Известия НАН Армении. Математика, 37, № 5, 2002, 3–11

INTEGRAL REPRESENTATIONS OF SOLUTIONS OF THE HEAT EQUATION

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Abstract. The paper defines new classes of solutions of the heat equation in half-spaces and presents a series of integral representations for them. On the basis of these formulas boundedness of corresponding integral operators is proved.

INTRODUCTION

Consider the heat equation

$$\frac{\partial u}{\partial t} = \Delta u, \tag{1}$$

where $u = u(x, t)$, $x \in \mathbb{R}^n$, $n \geq 1$, and $t > 0$. The classical Cauchy problem for (1) asks for a solution satisfying the initial data $u(x, 0) = f(x)$, where f is a bounded and continuous function.

A well-known solution is given by the formula :

$$u(x,t) = c_n \int_{\mathbb{R}^n} t^{-n/2} e^{-|x-y|^2/4t} f(y) dy, \tag{2}$$

where $c_n = (4\pi)^{-n/2}$. Solutions of the heat equation are commonly called **caloric functions**, and we will use that term throughout this paper.

Formula (2) suggests further generalizations. In fact, if we have

$$|f(x)| \leq C_1 \exp\{C_2 |x|^\alpha\}, \quad \alpha < 2,$$

then the integral (2) will still be convergent and the resulting function u will satisfy the heat equation (1). In other words, formula (2) generates solutions to the heat equation in a wider class of initial value functions. A standard reference on second order parabolic equations is the book by A. Friedman [6].

In the present paper we find some integral formulas providing solutions to the equation (1). Our starting point is the case of harmonic functions [4]. In [4] we constructed the classes A^p_α based on an integral representation formula. Using the same idea of the classes A^p_α and well-known parallels between harmonic and caloric functions, we provide similar representations for the case of equation (1). Then we use these integral representations to prove boundedness of certain integral operators in weighted Lebesgue functional spaces.

Note that spaces of solutions of more general equations (elliptic and parabolic) were introduced in [2]. For some particular defining functions and certain values of p the spaces $\tilde{H}^p$ considered in [2] intersect with our A^p_α spaces of solutions of the heat equation. However, our methods and results appear totally different.

A preliminary version of this work was reported at Joint Mathematical Meeting of AMS in Baltimore (see Abstracts of Papers Presented to AMS, Issue 79, 1992, p. 93).

§1. CLASSES OF CALORIC FUNCTIONS AND THE REPRESENTING KERNEL

We consider the equation (1) in the upper half-space $(x,t) \in \mathbb{R}^{n+1}_+$, $t > 0$. It is well-known that any caloric function is an infinitely differentiable function in any variable (see, e.g. [6]).

In analogy with the case of harmonic functions we define Hardy spaces of caloric functions as follows : a caloric function $u(x,t)$ in the half-space $\mathbb{R}^{n+1}_+$ belongs to the Hardy space CH^p, $0 < p \leq \infty$ (C stands for caloric), if

$$\sup_{t>0} \int_{\mathbb{R}^n} |u(x,t)|^p dx < \infty. \tag{3}$$

It is not difficult to see that CH^p functions possess boundary values and have integral representations given by the formula (2), where the boundary value of the given CH^p function stands for f. Moreover, the boundary function is in L^p and its L^p norm ($p > 1$) is estimated by the CH^p norm of the given function (see [2], [8] for details). The converse is also true. Indeed, let $f \in L^p$, $1 \leq p \leq \infty$, and

$$u(x,t) = \int_{\mathbb{R}^n} K(x-y,t) f(y) dy, \tag{4}$$

where $K(x,t) = c_n t^{-n/2} \exp\{-|x|^2/4t\}$ is the heat kernel. The choice of c_n implies $\int K\, dx = 1$, hence by Hölder's inequality,

$$|u(x,t)|^p \leq \int_{\mathbb{R}^n} K(x-y,t) |f(y)|^p dy,$$

and our assertion is proved.
We observe, that if $u \in CH^p$, $1 \leq p \leq \infty$, then u is bounded in any half-space $t \geq c > 0$ in $\mathbb{R}^{n+1}_+$. Indeed, as it is mentioned above, u has the representation (4) and hence, again using Hölder's inequality we will find

$$|u(x,t)|^p \leq \int_{\mathbb{R}^n} K(x-y,t)|f(y)|^p dy =$$

$$= Ct^{-n/2} \int_{\mathbb{R}^n} |f(y)|^p \exp\{-|x-y|^2/4t\}\, dy \leq Ct^{-n/2}\|f\|_p^p$$

or

$$|u(x,t)| \leq C\|f\|_p^{-n/2p}. \tag{5}$$

Hence, u is bounded in any half-space $t \geq c$ for any $c > 0$.
Now let us consider different spaces of caloric functions. A caloric function $u(x,t)$ in $\mathbb{R}^{n+1}_+$ $(n \geq 1)$ belongs to the class CA^p_α, $0 < p < \infty$, $-1 < \alpha < \infty$ (C again comes from caloric), if

$$\|u\|^p_{p,\alpha} = \int_0^\infty \int_{\mathbb{R}^n} |u(x,t)|^p t^\alpha \, dx\, dt < \infty.$$

Recall, that A^p_α functions have in general no boundary values (see [1] and [5]). The same is true for corresponding classes of caloric functions defined above, meaning that we cannot expect any integral formulas similar to (2).
One of the goals of this paper is to construct kernel functions, that produce formulas, which play a similar role. For representation formulas in A^p_α spaces of harmonic functions see [3], [4], [7].
There is no inclusion relationship between classes CH^p and CA^p_α. Indeed, let f be the characteristic function of the interval $[0,1]$ on the real line and

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(y) \exp\left\{-\frac{|x-y|^2}{4t}\right\} dy = \frac{1}{\sqrt{4\pi t}} \int_0^1 \exp\left\{-\frac{|x-y|^2}{4t}\right\} dy.$$

In that case

$$\int_{-\infty}^{\infty} u(x,t)\, dx = \int_0^1 dy \int_{-\infty}^{\infty} K(x-y,t)\, dx = \int_0^1 dy = 1,$$

and $f \in CH^1$. At the same time,

$$\int_0^\infty \int_{-\infty}^{\infty} u(x,t) t^\alpha \, dx\, dt = \int_0^\infty t^\alpha \, dt = \infty.$$

Construction of an example for any p is similar. The proof of the converse is more complicated and the example will be provided by Theorem 2 later. The same theorem will also show that the classes CA^p_α are not trivial.

After these preliminary observations we turn to introduction of the main objects of the paper. For any integer $m \geq 0$ and $(x,t) \in \mathbb{R}^{n+1}_+$ we denote

$$K_m(x,t) = \frac{(-2)^m}{m!} \frac{\partial^{m+1}}{\partial t^{m+1}} K(x,t). \tag{6}$$

where $K(x,t)$ is the classical heat kernel, as before.

We will need effective estimates of this kernel in order to use it for integral representation formulas. As a first step, we calculate :

$$K(x,t) = \frac{c}{t^{n/2}} \exp\{-|x|^2/4t\}, \qquad K_0(x,t) = c_1 \frac{e^{-|x|^2/4t}}{t^{n/2+2}}(|x|^2 - t),$$

$$K_1(x,t) = c_2 \frac{e^{-|x|^2/4t}}{t^{n/2+4}}(|x|^4 - 3t|x|^2 + 2t^2).$$

Using induction, we get the following result.

Lemma 1. *For any integer $m \geq 0$ and $(x,t) \in \mathbb{R}^{n+1}_+$ the kernels K_m satisfy the estimates*

$$|K_m(x,t)| \leq C \frac{e^{-|x|^2/4t}}{t^{n/2+2m+2}} \sum_{j=0}^{m+1} |x|^{(2m-j+1)} t^j. \tag{7}$$

To prove the integral representation formula, we need the following assertion.

Lemma 2. *For any $u \in CA^1_\alpha$, $\alpha > -1$ and any integer $m \geq 0$,*

$$N^m \frac{\partial^m}{\partial t^m} u(x, N+t) \to 0$$

uniformly for x as $N \to \infty$.

Proof : If $m = 0$ we just use (5) and the observation that functions from CA^1_α belong to CH^1 in any sub-half-space. So $u(x,t) \to 0$ uniformly for x as $t \to \infty$.

Assume now that $m > 0$. Applying integral formula (2) for the half-space $t > c > 0$ (here c is any positive constant), we get

$$u(x,t) = \int_{\mathbb{R}^n} u(y,c) K(x-y, t-c)\, dy.$$

This integral converges absolutely, because of our assumption about the function u. Because u and K are smooth, we can differentiate under the integral sign and get

$$\frac{\partial^m}{\partial t^m} u(x,t) = \int_{\mathbb{R}^n} u(y,c) \frac{\partial^m}{\partial t^m} K(x-y, t-c) dy.$$

Now, using estimates (7) from Lemma 1 we will get the desired result.

§2. INTEGRAL REPRESENTATIONS AND BOUNDED PROJECTIONS

In this section we formulate and prove our main results. As we have mentioned above, the classes CA^p_α cannot have integral representations similar to those for the Hardy type spaces.

Theorem 1. *Let u be caloric in the upper half-space and let $u \in CA^p_\alpha$, $1 \le p < \infty$, $-1 < \alpha < \infty$. For any integer m satisfying*

$$m \ge \alpha \quad \text{for} \quad p = 1; \quad m > (1+\alpha)/p - 1 \quad \text{for} \quad 1 < p < \infty, \tag{8}$$

the following integral representation formulas hold

$$u(x,t) = \int_{\mathbb{R}^{n+1}_+} u(y,\tau) K_m(x-y, t+\tau)\tau^m \, dy \, d\tau, \tag{9}$$

where

$$K_m(x,t) = \frac{(-2)^{m+1}}{m!} \frac{\partial^{m+1}}{\partial t^{m+1}} K(x,t).$$

Proof : First of all we show that integrals (9) are well defined. If $p = 1$ and $u \in CA^1_\alpha$, by Lemma 1,

$$\int_{\mathbb{R}^{n+1}_+} |u(y,\tau)||K_m(x-y,t+\tau)|\tau^m \, dy \, d\tau \le C \int_{\mathbb{R}^{n+1}_+} |u(y,\tau)| \times$$

$$\times \left\{ \frac{\exp[-|x-y|^2/4(t+\tau)]}{(t+\tau)^{n/2+2m+2}} \sum_{j=0}^{m+1} |x-y|^{2(m-j+1)}(t+\tau)^j \right\} \tau^m \, dy \, d\tau.$$

Now, considering separately the possible cases $|x-y| \le 1$, $|x-y| \ge 1$, $t+\tau \le 1$ and $t+\tau \ge 1$, we find that

$$|K_m(x-y,t+\tau)|\tau^m \le C \frac{\tau^m}{(t+\tau)^{n/2+m+1}},$$

where C depends on x only. It is easy to see, that if $m \ge \alpha$, then the right side of the last inequality is estimated by $C\tau^\alpha$, where C now depends on both x and t. If $1 < p < \infty$, then using Hölder's inequality and estimates like above, we see that integral (9) is finite if $m > (1+\alpha)/p - 1$.

We turn to the proof of the formula (9). Using, as in the proof of Lemma 2, the semigroup property of heat kernel we can write for any $t > 0$

$$u(x,t) = \int_{\mathbb{R}^n} u(y, t/2) K(x-y, t/2) \, dy. \tag{10}$$

and the integral converges absolutely. For any fixed $N > 0$ integration by parts gives

$$\int_{\mathbb{R}^n} u(y,t/2)K(x-y,t/2)\,dy = \frac{(-1)^m}{m!}\int_{\mathbb{R}^n} u(y,t/2)\times$$

$$\times \int_0^N \tau^m \frac{\partial^{m+1}}{\partial t^{m+1}} K(x-y,t/2+\tau)\,d\tau\,dy + u(x,N+t)+\ldots+\frac{N^m}{m!}\frac{\partial^m}{\partial t^m}u(x,N+t). \quad (11)$$

Now we can use Lemma 2 to find the limit as $N \to \infty$. Thanks to the observations at the beginning of the proof, change of order of integration is justified because of absolute integrability of all integrals.

$$u(x,t) = \frac{(-1)^{m+1}}{m!}\int_{\mathbb{R}^n}\int_0^\infty u(y,t/2)\tau^m \frac{\partial^{m+1}}{\partial t^{m+1}} K(x-y,t/2+\tau)\,d\tau\,dy =$$

$$= \frac{(-1)^{m+1}}{m!}\int_0^\infty\int_{\mathbb{R}^n} u(y,t/2)\tau^m \frac{\partial^{m+1}}{\partial t^{m+1}} K(x-y,t/2+\tau)\,d\tau\,dy =$$

$$= \frac{(-2)^{m+1}}{m!}\int_0^\infty \tau^m \left\{\frac{\partial^{m+1}}{\partial \tau^{m+1}}\int_{\mathbb{R}^n} u(y,t/2)K(x-y,t/2+\tau)\,dy\right\} d\tau =$$

$$= \frac{(-2)^{m+1}}{m!}\int_0^\infty \tau^m \left\{\frac{\partial^{m+1}}{\partial t^{m+1}}\int_{\mathbb{R}^n} u(y,\tau)K(x-y,t+\tau)\,dy\right\} d\tau = \quad (12)$$

$$= \frac{(-2)^{m+1}}{m!}\int_0^\infty \tau^m d\tau \int_{\mathbb{R}^n} u(y,\tau)\frac{\partial^{m+1}}{\partial t^{m+1}} K(x-y,t+\tau)\,dy =$$

$$= \frac{(-2)^{m+1}}{m!}\int_{\mathbb{R}^{n+1}_+} u(y,\tau)\tau^m \frac{\partial^{m+1}}{\partial t^{m+1}} K(x-y,t+\tau)\,dy\,d\tau =$$

$$= \int_{\mathbb{R}^{n+1}_+} u(y,\tau)\tau^m K_m(x-y,t+\tau)\,dy\,d\tau.$$

This last chain of equalities proves our Theorem.

The integral representation formula proved in Theorem 1 allows to define certain projection operators in appropriate functional spaces. We define that spaces as follows : For $1 \le p < \infty$, $-1 < \alpha < \infty$ denote by $L^p_\alpha = L^p_\alpha(\mathbb{R}^{n+1}_+)$ the class of all measurable functions, such that

$$\|g\|^p_{p,\alpha} = \int_{\mathbb{R}^{n+1}_+} |g(x,t)|^p t^\alpha\,dx\,dt < \infty.$$

It is obvious that $CA^p_\alpha \subset L^p_\alpha$. Also, it follows from the proof of the Theorem 1, that the integral in (9) remains well defined, if we replace $u \in CA^p_\alpha$, for $p \ge 1$ by any L^p_α function. Further, it is clear that

$$T_m g(x,t) = \int_{\mathbb{R}^{n+1}_+} g(y,t)K_m(x-y,t+\tau)\tau^m\,dy\,d\tau \quad (13)$$

is a caloric function, provided $g \in L^p_\alpha$, and the integer $m \ge 0$ satisfies certain conditions. Our next result describes the class of functions, to which the integral (13) belongs.

Theorem 2. *Assume $g \in L^p_\alpha(\mathbb{R}^{n+1}_+)$, $1 \le p < \infty$, $-1 < \alpha < \infty$, and let the operator T_m be defined by the formula (13). If $m > (1+\alpha)/p - 1$, then the function $T_m g$ is caloric, $T_m g \in CA^p_\alpha$ and there exists a constant $C > 0$ depending only on p and α, such that*

$$\|T_m g\|_{p,\alpha} \le C\|g\|_{p,\alpha}. \tag{14}$$

Proof : First we treat the case $p = 1$. Let $g \in L^p_\alpha$ and denote $u = T_m g$, where $m > \alpha$. Using Fubini's theorem (see remarks at the beginning of the proof of Theorem 1), we get

$$\int_{\mathbb{R}^{n+1}_+} |u(x,t)| t^\alpha \, dx \, dt \le \int_{\mathbb{R}^{n+1}_+} t^\alpha \, dx \, dt \int_{\mathbb{R}^{n+1}_+} |g(y,\tau)| \cdot |K_m(x-y, t+\tau)| \tau^m \, dy \, d\tau =$$

$$= \int_{\mathbb{R}^{n+1}_+} |g(y,\tau)| \tau^m \, dy \, d\tau \int_{\mathbb{R}^{n+1}_+} |K_m(x-y, t+\tau)| t^\alpha \, dx \, dt.$$

To estimate the inner integral we use Lemma 1 :

$$\int_{\mathbb{R}^{n+1}_+} |K_m(x-y, t+\tau)| t^\alpha \, dx \, dt \le$$

$$\le C \sum_{j=0}^{m+1} \int_0^\infty \int_{\mathbb{R}^n} \frac{\exp\{-|x-y|^2/4(t+\tau)\}}{(t+\tau)^{n/2+2+2m}} |x-y|^{2(m+1-j)} (t+\tau)^j t^\alpha \, dx \, dt =$$

$$= C \sum_{j=0}^{m+1} \int_0^\infty \frac{(t+\tau)^j t^\alpha \, dt}{(t+\tau)^{n/2+2+2m}} \int_{\mathbb{R}^n} \exp\left\{-\frac{|x-y|^2}{4(t+\tau)}\right\} |x-y|^{2(m+1-j)} dx. \tag{15}$$

The inner integral on the right-hand side of (15) can be estimated using change of variable :

$$\int_{\mathbb{R}^n} \exp\left\{-\frac{|x-y|^2}{4(t+\tau)}\right\} |x-y|^{2(m+1-j)} \, dx =$$

$$= (t+\tau)^{n/2+m+1-j} \int_{\mathbb{R}^n} \exp\left\{-\frac{|x-y|^2}{4(t+\tau)}\right\} \frac{|x-y|^{2(m+1-j)}}{(t+\tau)^{n/2+m+1-j}} dx =$$

$$= (t+\tau)^{n/2+m+1-j} \int_{\mathbb{R}^n} \exp\{-|z|^2/4\} |z|^{2(m+1-j)} \, dz = C(t+\tau)^{n/2+m+1-j}.$$

So, (15) extends to

$$\int_{\mathbb{R}^{n+1}_+} |K_m(x-y, t+\tau)| t^\alpha \, dx \, dt \le C \int_0^\infty \frac{t^\alpha \, dt}{(t+\tau)^{m+1}} \le$$

$$\le C \int_0^\infty \frac{dt}{(t+\tau)^{m+1-\alpha}} = C\tau^{-m+\alpha}.$$

Finally, combining all these estimates yields

$$\int_{\mathbb{R}_+^{n+1}} |u(x,t)|t^{\alpha}\, dx\, dt \le C \int_{\mathbb{R}_+^{n+1}} |g(y,\tau)|\tau^{m}\tau^{-m+\alpha}\, dy\, d\tau = C\|g\|_{1,\alpha},$$

which proves Theorem in case $p = 1$. The proof of the case $1 < p < \infty$ is standard and based on the classical Schur lemma, which can be found in many books (see e.g. [5], p. 34, Lemma 2.2).

Lemma 3 (Schur). *Let μ be a positive measure on some σ-algebra of a set X, $K : X \times X \longmapsto [0,\infty)$ be a measurable function. Assume that there exists a measurable function $g : X \longmapsto [0,\infty)$ and constants a and b, such that for $1 < p < \infty$, $q = p/(p-1)$,*

$$\int_X K(x,y)[g(y)]^q d\mu(y) \le (ag(x))^q, \quad x \in X,$$

$$\int_X K(x,y)[g(y)]^p d\mu(x) \le (bg(y))^p, \quad y \in X.$$

Then the equation

$$Tf(x) = \int_X K(x,y)f(y)\, d\mu(y)$$

defines a continuous operator on $L^p(d\mu)$ with $\|T\| \le ab$.

To conclude the proof of the Theorem, we need only to check the conditions of the Lemma, where $X = \mathbb{R}_+^{n+1}$, $d\mu$ is the Lebesgue measure in $\mathbb{R}_+^{n+1}$, $K = |K_m|$. As a test function g we can take t^{δ} with sufficiently small $\delta > 0$ and use estimates from the first part of the proof of the Theorem. The proof is now complete.

Concluding Remarks. 1) Theorem 2 shows that CA^p_α spaces are not empty. Indeed, any function of the form (13) with any $g \in L^p_\alpha$ is a function from CA^p_α.
2) Theorem 2 provides an example of a function, which belongs to the class CA^p_α but does not belong to Hardy-type class CH^p. Indeed, let us take a function (call it g) supported on the square $0 \le x \le 1$, $0 < t \le 1$ in the upper half-plane and zero everywhere else, and let it grow near the boundary $t = 0$ faster than t^{-1}. The function defined by formula (13) with m big enough, will obviously belong to CA^p_α (thanks to Theorem 2), but it will not belong to CH^p. We leave the details to the reader.

Резюме. В статье определяются новые классы решений уравнения теплопроводности в полупространствах и для них приводятся ряд интегральных представлений. На основе этих формул доказана ограниченность соответствующих интегральных операторов.

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Поступила 29 марта 2002