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A GENERALIZATION OF FRAMES IN BANACH SPACES

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Аннотация. Fusion Banach frames satisfying property $\mathcal{S}$ have been studied. A sufficient condition for the existence of a fusion Banach frame satisfying property $\mathcal{S}$ in weakly compactly generated Banach spaces has been given. Also, a necessary and sufficient condition for a fusion Banach frame to satisfy property $\mathcal{S}$ has been given. Finally, fusion Banach frames satisfying property $\mathcal{S}$ have been characterized in terms of closeness of certain subspaces of the dual spaces in the weak*-topology.

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1. INTRODUCTION

In 1952 Duffin and Schaeffer [9] introduced frames and used them as a tool in the study of non-harmonic Fourier series. It took more than 30 years to realize the importance of frames. In 1986, Daubechies, Grossmann and Meyer [8] reintroduced frames and thereafter a wider study of the theory of frames began. Frames have many properties that make them useful in the study of function spaces, signal and image processing, filter banks, wireless communications etc. One of these intrinsic properties of a frame is that, given a frame, we can get properties of the function and reconstruct it only from the frame co-efficients. An introduction to the frame theory and its applications can be found in [1, 2, 6, 7, 11, 17].

A number of new applications cannot be modeled naturally by one single frame system. In such cases, data assigned to one single frame system becomes too large to be handled numerically. So, it would be beneficial to split large frame system into a set of smaller systems and then to process the data locally within each subsystem effectively. This requires "distributed" frame theory for a set of local frame systems. In this direction, a theory based on fusion frames was developed in [3, 4, 5] which provides a framework to deal with these applications.

The concept of frames in Hilbert spaces was extended to Banach spaces by Feichtinger and Gröchenig [10] who introduced the concept of atomic decompositions in Banach spaces. This concept was further generalized by Gröchenig [12] who introduced the notion of Banach frames for Banach spaces. Banach frames were also studied in [13, 14, 15]. Jain *et al.* [16], generalized Banach frames in Banach spaces and introduced frames of subspaces (Fusion Banach frames) for Banach spaces.

In the present paper, fusion Banach frames satisfying property $\mathcal{S}$ are introduced and studied. We prove that a weakly compactly generated Banach space has a fusion Banach frame satisfying property $\mathcal{S}$ if the Banach space contains a subspace isomorphic to c_0. A sufficient condition under which a fusion Banach frame satisfies property $\mathcal{S}$ is given together with a necessary and sufficient condition for a fusion Banach frame to satisfy property $\mathcal{S}$. Finally, fusion Banach frames satisfying property $\mathcal{S}$ are characterized in terms of closedness of certain subspaces of the dual space in the weak*-topology.

2. PRELIMINARIES

Throughout this paper, E will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* the conjugate space of E, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E, E_d an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$.

A sequence $\{x_n\}$ in E is said to be *complete* if $[x_n] = E$ and a sequence $\{f_n\}$ in E^* is said to be *total* over E if $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$. A sequence of projections $\{v_n\}$ on E is *total* on E if $x \in E$, $v_n(x) = 0$, for all $n \in \mathbb{N}$ imply $x = 0$.

A sequence $\{x_n\}$ in E is called a *Markushevic-basis* for E if there exists a sequence $\{f_n\} \subset E^*$ such that $f_i(x_j) = \delta_{ij}$ (Kronecker delta), for all $i, j \in \mathbb{N}$, $[x_n] = E$ and $\{f_n\}$ is total over E. A Banach space E is said to be an *Asplund* space if every continuous convex function defined on an open set U of E is Fréchet differentiable over a G_δ-set dense in U. A Banach space E is called *weakly compactly generated* if there is a weakly compact subset W such that the span of W is dense in E. A Banach space E is called weakly sequentially complete if weakly Cauchy sequence is weakly convergent in E.

Definition 1. [12] *Let E be a Banach space and E_d be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$. Given $\{f_n\} \subset E^*$ and $S : E_d \to E$, the pair $(\{f_n\}, S)$ is called a* Banach frame *for E with respect to E_d if*

(i) $\{f_n(x)\} \in E_d$, *for each $x \in E$,*

(ii) *there exist positive constants* A *and* B *with* $0 < A \leq B < \infty$ *such that*

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E, \tag{2.1}$$

(iii) S *is a bounded linear operator such that*

$$S(\{f_n(x)\}) = x, \quad x \in E.$$

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the Banach frame $(\{f_n\}, S)$. The operator $S : E_d \to E$ is called the *reconstruction operator* (or, the *pre-frame operator*). The inequality (2.1) is called the *frame inequality.*

The Banach frame $(\{f_n\}, S)$ is called *tight* if $A = B$ and *normalized tight* if $A = B = 1$. If removal of one f_n renders the collection $\{f_n\} \subset E^*$ no longer a Banach frame for E, then $(\{f_n\}, S)$ is called an *exact* Banach frame.

Definition 2. [16] *Let* E *be a Banach space.* $\{G_n\}$ *be a sequence of subspaces of* E *and* $\{v_n\}$ *be a sequence of non-zero linear projections such that* $v_n(E) = G_n, n \in \mathbb{N}$, $\mathcal{A}$ *be a Banach space associated with* E *and* $S : \mathcal{A} \to E$ *be an operator. Then* $(\{G_n, v_n\}, S)$ *is called a* frame of subspaces *(or,* fusion Banach frame*) for* E *with respect to* $\mathcal{A}$ *if*

(i) $\{v_n(x)\} \in \mathcal{A}$*, for each* $x \in E$*,*

(ii) *there exist positive constants* A, B $(0 < A \leq B < \infty)$ *such that*

$$A\|x\|_E \leq \|\{v_n(x)\}\|_{\mathcal{A}} \leq B\|x\|_E, \quad x \in E, \tag{2.2}$$

(iii) S *is a bounded linear operator such that*

$$S(\{v_n(x)\}) = x, \quad x \in E.$$

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the frame of subspaces $(\{G_n, v_n\}, S)$. The operator $S : \mathcal{A} \to E$ is called the *reconstruction operator* (or, the *pre-frame operator*). The inequality (2.2) is called the *fusion Banach frame inequality.*

Definition 3. *Let* E *be a Banach space and* E^* *be its conjugate space,* $(E^*)_d$ *be a Banach space of scalar-valued sequences associated with* E^**, indexed by* $\mathbb{N}$*. Let* $\{x_n\} \subset E$*. Given* $T : (E^*)_d \to E^*$*, the pair* $(\{x_n\}, T)$ *is called a* retro Banach frame *for* E^* *with respect to* $(E^*)_d$ *if*

(i) $\{f(x_n)\} \in (E^*)_d$*, for each* $f \in E^*$*.*

(ii) *there exist positive constants* A *and* B *with* $0 < A \leq B < \infty$ *such that*

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B\|f\|_{E^*}, \quad f \in E^* \tag{2.3}$$

(iii) T *is a bounded linear operator such that* $T(\{f(x_n)\}) = f$*,* $f \in E^*$*.*

The positive constants A *and* B*, respectively, are called* lower *and* upper frame bounds *of the retro Banach frame* $(\{x_n\}, T)$*. The operator* $T : (E^*)_d \to E^*$ *is called the* reconstruction operator *(or, the* pre-frame operator*). The inequality (2.3) is called the* retro frame inequality.

The following lemma is proved in [16].

Lemma 1. *Let* $\{G_n\}$ *be a sequence of subspaces of* E *and* $\{v_n\}$ *be a sequence of non-zero linear projections with* $v_n(E) = G_n$*,* $n \in \mathbb{N}$*. If* $\{v_n\}$ *is total over* E*, then* $\mathcal{A} = \{\{v_n(x)\} : x \in E\}$ *is a Banach space with norm* $\|\{v_n(x)\}\|_{\mathcal{A}} = \|x\|_E$*,* $x \in E$.

3. MAIN RESULTS

A fusion Banach frame $(\{G_n, v_n\}, S)$ is said to satisfy property $\mathcal{S}$ if

$$\left[\bigcup_{n\in\mathbb{N}} v_n^*(E^*)\right] = E^*.$$

Theorem 1. *Let* E *be a Banach space having a separable and complemented subspace* G*. Then* E *has a fusion Banach frame satisfying property* $\mathcal{S}$ *if*

$$\lim_{n\to\infty} \left\| f|_{[x_n, x_{n+1}, \ldots]} \right\| = 0,$$

for all $f \in E^*$*, where* $\{x_n\}$ *is some suitably chosen sequence in* E.

Proof: Let $\{x_n\} \subset G$ be a Markushevic basis with associated sequence of functionals $\{f_n\} \subset G^*$. Since G is complemented in E, we may write $E = G_1 \oplus [x_n]$, where G_1 is a closed linear subspace of E. Let v_1 and v be the continuous linear projections of E onto G_1 and G, respectively. Putting $G_n = [x_{n-1}]$, $n = 2, 3, \ldots$, we define $v_n(x) = f_{n-1}(v(x))x_{n-1}$, $x \in E$, $n \geq 2$. Then, for each n, v_n is a linear projection of E onto G_n. Also, $\{v_n\}$ is total over E. Therefore, by Lemma 1, there exists an associated Banach space $\mathcal{A} = \{\{v_n(x)\} : x \in E\}$ together with a reconstruction operator $S : \mathcal{A} \to E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to $\mathcal{A}$. Further, since $[\bigcup_{i=1}^{\infty} G_i] = E$,

$$\text{dist}\left(f, \left[\bigcup_{i=1}^{n} v_i^*(E^*)\right]\right) = \|f|_{[\bigcup_{i=n+1}^{\infty} G_i]}\|, \quad n \in \mathbb{N},\ f \in E^*.$$

Therefore, by hypotheses, for all $f \in E^*$,

$$\text{dist}\left(f, \left[\bigcup_{i=1}^{\infty} v_i^*(E^*)\right]\right) \to 0 \text{ as } n \to \infty.$$

Hence $f \in [\bigcup_{n=1}^{\infty} v_i^*(E^*)]$.

The following corollaries can be deduced from Theorem 1.

Corollary 1. *A weakly compactly generated Banach space E has a fusion Banach frame satisfying property $\mathcal{S}$ if E contains a subspace isomorphic to c_0.*

Corollary 2. *If an Asplund space E has a separable complemented subspace, then E has a fusion Banach frame satisfying property $\mathcal{S}$.*

In the following result, we show that if a given Banach space has fusion Banach frame satisfying property $\mathcal{S}$, then one can construct a new Banach space having a fusion Banach frame satisfying property $\mathcal{S}$.

Theorem 2. *Let a Banach space E has a fusion Banach frame $(\{G_n, v_n\}, S)$ satisfying property $\mathcal{S}$. If, for each n, F_n is a non-zero closed linear subspace of G_n, then there exist a sequence $\{w_n\}$ of projections of F onto F_n, where $F = [\bigcup_{n=1}^{\infty} F_n]$ and an associated Banach space $\mathcal{A}_0$ together with a reconstruction operator $S_0 : \mathcal{A}_0 \to F$ such that $(\{F_n, w_n\}, S_0)$ is a fusion Banach frame for F with respect to $\mathcal{A}_0$ satisfying property $\mathcal{S}$.*

Proof: Let $x \in F$. Then, for some $k \in \mathbb{N}$, there exist $y \in G_k$, $z \in \left[\bigcup_{\substack{i=1 \\ i \neq k}}^{\infty} G_i\right]$ and a sequence $\{x_n\}$ in F such that $x = y + z$ and $\lim_{n\to\infty} x_n = x$. So $\lim_{n\to\infty} v_k(x_n) = y \in F_k$. Also, if $x \in G_j \setminus F_j$, then $x \notin F$. For each $n \in \mathbb{N}$, set $w_n = v_n|_F$. Then $\{w_n\}$ is total over F. Therefore, by Lemma 1, there exist an associated Banach space $\mathcal{A}_0 = \{\{w_n(x)\} : x \in F\}$ and a reconstruction operator $S_0 : \mathcal{A}_0 \to F$ such that $(\{F_n, w_n\}, S_0)$ is a fusion Banach frame for F with respect to $\mathcal{A}_0$. Let $g \in F^*$ and $f \in E^*$ be such that $f|_F = g$. Then

$$\operatorname{dist}\left(g, \left[\bigcup_{n=1}^{\infty} w_n^*(F^*)\right]\right) = \operatorname{dist}\left(f, \left[\bigcup_{n=1}^{\infty} v_n^*(E^*)\right]\right)$$

Since $(\{G_n, v_n\}, S)$ is a fusion frame satisfying property $\mathcal{S}$,

$$\operatorname{dist}\left(g, \left[\bigcup_{n=1}^{\infty} w_n^*(F^*)\right]\right) \to 0 \quad \text{as} \quad n \to \infty\,.$$

Therefore, $g \in [\bigcup_{n=1}^{\infty} w_n^*(F^*)]$. Hence $[\bigcup_{n=1}^{\infty} w_n^*(F^*)] = F^*$. □

Next we give a sufficient condition under which a fusion Banach frame satisfies property $\mathcal{S}$.

Theorem 3. *Let $(\{G_n, v_n\}, S)$ be a fusion Banach frame for E with*

$$\left[\bigcup_{n=1}^{\infty} G_n\right] = E.$$

If for any sequence $\{x_n\}$ with $0 \neq x_n \in G_n$, $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \left\|f|_{[x_n, x_{n+1}, \ldots]}\right\| = 0, \quad f \in [x_n]^*,$$

then $(\{G_n, v_n\}, S)$ *satisfies property* $\mathcal{S}$.

Proof: Suppose $(\{G_n, v_n\}, S)$ does not satisfy property $\mathcal{S}$. Then, there exist an $f \in E^*$ with $\|f\| = 1$, an $\varepsilon > 0$ and a positive integer m such that

$$\left\| f|_{[\bigcup_{i=n+1}^{\infty} G_i]} \right\| > \varepsilon, \quad \text{for all} \quad n \geq m\,.$$

Choose a positive integer $m_1 > m$ and $y_i \in G_i$ $(m+1 \leq i \leq m_1)$ with $z_1 = \sum_{i=m+1}^{m_1} y_i$ such that $\varepsilon < |f(z_1)| \leq 1$. Again, since

$$\left\| f|_{[\bigcup_{i=m_1+1}^{\infty} G_i]} \right\| > \varepsilon,$$

one can choose a positive integer $m_2 > m_1$ and $y_i \in G_i$ $(m_1 + 1 \leq i \leq m_2)$ with $z_2 = \sum_{i=m_1+1}^{m_2} y_i$ such that $\varepsilon < |f(z_2)| \leq 1$. Continuing this way, we get an increasing sequence of positive integers $\{m_n\}_{n=0}^{\infty}$ with $m_0 = m + 1$ and $y_i \in G_i$ $(i \geq m+1)$ with $z_n = \sum_{i=m_{n-1}+1}^{m_n} y_i$, $n \in \mathbb{N}$ such that $\varepsilon < |f(z_n)| \leq 1$, $n \in \mathbb{N}$. Now, define a sequence $\{x_n\}$ in E with $0 \neq x_n \in G_n$, $n \in \mathbb{N}$ such that whenever $y_n \neq 0$, $x_n = z_n$, for $n \geq m+1$. Put $g = f|_{[x_n]}$. Then, $g \neq 0$ and

$$\left| g\left(\sum_{i=m_{n-1}+1}^{m_n} y_i \right) \right| > \varepsilon\,, \quad n \in \mathbb{N}\,.$$

Therefore, we have a contradiction:

$$\left\| g|_{[x_{n+1}, x_{n+2}, \ldots]} \right\| > \varepsilon, \quad \text{for all} \quad n \geq m\,.$$

Hence $(\{G_n, v_n\}, S)$ satisfies property $\mathcal{S}$.

The following result gives a necessary and sufficient condition for a fusion Banach frame to satisfy property $\mathcal{S}$.

Theorem 4. *Let* $(\{G_n, v_n\}, S)$ *be a fusion Banach frame for a Banach space* E *with*

$$\left[\bigcup_{n=1}^{\infty} G_n \right] = E.$$

Then $(\{G_n, v_n\}, S)$ *satisfies property* $\mathcal{S}$ *if and only if* $y_n \xrightarrow{w} 0$, *where* $\{y_n\}$ *is a sequence in* E *such that* $\lim_{n\to\infty} v_i(y_n) = 0$, *for all* $i \in \mathbb{N}$.

Proof: First suppose that the fusion Banach frame $(\{G_n, v_n\}, S)$ satisfies property $\mathcal{S}$. Define $u_n = \sum\limits_{i=1}^{n} v_i$, $n \in \mathbb{N}$. Then, for each n, u_n is a projection of E onto $[\bigcup_{i=1}^{n} G_i]$. By hypothesis, we may write

$$E = \left[\bigcup_{i=1}^{n} G_i \right] \oplus \left[\bigcup_{i=n+1}^{\infty} G_i \right].$$

Since $(\{G_n, v_n\}, S)$ satisfies property $\mathcal{S}$, for each $f \in E^*$ and $\varepsilon > 0$, there exists a positive integer m such that

$$\left|f(x - u_n(x))\right| < \frac{\varepsilon}{2}, \quad (\|x\| \leq 1, n \geq m)\,.$$

Let $\{y_n\}$ be a sequence in E such that $\|y_n\| \leq 1$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} v_k(y_n) = 0$, $k \in \mathbb{N}$. Then, there exists a positive integer $m_0 > 0$ such that

$$\|v_k(y_n)\| < \frac{\varepsilon}{2m\|f\|}\,, \quad n \geq m_0,\, k = 1, 2, \ldots, m.$$

So

$$|f(y_n)| \leq |f(y_n - u_m(y_n))| + \|f\| \left\| \sum_{i=1}^{m} v_i(y_n) \right\| < \varepsilon, \quad \text{for all} \;\; n \geq m_0.$$

Conversely, suppose $(\{G_n, v_n\}, S)$ does not satisfy property $\mathcal{S}$. Then, there exists an $\varepsilon > 0$, an $f \in E^*$, an increasing sequence $\{m_k\} \subset \mathbb{N}$ and a bounded sequence $\{y_n\}$ in E such that $y_n \in \left[\bigcup_{i=m_n+1}^{m_{n+1}} G_i\right]$ and $|f(y_n)| > \varepsilon, n \in \mathbb{N}$. Thus $\lim_{n\to\infty} v_k(y_n) = 0$, $k \in \mathbb{N}$. But $y_n \overset{w}{\not\to} 0$. □

Next, we characterize fusion Banach frames satisfying property $\mathcal{S}$ in terms of closeness of certain subspaces of the dual space in the weak*-topology of the dual space.

Theorem 5. *A fusion Banach frame $(\{G_n, v_n\}, S)$ for a Banach space E with respect to $\mathcal{A}$ with*

$$E = \left[\bigcup_{n=1}^{\infty} G_n\right],$$

where each G_n is closed, satisfies property $\mathcal{S}$ if and only if for each $n \in \mathbb{N}$,

$$\left[\bigcup_{\substack{i=1\\ i\neq n}}^{\infty} v_i^*(E^*)\right]$$

is a closed subspace of E^* in the weak-topology.*

Proof: First suppose that $(\{G_n, v_n\}, S)$ satisfies property $\mathcal{S}$. Then, by hypotheses, $\{v_n^*\}$ is total over E^*. Therefore, by Lemma 1, there exists an associated Banach space $\mathcal{A}_0 = \{\{v_n^*(f)\} : f \in E^*\}$ together with a reconstruction operator $S_0 : \mathcal{A}_0 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, S_0)$ is a fusion Banach frame for E^* with respect to $\mathcal{A}_0$. Further, $[\bigcup_{n=1}^{\infty} v_n^*(E^*)] = E^*$. Since each v_n^* is weak*-continuous, $\left[\bigcup_{\substack{i=1\\ i\neq n}}^{\infty} v_i^*(E^*)\right]$ is a closed subspace of E^* in the weak*-topology of E^*.

Conversely, let for each k, $\left[\bigcup_{\substack{i=1\\ i\neq k}}^{\infty} v_i^*(E^*)\right]$ be a closed subspace of E^* in the weak*-topology. Note that $f \in E^*$ has representation of the type

$$f = w^*\text{-}\lim_{n\to\infty} \sum_{i=1}^{m_n} f_i^{(n)}, \quad f_i^{(n)} \in v_i^*(E^*), \quad 1 \le i \le m_n, \quad n \in \mathbb{N}.$$

So, we have

$$f - v_k^*(E^*) = f - w^*\text{-}\lim_{n\to\infty} f_k^{(n)} \in \left[\bigcup_{\substack{i=1\\ i\neq k}}^{\infty} v_i^*(E^*)\right].$$

Hence $E^* = [\bigcup_{i=1}^{\infty} v_i^*(E^*)]$. □

Observation. *If E is a reflexive Banach space, then every fusion Banach frame for E satisfies property $\mathcal{S}$.*

We finish with the following problem.

Problem. *Let $(\{G_n, v_n\}, S)$ be a fusion Banach frame for E satisfying property $\mathcal{S}$. Under which conditions:*

(a) E is weakly sequentially complete?

(b) E is reflexive?

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