ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱ
НАЦИОНАЛЬНАЯ АКАДЕМИЯ НАУК АРМЕНИИ
NATIONAL ACADEMY OF SCIENCES OF ARMENIA
ДОКЛАДЫ ԶԵԿՈՒՅՑՆԵՐ REPORTS

Հատոր Том Volume 119 2019 № 1

MATHEMATICS

УДК 517.9

H. A. Asatryan[1], A. H. Kamalyan[2], M. I. Karakhanyan[2]

On $\mathcal{L}$-convolution Type Operators with Semi-Almost Periodic Symbols

(Submitted by academician A. B. Nersessian 6/II 2019)

Keywords: *semi-almost periodic functions, $\mathcal{L}$-Wiener-Hopf operator, Fredholm operator.*

1. Introduction. The semi-Fredholm theory of Wiener-Hopf operators with semi-almost periodic symbols was developed in works [1] and [2] (see also [3]).

In works [4] and [5] the concept of the $\mathcal{L}$-convolution type operator is introduced by replacing the Fourier transform in the definition of the convolution operator by the operator which transforms the Sturm-Liouville operator on the whole axis into the operator of multiplication by an independent variable. The concept of the $\mathcal{L}$-Wiener-Hopf operator is also introduced in a natural way. In the case of zero potential these two notions coincide with classical convolution operators and Wiener-Hopf operators, respectively. The precise definitions of $\mathcal{L}$-convolution type and $\mathcal{L}$-Wiener-Hopf operators in the case of reflectionless potentials will be given in the next section. In this work we extend the results of [1, 2] to the case of $\mathcal{L}$-Wiener-Hopf operator with a reflectionless potential.

2. The $\mathcal{L}$-Wiener-Hopf operator. Let $c: \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function satisfying the condition $\int_{-\infty}^{\infty} (1 + |x|)c(x)\,dx < \infty$. An important role in the spectral theory of the Sturm-Liouville equation

$$-y''(x) + c(x)y(x) = \lambda^2 y(x), \quad x \in \mathbb{R}, \lambda \in \mathbb{C}$$

is played by the Jost solutions $e_+(x,\lambda)$ ($x \in \mathbb{R}$, $\operatorname{Im}\lambda \geq 0$) and $e_-(x,\lambda)$ ($x \in \mathbb{R}$, $\operatorname{Im}\lambda \leq 0$) defined by boundary conditions

$$\lim_{x\to\pm\infty} e^{-i\lambda x} e_\pm(x,\lambda) = 1, \quad \lim_{x\to\pm\infty} e^{-i\lambda x} e'_\pm(x,\lambda) = i\lambda$$

(see, e.g., [6]). For $\lambda \in \mathbb{R} \setminus \{0\}$ the pairs of functions $e_+(x,\lambda)$, $e_+(x,-\lambda)$ and $e_-(x,\lambda)$, $e_-(x,-\lambda)$ form fundamental systems of solutions of the Sturm-Liouville equation (see [6]) and hence $e_+(x,\lambda)$ can be represented as

$$e_+(x,\lambda) = b(\lambda)e_-(x,-\lambda) + b_0(\lambda)e_-(x,\lambda).$$

If the reflection coefficients $r_\pm(\lambda) := \mp b(\mp\lambda)/b_0(\lambda)$ vanish identically, the potential c is called reflectionless (see, e. g., [7], [8]).

It is known (see [7]-[9]) that every reflectionless potential has a representation of the form

$$c(x) = -2\frac{d^2}{dx^2}(\ln \Delta(x)) \tag{2.1}$$

where

$$\Delta(x) = \det\left[\delta_{ij} + \frac{m_j \exp(-(\mu_i+\mu_j)x)}{\mu_i+\mu_j}\right] \quad i,j = 1,\dots,N, \tag{2.2}$$

δ_{ij} is the Kronecker delta, μ_k, m_k $(k = 1,\dots,N)$ are positive numbers such that $\mu_k \neq \mu_j$ for $k \neq j$. Reflectionless potentials are connected with a family of explicit solutions of the Korteweg–de Vries equation, the so-called $\mathcal{N}$-soliton solutions (see [8], [9]). Let the potential c be given by (2.1), (2.2). The operators $\mathcal{L}$ and $\mathcal{L}_0$, defined on the Sobolev space $\mathcal{W}_2^2(\mathbb{R})$ by the formulas $\mathcal{L}y = -y'' + cy$, $\mathcal{L}_0 y = -y''$, are self-adjoint (see [10]). Let H_d be the direct sum of all eigenspaces of the operator $\mathcal{L}$, and let $\varphi_1,\dots,\varphi_N$ be the orthonormal basis of H_d, uniquely determined by the system of linear equations

$$\varphi_k(x) + \sum_{s=1}^{N} \frac{m_k m_s e^{-(\mu_k+\mu_s)x}}{\mu_k+\mu_s}\varphi_s(x) = m_k e^{-\mu_k x}, \qquad k = 1,\dots,N$$

(see [7]-[9]). Consider the functions

$$u^-(x,\lambda) := t(\lambda)e^{i\lambda x}\left(1 - \sum_{k=1}^{N}\frac{m_k e^{-\mu_k x}}{\mu_k - i\lambda}\varphi_k(x)\right),$$

$$u^+(x,\lambda) := e^{-i\lambda x}\left(1 - \sum_{k=1}^{N}\frac{m_k e^{-\mu_k x}}{\mu_k + i\lambda}\varphi_k(x)\right),$$

where the transmission coefficient $t(\lambda)$ is defined by

$$t(\lambda) = b_0^{-1}(\lambda) := \prod_{k=1}^{N}\frac{\lambda + i\mu_k}{\lambda - i\mu_k}.$$

Note that in the case of the zero potential the subspace H_d coincides with the zero subspace, $t(\lambda) \equiv 1$ and hence $u^\mp(x,\lambda) = e^{\pm i\lambda x}$.

Further, $m(g)$ will denote the operator of multiplication by a function (or a matrix-function) g, i.e., $(m(g)y)(x) = g(x)y(x)$. The operators $J: L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $\pi_\pm: L_p(\mathbb{R}) \to L_p(\mathbb{R}_\pm)$, $\pi_\pm^0: L_p(\mathbb{R}_\pm) \to L_p(\mathbb{R})$ $(1 \le p \le \infty$, $\mathbb{R}_+ = (0,\infty)$, $\mathbb{R}_- = \mathbb{R} \setminus \mathbb{R}_+)$ are given by the formulas

$$(Jy)(x) = y(-x), \qquad (\pi_\pm y)(x) = y(x),$$

$$(\pi_+^0 y)(x) = \begin{cases} y(x), & x \in \mathbb{R}_+ \\ 0, & x \in \mathbb{R}_-, \end{cases} \qquad (\pi_-^0 y)(x) = \begin{cases} 0, & x \in \mathbb{R}_+ \\ y(x), & x \in \mathbb{R}_-. \end{cases}$$

Consider the space $L_2(\Delta, \delta)$ where $\Delta = \{1, \dots, N\}$ and δ is the Dirac measure on Δ. The unitary operator $\widetilde{U}: H_d \to L_2(\Delta, \delta)$ is defined by the formula $\widetilde{U}\varphi_k = \xi_k$ where $\xi_k(j) = \delta_{kj}$. Consider also the operators $U_\mp$, $U: L_2(\mathbb{R}) \to L_2(\mathbb{R})$, $\widehat{U}: H_c \oplus H_d \to L_2(\mathbb{R}) \oplus L_2(\Delta, \delta)$ defined by the formulas

$$(U_\mp y)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^\mp(x,\lambda) y(x) dx \quad (\lambda \in \mathbb{R}),$$

$$U = m(\chi_+)U_- + m(\chi_-)JU_+,$$

$$\widehat{U} = \begin{pmatrix} U|_{H_c} & 0 \\ 0 & \widetilde{U} \end{pmatrix},$$

where $H_c = L_2(\mathbb{R}) \ominus H_d$, $\chi_\pm$ is the characteristic function of the set $\mathbb{R}_\pm$, and the integrals converge in the norm of the space $L_2(\mathbb{R})$.

Note that in the case $c = 0$ the equalities $U_- = U = \widehat{U} = F$, $U_+ = F^{-1}$ hold, where F denotes the Fourier transform:

$$(Fy)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} y(x)\, dx, \qquad y \in L_2(\mathbb{R}).$$

The following statement is true:

Theorem 2.1. The operators $U_\mp$, U are bounded, the operator $\widehat{U}$ is unitary and on a dense subspace of $L_2(\mathbb{R})$ the equality $\widehat{U}\hat{\mathcal{L}}\widehat{U}^* = m(\lambda^2)$ holds.

Let $d = (d_1, \dots, d_N)^T \in \mathbb{C}^N$. Define the operator $\widehat{m}(d): L_2(\Delta, \delta) \to L_2(\Delta, \delta)$ by the formula $\widehat{m}(d)(\xi) = (d_1\xi_1, \dots, d_N\xi_N)^T$.

Let $\mathcal{M}_{p,\mathcal{L}}$ $(1 \le p < \infty)$ denote the set of all functions $a \in L_\infty(\mathbb{R})$ such that the operator

$$\widehat{U}^* \begin{pmatrix} m(a) & 0 \\ 0 & \widehat{m}(d) \end{pmatrix} \widehat{U}$$

is $L_p(\mathbb{R})$-bounded on the subspace $L_p(\mathbb{R}) \cap L_2(\mathbb{R})$. The continuous extension of this operator to $L_p(\mathbb{R})$ will be denoted by $W_\mathcal{L}^0(a,d)$ and will be called the *$\mathcal{L}$-convolution type operator* with an $\mathcal{L}$-symbol (a,d) on the space $L_p(\mathbb{R})$.

The operator $W_\mathcal{L}(a,d) = \pi_+ W_\mathcal{L}^0(a,d)\pi_+^0$ will be called the *$\mathcal{L}$-Wiener-Hopf operator*. In the case $\mathcal{L} = \mathcal{L}_0$ we will use the notations $\mathcal{M}_p$, $W^0(a)$,

$W(a)$ instead of $\mathcal{M}_{p,\mathcal{L}_0}$, $W^0_{\mathcal{L}_0}(a,d)$, $W_{\mathcal{L}_0}(a,d)$. Obviously, $W^0(a)$ and $W(a)$ are the convolution integral operator and the Wiener-Hopf operator, respectively, and $\mathcal{M}_p$ is the set of all Fourier multipliers (see [11], [3]). It is known (see [11]) that $\mathcal{M}_p$ is a Banach algebra with the norm $\|a\|_{\mu_p} = \|W^0(a)\|_{B(L_p(\mathbb{R}))}$. In particular, $\mathcal{M}_2$ coincides with $L_\infty(\mathbb{R})$.

Theorem 2.2. *The inclusion* $\mathcal{M}_p \subset \mathcal{M}_{p,\mathcal{L}}$ $(1 \le p < \infty)$ *holds.*

3. Main results. The closure of the algebra of all piecewise constant functions in $\mathcal{M}_p$ $(1 \le p < \infty)$ will be denoted by PC_p. It is well known (see [11]) that $PC_p = PC_2 = PC$, where PC is the class of functions having finite one-sided limits $a(x \pm 0)$ at each point $x \in \mathbb{R}$ and also at $x = \pm\infty$. Further, it is known that $PC_p = \mathcal{M}_p \cap PC$ (see [3]).

Let $\overline{\mathbb{R}} := [-\infty, \infty]$, and let $C_p(\overline{\mathbb{R}}) := PC \cap C(\mathbb{R})$ with $C(\mathbb{R})$ being the set of all continuous complex-valued functions on $\mathbb{R}$. In particular, $C(\overline{\mathbb{R}}) := C_2(\overline{\mathbb{R}}) = PC \cap C(\mathbb{R})$. $C_0(\mathbb{R})$ will denote the subalgebra of $C(\overline{\mathbb{R}})$, consisting of all $a \in C(\overline{\mathbb{R}})$ with $a(\pm\infty) = 0$.

Further, let AP^0 be the algebra of all almost periodic polynomials, i.e., the algebra of all functions $p\colon \mathbb{R} \to \mathbb{C}$ which can be written as a finite sum

$$p(x) = \sum \alpha_j e^{i\lambda_j x}, \quad \alpha_j \in \mathbb{C},\ \lambda_j \in \mathbb{R}.$$

Let AP_p denote the smallest closed subalgebra of $\mathcal{M}_p$ $(1 < p < \infty)$ containing AP^0, and let SAP_p denote the smallest closed subalgebra of $\mathcal{M}_p$ containing $\mathbb{C}_p(\mathbb{R})$ and AP_p. The algebra AP_p (SAP_p) lies in AP_2 (SAP_2) which itself coincides with the algebra AP (SAP) of all Bohr almost periodic functions (semi-almost periodic functions). Every function $a \in SAP_p$ has a representation of the form

$$a = (1-u)a_\ell + ua_r + a_0 \tag{3.1}$$

where $a_\ell, a_r \in AP_p$, $a_0 \in \mathcal{M}_p \cap C_0(\mathbb{R})$ and $u \in C(\overline{\mathbb{R}})$ is a fixed increasing function satisfying conditions $u(-\infty) = 0$, $u(+\infty) = 1$ (see [2], [3]). The functions a_ℓ, a_r do not depend on the choice of u and are uniquely determined by the function a (see [1]-[3]).

The group of all invertible elements of an algebra A will be denoted by GA.

It is well known that a function $a \in SAP_p$ ($a \in AP_p$) belongs to $GSAP_p$ (GAP_p) whenever $a \in GL_\infty(\mathbb{R})$, i.e.,

$$\inf|a(\lambda)| > 0, \quad \lambda \in \mathbb{R}. \tag{3.2}$$

According to the Bohr theorem on the argument of an almost periodic function (see [12]), for $a \in GAP$ there exist a real number $\varkappa(a)$ and a function $\psi \in AP$ such that

$$a(x) = e^{i\varkappa(a)x}e^{\psi(x)} \text{ for all } x \in \mathbb{R}.$$

The uniquely determinable number $\varkappa(a)$ is called the *mean motion* of the function a and can be computed by the formula

$$\varkappa(a) = \lim_{\ell\to\infty} \frac{1}{2\ell}[(\arg a)(\ell) - (\arg a)(-\ell)].$$

Here $\arg a$ is to be understood as an arbitrary fixed function from $C(\mathbb{R})$, satisfying the equality $a = |a|e^{i\arg a}$.

Let $M(\psi) := \lim_{\ell\to\infty} \frac{1}{2\ell}\int_{-\ell}^{\ell}\psi(x)dx$ be the Bohr mean value of the function ψ. The number $\xi(a) := e^{M(\psi)}$ is uniquely determined by the function $a \in GAP$; it is called the *geometric mean value* of the function a. For any $a \in GSAP_p$, the functions a_ℓ and a_r, determined from (3.1), belong to GAP_p (see [2, 3]). Furthermore, the following equalities hold (see [3]):

$$\xi(a_r) = \exp \lim_{\ell\to\infty} \frac{1}{\ell}\int_0^{\ell} [\log|a(x)| + i(\arg a)(x) - i\,\varkappa(a_r)x]\,dx,$$

$$\xi(a_\ell) = \exp \lim_{\ell\to\infty} \frac{1}{\ell}\int_{-\ell}^{0} [\log|a(x)| + i(\arg a)(x) - i\,\varkappa(a_\ell)x]\,dx.$$

The following two theorems describe the semi-Fredholm properties of the operator $W_{\mathcal{L}}(a,d)$.

Theorem 3.1. *Let $a \in SAP_p \setminus \{0\}$ with $1 < p < \infty$. Condition (3.2) is necessary for the normal solvability of the operator $W_{\mathcal{L}}(a,d)$ in the space $L_p(\mathbb{R}_+)$. In order that the operator $W_{\mathcal{L}}(a,d)$ be normally solvable, it is necessary and sufficient that along (3.2) one of these two conditions hold:*

1. $\varkappa(a_\ell)\,\varkappa(a_r) \geq 0$ and $\varkappa(a_\ell) + \varkappa(a_r) \neq 0$.
2. $\varkappa(a_\ell) = \varkappa(a_r) = 0$ and

$$\inf_{x\in\mathbb{R}_+} \left|\frac{1}{2}\big(\xi(a_r) + \xi(a_\ell)\big) - \frac{1}{2}\big(\xi(a_r) - \xi(a_\ell)\big)\operatorname{cth}\pi\left(\frac{i}{p} + x\right)\right| > 0 \qquad (3.3)$$

Theorem 3.2. *Let the operator $W_{\mathcal{L}}(a,d)$ be normally solvable in the space $L_p(\mathbb{R}_+)$ with $1 < p < \infty$. Then the following assertions are true:*

1) *If* $\varkappa(a_\ell) + \varkappa(a_r) > 0$, *then* $\dim\ker W_{\mathcal{L}}(a,d) < \infty$, $\dim\operatorname{Coker} W_{\mathcal{L}}(a,d) = \infty$ and $\dim\ker W_{\mathcal{L}}(a,0) = 0$. *In the case $p = 2$ the operator $W_{\mathcal{L}}(a,0)$ is left invertible.*

2) *If* $\varkappa(a_\ell) + \varkappa(a_r) < 0$, *then* $\dim\ker W_{\mathcal{L}}(a,d) = \infty$, $\dim\operatorname{Coker} W_{\mathcal{L}}(a,d) < \infty$ and $\dim\operatorname{Coker} W_{\mathcal{L}}(a,0) = 0$. *In the case $p = 2$ the operator $W_{\mathcal{L}}(a,0)$ is right invertible.*

3) If $\varkappa(a_\ell) = \varkappa(a_r) = 0$ and condition (3.3) is satisfied, then the $W_{\mathcal{L}}(a,d)$ is an Fredholm operator and

$$\operatorname{Ind} W_{\mathcal{L}}(a,d) = -\frac{1}{2\pi}[(\arg a)(+\infty) - (\arg a)(-\infty)] + \frac{1}{p} - \left\{\frac{1}{p} + \frac{1}{2\pi}\left(\arg\frac{\xi(a_\ell)}{\xi(a_r)}\right)\right\}$$

where $\{s\}$ denotes the fractional part of the real number s.

[1] FernUniversität in Hagen, Germany
e-mail: hayk.asatryan@fernuni-hagen.de, asatrianh@gmail.com
[2] Yerevan State University;
e-mail: armen.kamalyan@ysu.am, kamalyan_armen@yahoo.com
m_karakhanyan@ysu.am, m_karakhanyan@yahoo.com

H. A. Asatryan, A. H. Kamalyan, M. I. Karakhanyan

On $\mathcal{L}$-convolution Type Operators with Semi-Almost Periodic Symbols

The notions of the $\mathcal{L}$-convolution operator and the $\mathcal{L}$-Wiener-Hopf operator are introduced by replacing the Fourier transform in the definition of the convolution operator by a unitary operator which transforms the Sturm-Liouville operator $\mathcal{L}$ on the whole axis to the operator of multiplication by an independent variable. It is considered the case when the potential of the operator is reflectionless and the symbol of the $\mathcal{L}$-Wiener-Hopf operator is a semi-almost periodic function. Criteria for semi-Fredholm and Fredholm properties of the $\mathcal{L}$-Wiener-Hopf operator are revealed. In the Fredholm case a formula for the index is obtained.

Հ. Ա. Ասատրյան, Ա. Հ. Քամալյան, Մ. Ի. Կարախանյան

Կիսա-համարյա պարբերական սիմվոլներով $\mathcal{L}$-փաթեթի տիպի օպերատորների մասին

Փաթեթի օպերատորի սահմանման մեջ Ֆուրիեի ձևափոխությունը փոխարինելով առանցքի վրա սահմանված Շտուրմ-Լիուվիլի օպերատորը անկախ փոփոխականով բազմապատկման օպերատորին բերող ունիտար օպերատորով ներմուծվել են $\mathcal{L}$-փաթեթի և $\mathcal{L}$-Վիներ-Հոպֆի օպերատորները: Դիտարկվել է այն դեպքը, երբ $\mathcal{L}$ օպերատորի պոտենցիալը չանդրադարձնող է, իսկ $\mathcal{L}$-Վիներ-Հոպֆի օպերատորի սիմվոլը կիսա-համարյա պարբերական ֆունկցիա է: Բացահայտվել են $\mathcal{L}$-Վիներ-Հոպֆի օպերատորի կիսա-ֆրեդհոլմյան և ֆրեդհոլմյան լինելու պայմանները: Ֆրեդհոլմյան դեպքում ստացվել է ինդեքսի բանաձև:

А. А. Асатрян, А. Г. Камалян, М. И. Караханян

Об операторах типа -свертки с полу-почти периодическими символами

Заменой в определении оператора свертки преобразования Фурье на унитарный оператор, приводящий оператор $\mathcal{L}$ Штурма – Лиувилля на оси к оператору умножения на независимую переменную, введены понятия оператора $\mathcal{L}$-свертки и оператора $\mathcal{L}$-Винера–Хопфа. Рассмотрен случай, когда потенциал оператора является безотражательным, а символ оператора $\mathcal{L}$-Винера–Хопфа – полу-почти периодической функцией. Выявлены условия полу-фредгольмовости и фредгольмовости оператора $\mathcal{L}$-Винера–Хопфа. В фредгольмовом случае получена формула для индекса.

References

1. *Sarason D.* – Duke. Math. J. 1977. V. 44. № 2. P. 356-364. DOI:10.1215/S0012-7094-77-04415-5.
2. *Duduchava R.V., Saginashvili A.I.* – Differ. Uravn. 1981. V. 17. № 2. P. 301-312.
3. *Böttcher A., Karlovich Yu.I., Spitkosky I.M.* Convolution Operators and Factorization of Almost Periodic Matrix Functions. Basel; Boston; Berlin: Birkhäuser. 2002.
4. *Kamalyan A.H., Spitkovsky I.M.* – Math Notes. 2018. V. 104. № 3. P. 404-416. DOI: 10.1134/S0001434618090080.
5. *Kamalyan A. H., Karakhanyan M. I., Hovhannisyan A. H.* – Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences). 2018. V. 53. № 3. P. 134-138. DOI: 10.3103/S1068362318030032.
6. *Marchenko V.A.* – Sturm-Liouville Operators and Applications. Basel; Boston; Stuttgart. Birkhäuser. 1986.
7. *Yurko V.A.* Introduction to the Theory of Inverse Spectral Problems [in Russian]. Fizmatlit. M. 2007.
8. *Novikov S., Manakov S.V., Pitaevskii L.P., Zakharov V.E.* Theory of Solitons: The Inverse Scattering Method. New York; London. Consultants Bureau. 1984.
9. *Bhatnagar P.L.* Nonlinear Waves in One-dimensional Dispersive Systems. Clarendon Press. Oxford. 1979.
10. *Faddeev L.D.* – Journal of Soviet Mathematics. 1976. V. 5. № 3. P. 334–396. DOI: 10.1007/BF01083780.
11. *Duduchava R.V.* Convolution Integral Equations with Discontinuous Presymbols, Singular Integral Equations with Fixed Singularities and Their Applications to Problems of Mechanics [in Russian]. Metsniereba. Tbilisi. 1973.
12. *Levitan B.M.* Almost Periodic Functions [in Russian]. Fizmatlit. Moscow. 1953.