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BOOL'S BLOCK–BY–BLOCK METHOD FOR SECOND KIND VOLTERRA INTEGRAL EQUATIONS

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The paper develops the block-by-block method by using Bool's quadrature rule with the order of convergence $O(h^6)$ and adaptation of the Bool's blocks for use on graded nodes. To illustrate the effectiveness of developed method, a number of numerical examples are presented containing a comparison with Simpson's block-by-block method.

§1. INTRODUCTION

Volterra equations arise most natural in certain types of time-dependent problems whose behavior at time t depends not only on the state at that time, but also on the states at pervious times as in renewal equation. Volterra integral equations have applications in history-dependent problems, in the system theory, in heat conduction and diffusion, see [18]. Nonlinear Volterra integral equation of the second kind has the form

$$\varphi(x) = f(x) + \int_0^x k\left(x, t, \varphi(t)\right)\, dt, \quad x \in R^+, \tag{1}$$

where $f(x)$ is the free term, $k\left(x, t, \varphi(t)\right)$ is the kernel of the integral equation and $\varphi(x)$ is the unknown function.

Various authors used Fredholm integral equation technique to solve Volterra integral equations. Orsi [17] introduced a numerical approach for solving Volterra integral equations of the second kind when the kernel contains a mild singularity. The idea of that approach was to use the technique of Fredholm equations to find the starting values of equation (1). A convergence result in [17] was illustrated by numerical examples.

The second-kind Volterra integral equations with weakly singular kernels typically have solutions which are not smooth near the initial point of the integration interval. Brunner [5] used an adaptation of the analysis originally developed for nonlinear weakly singular Fredholm integral equations, and presented a complete discussion of the optimal (global and local) order of convergence of piecewise polynomial collocation methods on graded grids for nonlinear Volterra integral equations with algebraic or logarithmic singularities in their kernels.

Different algorithms were used to solve Volterra integral equations of the second kind with weakly singular kernel. So in [2], [3] and [4] the authors studied some unbounded integral operators as Kober and Djrbashian and used the product block-by-block method for singular equations.

Blyth et el [6] used Walsh function method to find a numerical solution of the second kind Volterra integral equation rewritten as Fredholm integral equations with appropriately modified kernels. However, the Walsh function solution method has lower order of convergence as compared with the quadrature method.

Ford et al [8] considered nonlinear Volterra integral equations of the form

$$\varphi(x) = f(x) + \int_0^x k(x,t)\,\omega(\varphi(t))\,dt, \quad x \in R^+, \tag{2}$$

where $\omega(\varphi)$ is not Lipschitz continuous. They demonstrated that the solutions in that case may not be smooth. They also selected some cases, that can be reduced to ordinary differential equations, and compared the solutions obtained by standard numerical methods for ordinary differential equations with those obtained by quadrature methods.

Karapetyants et al [15] considered Volterra nonlinear integral equation of the form

$$\varphi^m(x) = f(x) + \alpha(x)\int_0^x k(x-t)b(t)\varphi(t)\,dt, \quad 0 < x < d \leq \infty, \tag{3}$$

where $m > 1$ and the functions $\alpha(x)$, $k(u)$, $b(t)$ and $f(x)$ are real-valued.

If $b(t) = 1$ and $m > 0$, this equation arises in applications, e.g. in water perlocation [11], [19] and in the nonlinear theory of wave propagation [16]. When $m > 1$ and $f(x) = 0$, equation (3) may have a nontrivial solution $\varphi(x)$, see [20]. Equation (3) with $\alpha(x) = b(t) = 1$, $0 < m < 1$, and continuous kernel $k(u)$ was considered in [1], where the results refer to the uniqueness of solution $\varphi(x)$ in some spaces of continuous or integrable functions. A similar problem for equation (3) with $m < 0$ and a non-increasing kernel $k(u)$ in the class of almost decreasing functions was studied in [13] with $\alpha(x) = b(t) = 1$. In [14] estimates from below and asymptotic properties near zero of the solution $\varphi(x)$ of the equation (3) with $m > 1$ were obtained for $\alpha(x)$, $k(u)$ and $f(x)$ possessing power asymptotic behavior near zero.

In the following Section 2 we state the basic theories of the existence and uniqueness of the solution of the nonlinear Volterra integral equation of the second kind on the interval $[0, T]$ for some $T > 0$. Section 3 gives a short survey on the literature on Simpson's block-by-block method and the Bool's block-by-block method for nonlinear Volterra equation of the second kind.

§2. PRELIMINARIES

This section is devoted to introduction of nonlinear Volterra integral equation of the second kind and discusses about existence, uniqueness of solutions and error bounds.

2.1. Existence and uniqueness. The nonlinear Volterra integral equation of the second kind is of the form

$$\varphi(x) = f(x) + \int_0^x k\left(x, t, \varphi(t)\right)\, dt, \quad 0 < x < d \leq \infty, \tag{4}$$

where $f(x)$, $k\left(x, t, \varphi(t)\right)$ and $\varphi(x)$ are the free term, the kernel of the integral equation and the unknown functions respectively. The following theorem establishes some conditions for unique solvability of the equation (4), see [12] and [18]. One of them is that the kernel $k(x, t, u)$ satisfies the simple Lipschitz condition with respect to the third argument :

$$\left|k\left(x, t, y\right) - k\left(x, t, z\right)\right| \leq L|y - z|, \tag{5}$$

where L is independent of x, t, y and z.

Theorem 1. *Let the functions $f(t)$ and $k(x,t,u)$ in equation (4) be continuous in $0 \le t \le x \le T$. If the kernel $k(x,t,u)$ satisfy the Lipschitz condition (5), then equation (4) has unique, continuous solution for any finite T.*

2.2. Error Bound. The following theorem estimates the value of the error bound for the approximate solution φ_n [18].

Theorem 2. *Let the conditions of the previous theorem be satisfied. Besides, let φ_n be an approximate solution of the equation (4), which is obtained by the quadrature technique and let*

$$r(x) = f(x) + \int_0^x k(x,t,\varphi(t))\, dt - \varphi_n(x). \tag{6}$$

Then

$$|\varphi(x) - \varphi_n(x)| \le Re^{Lx}, \tag{7}$$

where $\varphi(x)$ is the solution of equation (4) and $R = \max|r(x)|$.

§3. SOLUTION METHODS

One of the popular methods for solving nonlinear Volterra integral equation of the second kind is the quadrature method. The error of that method depends on the position of mesh points. The following technique is general for determining the nodes. For instance, if $\beta = 1$ (see (9)), then we have the case of equal space nodes (uniform nodes), and if $\beta \neq 1$, then we have the case of graded nodes.

3.1. Graded nodes for Simpson's block-by-block method. The interval $[0,T]$ is divided into $N = 2M$ subintervals, and the nodes are chosen to satisfy

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N \le T. \tag{8}$$

The even nodes are chosen as

$$t_{2k} = \left(\frac{2kT}{N}\right)^{\beta} = \left(\frac{k}{M}\right)^{\beta} T, \quad k = 0,1,\ldots,M, \tag{9}$$

and the odd nodes as

$$t_{2k+1} = \frac{1}{2}\left(t_{2k} + t_{2k+2}\right), \quad k = 0,1,\ldots,M-1. \tag{10}$$

The width of each subinterval is

$$h_k = (t_{k+1} - t_k), \quad k = 0, 1, \ldots, 2M - 1, \tag{11}$$

$$x_k = t_k, \quad k = 0, 1, \ldots, 2M. \tag{12}$$

3.2. Simpson's Block-by Block method. The block-by-block method essentially is an extrapolation procedure and has the advantage of being self-starting. This method can be adapted to graded nodes see [3]. The trapezoidal method is a low order method, the other methods as Simpson's require one or more starting values. We shall follow Linz [18] in description of the method. The so-called block-by-block methods generalize the well-known implicit Runge-Kutta methods for ordinary differential equations and the term is also used in connection with integral equations. The idea behind the block-by-block methods is quite general, but is most easily understood by considering a special case.

Let us use Simpson's rule as the numerical integration formula. If we know φ_1, then we can simply compute φ_2 by the formula

$$\varphi_2 = f(x_2) + \frac{h_1}{3}\left[k(x_2, t_0, \varphi(t_0)) + 4k(x_2, t_1, \varphi(t_1)) + k(x_2, t_2, \varphi(t_2))\right]. \tag{13}$$

To obtain a value of φ_1, we introduce another point $t_{1/2} = \frac{h_1}{2}$ and the corresponding value $\varphi_{1/2}$, then by Simpson's rule with $\varphi_0, \varphi_{1/2}$ and φ_1,

$$\varphi_1 = f(x_1) + \frac{h_1}{6}\left[k(x_1, t_0, \varphi(t_0)) + 4k\left(x_1, t_{1/2}, \varphi\left(t_{1/2}\right)\right) + k(x_1, t_1, \varphi(t_1))\right]. \tag{14}$$

The unknown value of $\varphi_{1/2}$ can be approximated by quadratic interpolation, using the values of φ_0, φ_1 and φ_2, that is $\varphi_{1/2}$ is being replaced by

$$\varphi_{1/2} \approx \frac{1}{8}\left[3\varphi_0 + 6\varphi_1 - \varphi_2\right]. \tag{15}$$

Equations (13) and (14) are a pair of simultaneous equations for φ_1 and φ_2. For sufficiently small h, unique solution exists and can be obtained by Newton's method.

The general process should now be clear : for $(k = 0, 1, \ldots, M-1)$ the approximate solution can be computed by

$$\varphi_{2k+1} = f(x_{2k+1}) + \sum_{j=1}^{k} h_j \sum_{l=0}^{2} \varpi_l k\left(x_{2k+1}, t_{2j+l}, \varphi(t_{2j+l})\right) +$$

$$+\frac{h_k}{2} \sum_{l=0}^{2} \varpi_l k\left(x_{2k+1}, t_{2j+\frac{l}{2}}, \varphi\left(t_{2j+\frac{l}{2}}\right)\right), \tag{16}$$

$$\varphi_{2k+2} = f(x_{2k+2}) + \sum_{j=1}^{k+1} h_j \sum_{l=0}^{2} \varpi_l k\left(x_{2k+2}, t_{2j+l}, \varphi(t_{2j+l})\right), \tag{17}$$

$$\varphi_{2k+1/2} \approx \frac{1}{8}\left[3\varphi_{2k} + 6\varphi_{2k+1} - \varphi_{2k+2}\right], \quad k = 0, 1, \ldots, M-1, \tag{18}$$

$$[\varpi_l] = \begin{bmatrix} \frac{1}{3} & \frac{4}{3} & \frac{1}{3} \end{bmatrix}, \quad l = 0, 1, 2. \tag{19}$$

3.3. Graded nodes for Bool's block-by-block method. In this subsection we make two suggestions. The first is on application of Bool's quadrature rule, which is considered as an extrapolation technique for Simpson's rule, and the second is on application of Bool's rule relating graded nodes. The Bool's quadrature formula states that

$$\int_0^{x_4} \varphi(x)dx = \frac{2h}{45}\left[7\varphi_0 + 32\varphi_1 + 12\varphi_2 + 32\varphi_3 + 7\varphi_4\right]. \tag{20}$$

Furthermore, there exists a value c with $c \in (0, x_4)$, such that the error term $E(\varphi, h)$ takes the form

$$E(\varphi, h) = -\frac{8h^7}{945}\varphi^{(6)}(c). \tag{21}$$

The interval $[0, T]$ is divided into $N = 4M$ subintervals, and the nodes are chosen to satisfy

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N \leq T. \tag{22}$$

Besides, the even nodes are found from

$$t_{4k} = \left(\frac{4k}{N}\right)^{\beta} T = \left(\frac{k}{M}\right)^{\beta} T, \quad k = 0, 1, \ldots, M, \tag{23}$$

and the width of each subinterval is

$$h_{4k} = \frac{1}{4}\left(t_{4k+4} - t_{4k}\right), \quad k = 0, 1, \ldots, M-1. \tag{24}$$

The other nodes are found from the formula

$$t_{4k+j} = t_{4k} + jh_{4k}, \quad j = 1, 2, 3 \quad \text{and} \quad k = 0, 1, \ldots, M-1, \tag{25}$$

$$x_k = t_k, \quad k = 0, 1, \ldots, 4M. \tag{26}$$

3.4. Bool's block-by-block Method. In the Simpson's block-by-block method, two equations in two unknowns are solved in each stage with an error of order $O(h^4)$. In the case of Bool's block-by-block method, actually we solve four equations in four unknowns but the error is of order $O(h^6)$ and the method uses the extrapolation technique of Simpson's block-by-block method. The equations of Bool's block-by-block method are

$$\varphi_{4k+1} = f\left(x_{4k+1}\right) + \sum_{j=1}^{k} 2h_{4j-4} \sum_{l=4}^{0} w_l k\left(x_{4k+1}, t_{4j-l}, \varphi\left(t_{4j-l}\right)\right) +$$

$$+\frac{h_{4k}}{2} \sum_{l=0}^{4} w_l k\left(x_{4k+1}, t_{4k+l/4}, \varphi\left(t_{4k+l/4}\right)\right), \tag{27}$$

$$\varphi_{4k+2} = f\left(x_{4k+2}\right) + \sum_{j=1}^{k} 2h_{4j-4} \sum_{l=4}^{0} w_l k\left(x_{4k+2}, t_{4j-l}, \varphi\left(t_{4j-l}\right)\right) +$$

$$+h_{4k} \sum_{l=0}^{4} w_l k\left(x_{4k+2}, t_{4k+l/2}, \varphi\left(t_{4k+l/2}\right)\right), \tag{28}$$

$$\varphi_{4k+3} = f(x_{4k+3}) + \sum_{j=1}^{k} 2h_{4j-4} \sum_{l=4}^{0} w_l k\left(x_{4k+3}, t_{4j-l}, \varphi\left(t_{4j-l}\right)\right) +$$

$$+\frac{3h_{4k}}{2}\sum_{l=0}^{4} w_l k\left(x_{4k+3}, t_{4k+3l/4}, \varphi\left(t_{4k+3l/4}\right)\right), \tag{29}$$

$$\varphi_{4k+4} = f\left(x_{4k+4}\right) + \sum_{j=1}^{k+1} 2h_{4j-4}\sum_{l=4}^{0} w_l k\left(x_{4k+4}, t_{4j-l}, \varphi\left(t_{4j-l}\right)\right), \tag{30}$$

where $k = 0, 1, 2, \ldots, M-1$.

$$[w_l] = \left[\frac{7}{45}\ \frac{32}{45}\ \frac{12}{45}\ \frac{32}{45}\ \frac{7}{45}\right], \quad l = 0, 1, \ldots, 4. \tag{31}$$

At each step, equations (27) – (30) have to be solved simultaneously for the unknowns $\varphi(t_{4k+1})$, $\varphi(t_{4k+2})$, $\varphi(t_{4k+3})$ and $\varphi(t_{4k+4})$, so that we obtain a block of unknowns at a time. The values of $\varphi\left(t_{4k+1/4}\right)$, $\varphi\left(t_{4k+1/2}\right)$, $\varphi\left(t_{4k+3/4}\right)$, $\varphi\left(t_{4k+3/2}\right)$ and $\varphi\left(t_{4k+9/4}\right)$ can be computed by interpolation of a fourth degree polynomial, using $\varphi(t_{4k}), \varphi(t_{4k+1})$, $\varphi(t_{4k+2})$, $\varphi(t_{4k+3})$ and $\varphi(t_{4k+4})$ as follows :

$$\varphi_{4k+1/4} = \frac{1}{2048}\left[1155\varphi_{4k} + 1540\varphi_{4k+1} - 990\varphi_{4k+2} + 420\varphi_{4k+3} - 77\varphi_{4k+4}\right], \tag{32}$$

$$\varphi_{4k+1/2} = \frac{1}{128}\left[35\varphi_{4k} + 140\varphi_{4k+1} - 70\varphi_{4k+2} + 28\varphi_{4k+3} - 5\varphi_{4k+4}\right], \tag{33}$$

$$\varphi_{4k+3/4} = \frac{1}{2048}\left[195\varphi_{4k} + 2340\varphi_{4k+1} - 702\varphi_{4k+2} + 260\varphi_{4k+3} - 45\varphi_{4k+4}\right], \tag{34}$$

$$\varphi_{4k+3/2} = \frac{1}{128}\left[-5\varphi_{4k} + 60\varphi_{4k+1} + 90\varphi_{4k+2} - 20\varphi_{4k+3} + 3\varphi_{4k+4}\right], \tag{35}$$

$$\varphi_{4k+9/4} = \frac{1}{2048}\left[35\varphi_{4k} - 252\varphi_{4k+1} + 1890\varphi_{4k+2} + 420\varphi_{4k+3} - 45\varphi_{4k+4}\right]. \tag{36}$$

§4. NUMERICAL EXAMPLES

Three examples are given to illustrate the effectiveness of the above.

Example 1. A linear example :

$$\varphi(x) = 1 + x^2 + \int_0^x \frac{1+x^2}{1+t^2}\varphi(t)\, dt. \tag{37}$$

This equation has an exact solution of the form

$$\varphi(x) = \left(1 + x^2\right) \exp(x) \tag{38}$$

Example 2. Equation with a convolution type kernel :

$$\varphi(x) = \sinh(x) - \int_0^x \cosh(x - t)\varphi(t)\, dt. \tag{39}$$

The exact solution is

$$\varphi(x) = \frac{2}{\sqrt{5}} \exp\left(\frac{-x}{2}\right) \sinh\left(\frac{\sqrt{5}x}{2}\right) \tag{40}$$

Example 3. A nonlinear example :

$$\varphi(x) = 1 + \sin^2(x) - \int_0^x 3\sin(x - t)\varphi^2(t)dt. \tag{41}$$

The exact solution is

$$\varphi(x) = \cos(x). \tag{42}$$

4.1. Results. The numbers 1,2,3 refer to the above examples. Table 1 contains the comparison between Max. relative (E_r) of Simpson's block and the Bool's block for same number of subintervals.

Ex.	*N*	E_r *of Simpson's block*	E_r *of Bool's block*
1	32	$4.813E-7$	$5.617E-10$
	64	$3.002E-8$	$8.728E-12$
2	32	$2.432E-6$	$2.537E-9$
	64	$3.108E-7$	$8.319E-11$
3	32	$9.866E-8$	$4.372E-10$
	64	$5.949E-9$	$6.224E-12$

Table 1. Results of relative error of Simpson and Bool blocks on uniform mesh.

Table (2) shows the effect of graded nodes in Bool's block by block method, which reduces the Max. relative error.

Ex.	*N*	E_r *of Bool's at* $\beta = 1$	E_r *of Bool's at* $\beta \neq 1$	β
1	16	$3.638E-8$	$3.298E-8$	0.90
	64	$8.728E-12$	$7.948E-12$	0.95
2	16	$6.710E-9$	$6.501E-9$	1.05
	64	$8.319E-11$	$1.544E-12$	1.05
3	16	$3.214E-8$	$1.147E-8$	0.95
	64	$6.224E-12$	$1.844E-12$	0.95

Table 2. The effect of graded nodes on relative error of Bool's block.

Conclusion. The block-by-block method is a self-starting method and it can be applied to solve Volterra integral equations as Fredholm integral equations. The method is developed in two directions. Firstly, we use Bool's quadrature rule which has order of convergence $O(h^6)$. Secondly, the Bool's block is adapted for use on graded nodes. The comparison of Simpson's block and Bool's block shows that the Max. relative error of Bool's block becomes about 2.907×10^{-4} times the value of Max. relative error of Simpson's block. Also, the Max. relative error when Bool's block is used on graded nodes reduces considerably.

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