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THE GENERALIZED ENTROPIC PROPERTY FOR THE PAIR OF OPERATIONS

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Abstract. In this paper, we define the generalized entropic property for a pair of operations. We show that for an idempotent algebra, $A = (A, f, g)$, with two ternary operations,if one of f or g is commutative and the pair of operations (f, g) satisfies the generalized entropic property, then (f, g) is entropic. Also we prove that every idempotent and commutative algebra $A = (A, f, g)$, with a ternary and a binary operation, satisfying the generalized entropic property, is entropic.

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1. Introduction

The algebra $A = (A, F)$ is called entropic or medial if it satisfies the identity of mediality

$$g(f(x_{11}, ..., x_{n1}), ..., f(x_{1m}, ..., x_{nm})) = f(g(x_{11}, ..., x_{1m}), ..., g(x_{n1}, ..., x_{nm}) \tag{1}$$

for every n-ary $f \in F$ and m-ary $g \in F$.
In other words, A is medial if it satisfies the hyperidentity of mediality ([1], [2]). Note that a groupoid is entropic if and only if it satisfies the identity of mediality: $xy.uv \approx xu.yv$ [3].
An algebra $A = (A, f)$ with a single ternary operation is entropic if it satisfies the following identity:

$$f(f(x_{11}, x_{21}, x_{31}), f(x_{12}, x_{22}, x_{32}), f(x_{13}, x_{23}, x_{33})) \approx$$

$$\approx f(f(x_{11}, x_{12}, x_{13}), f(x_{21}, x_{22}, x_{23}), f(x_{31}, x_{32}, x_{33})).$$

A variety V is called entropic (or medial) if every algebra in V is entropic. The Algebra A is called idempotent (commutative), if every operation of A is idempotent (commutative). An n-ary operation f is called commutative if

$$f(x_1, x_2, \ldots, x_n) = f(x_{\alpha(1)}, x_{\alpha(2)}, \ldots, x_{\alpha(n)}),$$

where $\alpha \in S_n$. An n-ary operation f is called idempotent if the identity $f(x, ..., x) = x$ is true. An idempotent entropic algebra is called a mode [4].
We say that the variety V (respectively, the algebra A) satisfies the generalized entropic property if for every n-ary operation f and m-ary operation g of V (of A) there exist m-ary terms $t_1, ..., t_n$ such that the identity:

$$g(f(x_{11}, ..., x_{n1}), ..., f(x_{1m}, ..., x_{nm})) = f(t_1(x_{11}, ..., x_{1m}), ..., t_n(x_{n1}, ..., x_{nm}) \tag{2}$$

holds in V (in A) [5].
For example, a groupoid satisfies the generalized entropic property if there are binary terms t and s such that the identity $xy.uv \approx t(x, u).s(y, v)$ holds.
An algebra $A = (A, f)$ with a ternary operation satisfies the generalized entropic property if there are ternary terms t, s and r such that

$$f(f(x_{11}, x_{21}, x_{31}), f(x_{12}, x_{22}, x_{32}), f(x_{13}, x_{23}, x_{33})) \approx$$

$$\approx f(t(x_{11}, x_{12}, x_{13}), s(x_{21}, x_{22}, x_{23}), r(x_{31}, x_{32}, x_{33})).$$

It was proved by T. Evans [6] that any gpoupoid in a variety V of groupoids has the complex algebra of subalgebras if and only if V satisfies the above identity for some t and s.
Every algebra in a variety V has a complex algebra of subalgebras if and only if the variety V satisfies the generalized entropic property [5] (see also [7-11]).

2. MAIN RESULTS

The immediate consequences of the generalized entropic property in the idempotent algebra $A = (A, f)$ with a ternary operation are the following identities, that can be treated as pseudo-distributivity laws:

$$f(t(x, y, z), \alpha, \beta) \approx f(f(x, \alpha, \beta), f(y, \alpha, \beta), f(z, \alpha, \beta),$$

$$f(\beta, s(x, y, z), \alpha) \approx f(f(\beta, x, \alpha), f(\beta, y, \alpha), f(\beta, z, \alpha)),$$

$$f(\alpha, \beta, r(x, y, z)) \approx f(f(\alpha, \beta, x), f(\alpha, \beta, y), f(\alpha, \beta, z)).$$

Theorem 2.1. *An idempotent and commutative algebra $A = (A, f)$ with a ternary operation satisfying the generalized entropic property is entropic.*

Proof. Using pseudo-distributivity and the commutativity, we obtain

$$f(t(x, y, z), \alpha, \beta) \approx f(f(x, \alpha, \beta), f(y, \alpha, \beta), f(z, \alpha, \beta)) \approx$$

$$\approx f(f(\alpha, \beta, x), f(\alpha, \beta, y), f(\alpha, \beta, z)) \approx f(\alpha, \beta, r(x, y, z)) \approx f(r(x, y, z), \alpha, \beta),$$

and also

$$f(t(x,y,z),\alpha,\beta) \approx f(f(x,\alpha,\beta), f(y,\alpha,\beta), f(z,\alpha,\beta)) \approx$$
$$\approx f(f(\beta,x,\alpha), f(\beta,y,\alpha), f(\beta,z,\alpha)) \approx f(\beta, s(x,y,z),\alpha).$$

Thus

$$f(t(x,y,z),\alpha,\beta) \approx f(\alpha,\beta,t(x,y,z)) \approx f(r(x,y,z),\alpha,\beta) \approx f(\beta,s(x,y,z),\alpha).$$

Now, using idempotency and the above identity we get

$$t(x,y,z) \approx f(t(x,y,z),t(x,y,z),t(x,y,z)) \approx f(r(x,y,z),t(x,y,z),t(x,y,z)) \approx$$
$$\approx f(t(x,y,z),s(x,y,z),r(x,y,z)) \approx f(x,y,z).$$

Similarly, for s and r we have $s(x,y,z) \approx f(x,y,z)$ and $r(x,y,z) \approx f(x,y,z)$. Thus, by the generalized entropic property and the last three identities we have

$$f(f(x_{11},x_{21},x_{31}), f(x_{12},x_{22},x_{32}), f(x_{13},x_{23},x_{33})) \approx$$
$$\approx f(t(x_{11},x_{12},x_{13}), s(x_{21},x_{22},x_{23}), r(x_{31},x_{32},x_{33})) \approx$$
$$\approx f(f(x_{11},x_{12},x_{13}), f(x_{21},x_{22},x_{23}), f(x_{31},x_{32},x_{33})).$$

Theorem 2.1 is proved. □

The generalized entropic property for an algebra $A = (A, f, g)$ with two ternary operations means that the following identities are true:

$$f(f(x_{11},x_{21},x_{31}), f(x_{12},x_{22},x_{32}), f(x_{13},x_{23},x_{33})) \approx$$
$$\approx f(t_1(x_{11},x_{12},x_{13}), s_1(x_{21},x_{22},x_{23}), r_1(x_{31},x_{32},x_{33})),$$
$$f(g(x_{11},x_{21},x_{31}), g(x_{12},x_{22},x_{32}), g(x_{13},x_{23},x_{33})) \approx$$
$$\approx g(t_2(x_{11},x_{12},x_{13}), s_2(x_{21},x_{22},x_{23}), r_2(x_{31},x_{32},x_{33})),$$
$$g(g(x_{11},x_{21},x_{31}), g(x_{12},x_{22},x_{32}), g(x_{13},x_{23},x_{33})) \approx$$
$$\approx g(t_3(x_{11},x_{12},x_{13}), s_3(x_{21},x_{22},x_{23}), r_3(x_{31},x_{32},x_{33})),$$
$$g(f(x_{11},x_{21},x_{31}), f(x_{12},x_{22},x_{32}), f(x_{13},x_{23},x_{33})) \approx$$
$$\approx f(t_4(x_{11},x_{12},x_{13}), s_4(x_{21},x_{22},x_{23}), r_4(x_{31},x_{32},x_{33})).$$

The immediate consequences of the generalized entropic property in an idempotent algebra $A = (A, f, g)$ with two ternary operations are the following identities that can be treated as pseudo-distributivity laws:

$$g(t(x,y,z),\alpha,\beta) \approx f(g(x,\alpha,\beta), g(y,\alpha,\beta), g(z,\alpha,\beta)),$$
$$g(\beta,s(x,y,z),\alpha) \approx f(g(\beta,x,\alpha), g(\beta,y,\alpha), g(\beta,z,\alpha)),$$
$$g(\alpha,\beta,r(x,y,z)) \approx f(g(\alpha,\beta,x), g(\alpha,\beta,y), g(\alpha,\beta,z)).$$

At last, the entropic law for an algebra $A = (A, f, g)$ with two ternary operations means the following identities:

$$(a) \qquad f(f(x_{11}, x_{21}, x_{31}), f(x_{12}, x_{22}, x_{32}), f(x_{13}, x_{23}, x_{33})) \approx$$
$$\approx f(f(x_{11}, x_{12}, x_{13}), f(x_{21}, x_{22}, x_{23}), f(x_{31}, x_{32}, x_{33}))$$
$$(b) \qquad g(g(x_{11}, x_{21}, x_{31}), g(x_{12}, x_{22}, x_{32}), g(x_{13}, x_{23}, x_{33})) \approx$$
$$\approx g(g(x_{11}, x_{12}, x_{13}), g(x_{21}, x_{22}, x_{23}), g(x_{31}, x_{32}, x_{33}))$$
$$(c) \qquad f(g(x_{11}, x_{21}, x_{31}), g(x_{12}, x_{22}, x_{32}), g(x_{13}, x_{23}, x_{33})) \approx$$
$$\approx g(f(x_{11}, x_{12}, x_{13}), f(x_{21}, x_{22}, x_{23}), f(x_{31}, x_{32}, x_{33})).$$

Definition 2.1. *Let g and f be an m-ary and an n-ary operation on the set A. We say that the pair of operations (f, g) satisfies the generalized entropic property if there exist terms $t_1, ..., t_n$ of the algebra $A = (A, f, g)$ such that identity (2) holds in the algebra, $A = (A, f, g)$. The pair of operations (f, g) is called entropic or medial, if identity (1) in the algebra $A = (A, f, g)$ is true. If $f = g$, then we say that the operation f satisfies the generalized entropic property.*

Theorem 2.2. *Let $A = (A, f, g)$ be an idempotent algebra with two ternary operations. If g is commutative and the pair (f, g) satisfies the generalized entropic property, then (f, g) is entropic.*

Proof. To prove (c), observe that by the generalized entropic property:

$$f(g(x_{11}, x_{21}, x_{31}), g(x_{12}, x_{22}, x_{32}), g(x_{13}, x_{23}, x_{33})) \approx$$
$$\approx g(t(x_{11}, x_{12}, x_{13}), s(x_{21}, x_{22}, x_{23}), r(x_{31}, x_{32}, x_{33})).$$

Using the pseudo-distributivities and the commutativity of g, we obtain:

$$g(t(x, y, z), \alpha, \beta) \approx f(g(x, \alpha, \beta), g(y, \alpha, \beta), g(z, \alpha, \beta)) \approx$$
$$\approx f(g(\alpha, \beta, x), g(\alpha, \beta, y), g(\alpha, \beta, z)) \approx g(\alpha, \beta, r(x, y, z)) \approx g(r(x, y, z), \alpha, \beta),$$

and also

$$g(t(x, y, z), \alpha, \beta) \approx f(g(x, \alpha, \beta), g(y, \alpha, \beta), g(z, \alpha, \beta)) \approx$$
$$\approx f(g(\beta, x, \alpha), g(\beta, y, \alpha), g(\beta, z, \alpha)) \approx g(\beta, s(x, y, z), \alpha).$$

But $g(t(x, y, z), \alpha, \beta) \approx b(\alpha, \beta, t(x, y, z)$. So, we have

$$g(t(x, y, z), \alpha, \beta) \approx g(\alpha, t(x, y, z), \beta) \approx g(r(x, y, z), \alpha, \beta) \approx g(\beta, s(x, y, z), \alpha).$$

By idempotency and the above identities we have

$$t(x, y, z) \approx g(t(x, y, z), t(x, y, z), t(x, y, z)) \approx g(r(x, y, z), t(x, y, z), t(x, y, z)) \approx$$

$$g(t(x,y,z),s(x,y,z),r(x,y,z)) \approx f(g(x,x,x,),g(y,y,y),g(z,z,z)) \approx f(x,y,z)$$

In the same manner $s(x,y,z) \approx f(x,y,z)$ and $r(x,y,z) \approx f(x,y,z)$. Thus, by the generalized entropic property and the last three identities we have

$$f(g(x_{11},x_{21},x_{31}),g(x_{12},x_{22},x_{32}),g(x_{13},x_{23},x_{33})) \approx$$

$$\approx g(t(x_{11},x_{12},x_{13}),s(x_{21},x_{22},x_{23}),r(x_{31},x_{32},x_{33})) \approx$$

$$\approx g(f(x_{11},x_{12},x_{13}),f(x_{21},x_{22},x_{23}),f(x_{31},x_{32},x_{33})). \qquad \square$$

The generalized entropic property for an algebra $A = (A,f,g)$ with a ternary and one binary operation (respectively f,g) means that the following identities are true

$$f(f(x_{11},x_{21},x_{31}),f(x_{12},x_{22},x_{32}),f(x_{13},x_{23},x_{33})) \approx$$

$$f(t_1(x_{11},x_{12},x_{13}),s_1(x_{21},x_{22},x_{23}),r_1(x_{31},x_{32},x_{33})),$$

$$g(g(x_{11},x_{21}),g(x_{12},x_{22})) \approx g(t_2(x_{11},x_{12}),s_2(x_{21},x_{22})),$$

$$f(g(x_{11},x_{21}),g(x_{12},x_{22}),g(x_{13},x_{23})) \approx g(t_3(x_{11},x_{12},x_{13}),s_3(x_{21},x_{22},x_{23})),$$

$$g(f(x_{11},x_{21},x_{31}),f(x_{12},x_{22},x_{32})) \approx f(t_4(x_{11},x_{12}),s_4(x_{21},x_{22}),r_4(x_{31},x_{32})).$$

The immediate consequences of the generalized entropic property in the idempotent algebra $A = (a,f,g)$ with a ternary and a binary operation (respectively f,g) are the following identities that can be treated as pseudo-distributivity laws:

$$g(t(x,y,z),\alpha) \approx f(g(x,\alpha),g(y,\alpha),g(z,\alpha)),$$

$$g(\beta,s(x,y,z)) \approx f(g(\beta,x),g(\beta,y),g(\beta,z)).$$

At last, the entropic law for an algebra $A = (A,f,g)$ with a ternary and a binary operation (respectively f,g) means the validity of the following identities

$$(d) \qquad f(f(x_{11},x_{21},x_{31}),f(x_{12},x_{22},x_{32}),f(x_{13},x_{23},x_{33})) \approx$$

$$\approx f(f(x_{11},x_{12},x_{13}),f(x_{21},x_{22},x_{23}),f(x_{31},x_{32},x_{33}))$$

$$(e) \qquad g(g(x_{11},x_{21}),g(x_{12},x_{22})) \approx g(g(x_{11},x_{12}),g(x_{21},x_{22}))$$

$$(f) \qquad f(g(x_{11},x_{21}),g(x_{12},x_{22}),g(x_{13},x_{23})) \approx g(f(x_{11},x_{12},x_{13}),f(x_{21},x_{22},x_{23})).$$

Theorem 2.3. *Let $A = (A,f,g)$ be an idempotent algebra with a ternary operation f and a binary operation g. If g is commutative and the pair (f,g) satisfies the generalized entropic property, then (f,g) is entropic.*

Proof. To prove (f) observe that by the generalized entropic property

$$f(g(x_{11},x_{21}),g(x_{12},x_{22}),g(x_{13},x_{23}))\approx g(t(x_{11},x_{12},x_{13}),s(x_{21},x_{22},x_{23})).$$

Using the pseudo-distributivities and the commutativity of g we obtain

$$g(t(x,y,z),\alpha)\approx f(g(x,\alpha),g(y,\alpha),g(z,\alpha))\approx$$
$$\approx f(g(\alpha,x),g(\alpha,y),g(\alpha,z))\approx g(\alpha,s(x,y,z)).$$

On the other hand, by idempotency and above identities we have

$$t(x,y,z)\approx g(t(x,y,z),t(x,y,z))\approx g(t(x,y,z),s(x,y,z))\approx$$
$$\approx f(g(x,x),g(y,y),g(z,z))\approx f(x,y,z).$$

In the same manner we get $s(x,y,z)\approx f(x,y,z)$. Thus, by the generalized entropic property and the last two identities we have

$$f(g(x_{11},x_{21}),g(x_{12},x_{22}),g(x_{13},x_{23}))\approx g(t(x_{11},x_{12},x_{13}),s(x_{21},x_{22},x_{23}))\approx$$
$$\approx g(f(x_{11},x_{12},x_{13}),f(x_{21},x_{22},x_{23})).$$ □

Corollary 2.1. *Every idempotent and commutative algebra $A=(A,f,g)$ with a ternary and a binary operations satisfying the generalized entropic property is entropic.*

Proof. We have to show that identities (d), (e) and (f) hold in the algebra $A=(A,f,g)$. The identity (d) is proved in Theorem 2.1, the identity (e) is proved analogically, and the identity (f) is proved in Theorem 2.3. □

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