $W(D)$ of domain D (see, for instance, [4]). It consists of mutually disjoint dyadic squares $Q \subset D$ such that

$$diam Q \leq dist(Q, \Gamma) \leq 4 diam Q.$$

We denote by m_n the number of squares from $W(D)$ with side 2^{-n}. Then

$$S_\alpha(D) = \sum_{Q \in W(D)} \iint_Q \frac{dxdy}{(dist(z,\Gamma))^\alpha} \leq C \sum_{2^{-n} \leq \omega(D)} 2^{-n(2-\alpha)} m_n.$$

Clearly, $m_n \leq \lambda(D)2^n$. Let n_0 be the least n satisfying inequality $2^{-n} \leq \omega(D)$. Then

$$S_\alpha(D) \leq C\lambda(D) \sum_{n=n_0}^{\infty} (2^{-n})^{1-\alpha} \leq C\lambda(D)2^{-n_0(1-\alpha)} \leq C\lambda(D)\omega^{1-\alpha}(D),$$

where C stands for various constants depending only on α. □

We consider contours $\Gamma = \bigcup_{j=1}^{\infty} \Gamma_j$ consisting of mutually disjoint rectifiable Jordan closed curves $\Gamma_j, j = 1, 2, 3, \ldots$, which are boundaries of finite domains D_j: $\Gamma_j = \partial D_j$, $\overline{D_j} \cap \overline{D_k} = \emptyset$ for $j \neq k$. Let $L(\Gamma)$ be the set of limit points of the contour Γ, i.e.,

$$L(\Gamma) = \{a \in \overline{\mathbb{C}} : a = \lim_{j\to\infty} z_j, z_j \in \Gamma_j\}.$$

Thus $D^- := \overline{\mathbb{C}} \setminus \bigcup_{j=1}^{\infty} \overline{D_j}$.

Definition 2.2. *The class $\mathfrak{F}$ consists of all contours $\Gamma = \bigcup_{j=1}^{\infty} \Gamma_j$ such that their limit sets $L(\Gamma)$ contain only finite number of points, and all these points are finite.*

For $\Gamma \in \mathfrak{F}$ we introduce $\alpha-$size by equality

$$S_\alpha(\Gamma) = \sum_{j=1}^{\infty} S_\alpha(D_j).$$

As we have shown,

$$S_\alpha(\Gamma) \leq C \sum_{j=1}^{\infty} \lambda(D_j)\omega^{1-\alpha}(D_j). \tag{2.1}$$

Below we denote $H_\nu(A)$ the set of all functions satisfying the Hölder condition with exponent ν on a set $A \subset \mathbb{C}$. This condition consists of the finiteness of value

$$h_\nu(g, A) := \sup\{\frac{|g(t') - g(t'')|}{|t' - t''|^\nu} : t', t'' \in A, t' \neq t''\} < \infty. \tag{2.2}$$

Definition 2.3. *The class $\mathfrak{H}_\nu(\Gamma)$ consists of functions $f(t)$ defined on $\Gamma \in \mathfrak{F}$ such that $f_j := f|_{\Gamma_j} \in H_\nu(\Gamma_j)$ for $j = 1, 2, \ldots$, and*

$$h_\nu(f, \Gamma) := \sup\{h_\nu(f_j, \Gamma_j) : j = 1, 2, \ldots\}, \|f\|_{C(\Gamma)} := \sup\{|f(t)| : t \in \Gamma\}$$

are finite.

Theorem 2.1. *Let $\Gamma \in \mathfrak{F}$, $f \in \mathfrak{H}_\nu(\Gamma)$ and $\nu > \frac{1}{2}$. If $S_{1-\nu}(\Gamma) < \infty$ then the series*

$$\Phi(z) = \sum_{j=1}^{\infty} \frac{1}{2\pi i} \int_{\Gamma_j} \frac{f(t)dt}{t - z} \tag{2.3}$$

converges in $\mathbb{C} \setminus \overline{\Gamma}$ to holomorphic function $\Phi(z)$. This function vanishes at ∞, and at any point $t \in \Gamma_j, j = 1, 2, \ldots$ it has limit values $\Phi^+(t) := \lim_{D_j \ni z \to t} \Phi(z)$ and

$\Phi^-(t) := \lim_{D^-\ni z\to t}\Phi(z)$ *related by equality*

$$\Phi^+(t) - \Phi^-(t) = f(t), t \in \Gamma. \tag{2.4}$$

In addition, if $S_{p(1-\nu)}(\Gamma) < \infty$ *for some* $p > 2$, *then* Φ *is bounded in the whole complex plane.*

Proof. We consider the Whitney continuation f_j^w of function f_j from the curve Γ_j into the whole complex plane (see [4]). Let $\phi(z) = \sum_{j=1}^{\infty}\chi_j(z)f_j^w(z)$, where $\chi_j(z)$ is characteristic function of the set D_j. According well known properties of the Whitney extension, ϕ is differentiable in $\mathbb{C}\setminus\overline{\Gamma}$ and $|\nabla\phi(z)| \le h_\nu(f,\Gamma)dist^{\nu-1}(z,\Gamma)$. Hence, $|\nabla\phi|^p$ is integrable in $\mathbb{C}$ if $p(1-\nu)-$size of Γ is finite. We apply to each term of the series (2.3) the Borel–Pompeju formula (see [5]). We obtain

$$\frac{1}{2\pi i}\int_{\Gamma_j}\frac{f(t)dt}{t-z} = \chi_j(z)f_j^w(z) + \frac{1}{\pi}\iint_{D_j}\frac{\partial\phi}{\partial\overline{\zeta}}\frac{d\xi d\eta}{\zeta - z}, \zeta = \xi + i\eta,$$

and

$$\Phi(z) = \phi(z) + \frac{1}{\pi}\iint_{D}\frac{\partial\phi}{\partial\overline{\zeta}}\frac{d\xi d\eta}{\zeta - z}. \tag{2.5}$$

If $S_{1-\nu}(\Gamma) < \infty$, then $\frac{\partial\phi}{\partial\overline{\zeta}}$ is integrable, i.e., the series converges to holomorphic in $\mathbb{C}\setminus\overline{\Gamma}$ function. The existence of boundary values $\Phi^\pm(t)$ and relation (2.4) follow from Dynkin–Salimov theorem of continuity of the Cauchy type integral over rectifiable curve for $\nu > \frac{1}{2}$ (see [6], [7]). The equality $\Phi(\infty) = 0$ is obvious. Finally, if $S_{p(1-\nu)}(\Gamma) < \infty$ for $p > 2$, then $\frac{\partial\phi}{\partial\overline{\zeta}}$ is integrable with power $p > 2$, and the last term of (2.5) is continuous in the whole complex plane (see [5]). Thus, under this restriction the function Φ is bounded. □

This theorem gives a sufficient condition for solvability of jump boundary value problem (2.4) on countable set of closed rectifiable curves. Another sufficient conditions for its solvability can be found in [8] (see also bibliography in this book) and [9].

The inequality (2.1) immediately implies

Corollary 2.1. *Let* $\Gamma \in \mathfrak{F}$, $f \in \mathfrak{H}_\nu(\Gamma)$ *and* $\nu > \frac{1}{2}$. *If the series*

$$\sum_{j=1}^{\infty}\lambda(D_j)\omega^\beta(D_j) \tag{2.6}$$

converges for $\beta = \nu$, *then the jump problem (2.4) is solvable, and if this series converges for some* $\beta < 2\nu - 1$, *then the jump problem (2.4) has a bounded solution.*

For $\nu = 1$ the sizes $S_{1-\nu}(\Gamma) = S_{p(1-\nu)}(\Gamma) = S_0(\Gamma)$ are equal to the sum of areas of domains D_j, and, consequently, are finite.

We come to the following corollary.

Corollary 2.2. *If* $\Gamma \in \mathfrak{F}$ *and* $f \in \mathfrak{H}_1(\Gamma)$*, then the series (2.3) converges to a bounded solution of the jump problem (2.4).*

If all curves Γ_j are piecewise smooth, then the Cauchy type integral over this curve has continuous boundary values for any positive exponent ν (see [10], [11]).

Corollary 2.3. *Let* $\Gamma \in \mathfrak{F}$ *consist of piecewise–smooth curves* $\Gamma_j, j = 1, 2, \dots,$ *and* $f \in \mathfrak{H}_\nu(\Gamma)$*. Then Theorem 2.1 keeps validity for any positive exponent* ν*.*

3. Non-smooths arcs with smooth skeleton

Let us consider the following example. We put $Y(x) = x\sin\frac{\pi}{x}$ for $0 < x \leq 1$, $Y(0) = 0$, $\Gamma = \{z = x + iy : 0 \leq x \leq 1, y = Y(x)\}$. The arc Γ begins at the point 0 and ends at the point 1; it loses smoothness at its starting and has infinite length. Let I be segment $[0, 1]$ with the same beginning and end points. We can represent the union $\Gamma \cup I$ in the form $\Gamma \cup I = \cup_{j=1}^{\infty}\Gamma_j$, where $\Gamma_j = \partial D_j$ and

$$D_j = \{z = x + iy : (j+1)^{-1} < x < j^{-1}, 0 < (-1)^j y < (-1)^j x \sin\frac{\pi}{x}\},$$

$j = 1, 2, 3, \dots$. The even curves Γ_{2j} are situated in upper half of the plane, and their intrinsic orientation is opposite to orientation of Γ. The odd curves Γ_{2j-1} are situated in lower half of the plane and directed along Γ. Obviously,

$$\int_\Gamma - \int_I = \sum_{j=1}^{\infty}\int_{\Gamma_{2j-1}} - \sum_{j=1}^{\infty}\int_{\Gamma_{2j}},$$

and both contours $\Gamma_{odd} := \{\Gamma_1, \Gamma_3, \Gamma_5, \dots\}$ and $\Gamma_{even} := \{\Gamma_2, \Gamma_4, \Gamma_6, \dots\}$ belong to the class $\mathfrak{F}$. Let us give general description of this phenomenon.

Definition 3.1. *Let* Γ *be a Jordan arc beginning at point* a_1 *and ending at point* a_2*. We say that arc* γ *with the same starting and end points is its skeleton if* $\overline{\Gamma\triangle\gamma} = \Gamma^+ \cup \Gamma^-$*, where* Γ^+ *(correspondingly,* Γ^-*) is a contour of class* $\mathfrak{F}$ *consisting of mutually disjoint closed curves oriented positively (correspondingly, negatively) with respect to their interior domains, and* $L(\Gamma^\pm) \subset \{a_1, a_2\}$*. We denote by* $\mathfrak{S}$ *the class of all arcs with smooth skeletons.*

We introduce $\alpha-$size of arc $\Gamma \in \mathfrak{S}$ by equality $S_\alpha(\Gamma) = S_\alpha(\Gamma^+) + S_\alpha(\Gamma^-)$. Generally speaking, an arc $\Gamma \in \mathfrak{S}$ has infinite length.

We consider the jump problem on arc Γ, i.e., the problem on evaluation of holomorphic function $\Phi(z)$ in $\overline{\mathbb{C}}\setminus\Gamma$ such that $\Phi(\infty)=0$, and

$$\Phi^+(t)-\Phi^-(t)=f(t),\quad t\in\Gamma\setminus\{a_1,a_2\}, \tag{3.1}$$

$$\Phi(z)=O(|z-a_j|^{-\gamma}),\ z\to a_j,\ j=1,2,\quad \gamma=\gamma(\Phi)\in[0,1). \tag{3.2}$$

Here $\Phi^+(t)$ and $\Phi^-(t)$ stand for limit values of desired function Φ on Γ from the left and from the right correspondingly.

Let $\Gamma\in\mathfrak{S}$, γ is its smooth skeleton, $f\in H_\nu(\Gamma)$. As above, we denote by f^w the Whitney extension of f from Γ onto the whole complex plane. Obviously, $f^w|_{\Gamma\cup\gamma}\in H_\nu(\Gamma\cup\gamma)$. Then

$$\frac{1}{2\pi i}\int_\Gamma\frac{f(t)dt}{t-z}=\frac{1}{2\pi i}\int_\gamma\frac{f^w(t)dt}{t-z}+\frac{1}{2\pi i}\int_{\Gamma^+}\frac{f^w(t)dt}{t-z}-\frac{1}{2\pi i}\int_{\Gamma^-}\frac{f^w(t)dt}{t-z}.$$

The first term in the right-hand side is the Cauchy type integral over smooth arc γ with density from the Hölder class $H_\nu(\gamma)$. It has jump f^w on $\gamma\setminus\{a_1,a_2\}$ and logarithmic singularities at the points $a_{1,2}$ (see [10], [11]). The second and third terms are series of the Cauchy type integrals over sets of closed curves of the class $\mathfrak{F}$. If $S_{p(1-\nu)}(\Gamma)<\infty$ for some $p>2$, then both series converge to bounded functions with jump f^w on these curves by virtue of Theorem 2.1. Thus, we proved the following assertion.

Theorem 3.1. *Let $\Gamma\in\mathfrak{S}$, $f\in H_\nu(\Gamma)$, $\nu>\frac{1}{2}$ and $S_{p(1-\nu)}(\Gamma)<\infty$ for some $p>2$. Then the Cauchy type integral in the right-hand side of equality*

$$\Phi(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f(t)dt}{t-z} \tag{3.3}$$

converges, and the function $\Phi(z)$ is solution of the jump problem (3.1) satisfying condition (3.2).

Analogs of Corollaries 2.1, 2.2, 2.3 are valid, too. We cite an analog of Corollary 2.2.

Corollary 3.1. *Let $\Gamma\in\mathfrak{S}$ and $f\in H_1(\Gamma)$. Then the Cauchy type integral in the right-hand side of equality (3.3) converges and gives a solution of the jump problem (3.1) satisfying condition (3.2).*

4. The Riemann–Hilbert boundary problem for matrices

Now we solve the problem (1.1) in the class of holomorphic matrices satisfying conditions (1.2) and (1.3). We assume that the arc Γ has smooth skeleton, i.e., belongs to the class $\mathfrak{S}$. The entries of desired matrix Y satisfy the following relations:

$$Y_{11}^{+}(t) = Y_{11}^{-}(t), t \in \Gamma \setminus \{a_1, a_2\}, \quad Y_{11}(z) = z^n + O(z^{n-1}), \tag{4.1}$$

$$Y_{12}^{+}(t) = Y_{12}^{-}(t) + Y_{11}^{-}(t)w(t), t \in \Gamma \setminus \{a_1, a_2\}, \quad Y_{12}(z) = O(z^{-n-1}),$$

$$Y_{21}^{+}(t) = Y_{21}^{-}(t), t \in \Gamma \setminus \{a_1, a_2\}, \quad Y_{21}(z) = O(z^{n-1}),$$

$$Y_{22}^{+}(t) = Y_{22}^{-}(t) + Y_{21}^{-}(t)w(t), t \in \Gamma \setminus \{a_1, a_2\}, \quad Y_{22}(z) = z^{-n} + O(z^{-n-1}),$$

where the second equalities concern behavior of desired functions near infinity, and

$$Y_{k,m}(z) = O(|z - a_j|^{-\gamma}), \gamma = \gamma(Y) < 1, z \to a_j, \quad j, k, m = 1, 2. \tag{4.2}$$

The equalities (4.1) and (4.2) imply that Y_{11} is polynomial of degree n with highest term z^n, i.e.,

$$Y_{11}(z) = \pi_n(z) = z^n + c_1 z^{n-1} + \cdots + c_{n-1} z + c_n. \tag{4.3}$$

Analogously, Y_{21} is polynomial

$$Y_{21}(z) = \tilde{\pi}_{n-1}(z) = \tilde{c}_0 z^{n-1} + \tilde{c}_1 z^{n-2} + \cdots + \tilde{c}_{n-2} z + \tilde{c}_{n-1}. \tag{4.4}$$

The functions Y_{12} and Y_{22} are solutions of the jump problems

$$Y_{12}^{+}(t) - Y_{12}^{-}(t) = \pi_n(t)w(t), t \in \Gamma \setminus \{a_1, a_2\},$$

$$Y_{22}^{+}(t) - Y_{22}^{-}(t) = \tilde{\pi}_{n-1}(t)w(t), t \in \Gamma \setminus \{a_1, a_2\},$$

in the class (4.2). Under assumptions of Theorem 2 these problems have unique solutions

$$Y_{12}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\pi_n(t)w(t)dt}{t - z}, \quad Y_{22}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\tilde{\pi}_{n-1}(t)w(t)dt}{t - z}.$$

But $Y_{12}(z) = O(z^{-n-1})$, $Y_{22}(z) = z^{-n} + O(z^{-n-1})$ for $z \to \infty$. The convergence of the Cauchy type integrals allows us to rewrite these conditions in the form (see [2])

$$\int_\Gamma \pi_n(t)w(t)t^j dt = 0, 0 \le j \le n-1, \tag{4.5}$$

$$\int_\Gamma \tilde{\pi}_{n-1}(t)w(t)t^j dt = 0, 0 \le j \le n-2, \quad \int_\Gamma \tilde{\pi}_{n-1}(t)w(t)t^{n-1} dt = -2\pi i. \tag{4.6}$$

Thus, we come to the following result.

Theorem 4.1. *Let $\Gamma \in \mathfrak{S}$, $w \in H_\nu(\Gamma)$, $\nu > \frac{1}{2}$ and $S_{p(1-\nu)}(\Gamma) < \infty$ for some $p > 2$. Then the matrix Riemann–Hilbert boundary value problem (1.1) has a unique solution satisfying conditions (1.2) and (1.3) if and only if there exist polynomials (4.3) and (4.4) satisfying conditions (4.5) and (4.6) correspondingly.*

Now we describe a simple case where that polynomials exist. Assume that one of skeletons γ of the arc Γ is a segment of real axis (we can say that it has right skeleton), and function $w(t)$ is restriction on Γ of a function $w(z)$ which is holomorphic in a simply connected domain containing $\Gamma \cup \gamma$ and positive on the segment γ. Then $w \in H_1(\Gamma)$. The conditions (4.5) and (4.6) are equivalent to equalities

$$\int_\gamma \pi_n(t) w(t)\, t^j\, dt = 0, \ \ 0 \le j \le n-1,$$

$$\int_\gamma \tilde{\pi}_{n-1}(t)\, w(t)\, t^j\, dt = 0, \ \ 0 \le j \le n-2, \quad \int_\gamma \tilde{\pi}_{n-1}(t) w(t)\, t^{n-1}\, dt = -2\pi i,$$

i.e. $\pi_n(z) = P_n(z)$ and $\tilde{\pi}_{n-1}(z) = bP_{n-1}(z)$, where P_n and P_{n-1} are monic orthogonal polynomials on the segment Γ with weight $w|\gamma$ of degrees n and $n-1$ correspondingly, and b is certain constant.

Corollary 4.1. *Let $\Gamma \in \mathfrak{S}$ have straight skeleton γ, and let $w(t)$ be restriction on Γ of a function $w(z)$ which is holomorphic in a simply connected domain containing $\Gamma \cup \gamma$ and positive on the segment γ. Then the matrix Riemann–Hilbert boundary value problem (1.1) has a unique solution satisfying conditions (1.2) and (1.3).*

Список литературы

[1] A. S. Fokas, A. R. Its and A. V. Kitaev, “Discrete Painleve equations and their appearance in quantum gravity”, Comm. Math. Phys., **142**, no. 2, 313 – 344 (1991).

[2] P. Deift, Orthogonal Polynomials and Random Matrices: A Riemann – Hilbert Approach, Courant Lecture Notes, **3**, New York University (1999).

[3] A. I. Aptekarev, “Exact constants for rational approximations of analytic functions”, Mathem. Sbornic, **193**, no. 9, 3 – 72 (2002).

[4] E. M. Stein, “Singular integrals and differential properties of functions”, Princeton University Press, Princeton (1970).

[5] I. N. Vekua, Generalized analytical functions, Nauka publishers, Moscow (1988).

[6] E. M. Dynkin, “Smoothness of the Cauchy type integral”, Zapiski nauchn. sem. Leningr. dep. mathem. inst. AN USSR, **92**, 115 – 133 (1979).

[7] T. S. Salimov, “A direct estimate for a singular Cauchy integral over a closed curve”, Azerbaidzhan. Gos. Univ. Uchen. Zap., no. 5 59 – 75 [in Russian] (1979).

[8] L. I. Chibrikova, “The main boundary value problems for analytical functions”, Kazan: Publishers of Kazan University (1977).

[9] B. A. Kats, “The Cauchy type integral and the Riemann problem on countable set of closed curves”, Izvest. vuzov. Mathem., no. 3, 20 – 29 (1985).

[10] F. D. Gakhov, Boundary Value Problems, Nauka publishers, Moscow (1977).

[11] N. I. Muskhelishvili, Singular Integral Equations, Nauka publishers, Moscow (1962).

Поступила 1 ноября 2011