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Foreign member of NAS RA A. P. Seyranian[1], Y. Wang[2]

On Stability of Periodic Solutions for Duffing's Equation

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Key words: *Duffing's equation, stability of periodic solutions, frequency-response curve, numerical simulation*

1. Introduction. General approach to the stability study of periodic solutions is related to a classical Lyapunov's theorem based on a linear approximation [1-3]. This theorem reduces the stability study of periodic solutions to the stability of the system linearized at the periodic motion. Since linearized systems contain periodic coefficients the theory of parametric resonance can be applied. Such approach with the analysis of Floquet multipliers is used in [4-6]. The other traditional approach to stability study of periodic solutions is related to approximate averaging and multiple scales methods which reduce original time-dependent dynamical systems to autonomous systems. In this case stability study is reduced to analysis of fixed points [7-9].

In the present paper we study the stability of periodic solutions of the harmonically excited Duffing's equation with the direct application of the Lyapunov theorem. The damping coefficient and excitation amplitude are assumed to be small. Periodic solutions are found with the use of approximate methods. We derive the stability conditions and find stable and unstable regimes on the frequency-response curve. Two types of detuning parameter are considered and corresponding frequency-response curves are compared with the results obtained by numerical simulation.

2. Duffing's equation. We consider the harmonically excited Duffing's equation

$$\ddot{u} + 2\mu\dot{u} + \omega_0^2 u + \alpha u^3 = f\cos\Omega t \tag{1}$$

Here it is assumed that the excitation frequency is close to the natural frequency $\Omega = \omega_0 + \sigma$ (primary resonance), and the constants μ, α, f and σ are small quantities of the same order $o(1)$. Equation (1) admits a periodic solution in the form [2]

$$u_0(t) = a\cos(\Omega t - \gamma) + \frac{\alpha a^3}{32\omega_0^2}\cos(3\Omega t - 3\gamma) + o(\alpha), \tag{2}$$

where the term $o(\alpha)$ contains higher harmonics, and the amplitude a and phase γ are constants of the order $O(1)$ satisfying in the first approximation the equations

$$\left[\left(\Omega - \omega_0 - \frac{3\alpha a^2}{8\omega_0}\right)^2 + \mu^2\right] a^2 = \frac{f^2}{4\omega_0^2}, \tag{3}$$

$$\gamma = \arctan\left(\frac{\mu a}{\frac{3\alpha a^3}{8\omega_0} - a(\Omega - \omega_0)}\right) \tag{4}$$

Formulas (3) and (4) were derived in [1, 2] using the method of multiple scales, and similar relations were obtained in [7] with the use of the averaging method. For given constants $\Omega, \omega_0, \alpha, \mu, f$ (3) is a cubic equation on the unknown squared amplitude a^2.

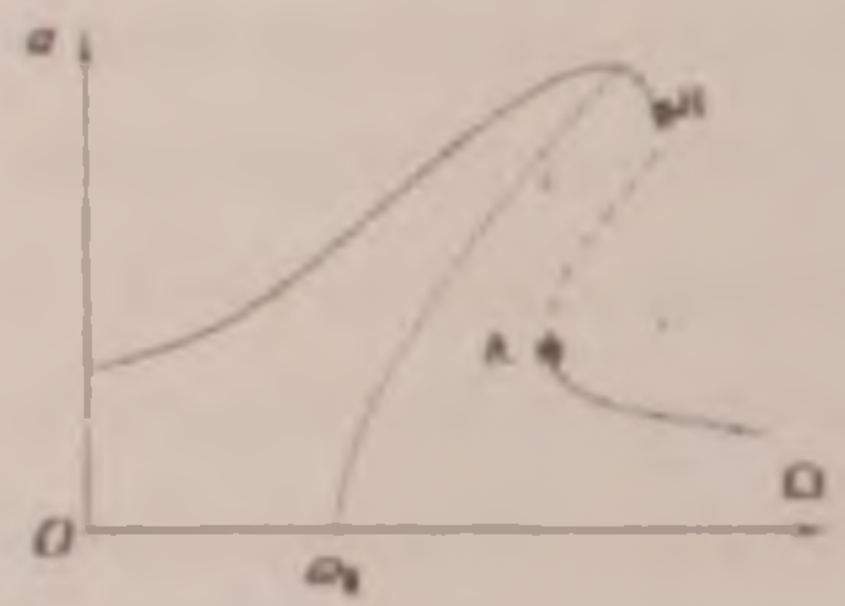

Fig 1. Frequency-response curve of a Duffing's oscillator near the primary resonance

The plot of a as a function of Ω for given other quantities is called a *frequency-response curve*. To draw this curve it is convenient to express Ω from (3) as a function of a

$$\Omega = \omega_0 + \frac{3\alpha a^2}{8\omega_0} \pm \sqrt{\frac{f^2}{4\omega_0^2 a^2} - \mu^2}. \tag{5}$$

This formula gives two branches Ω_+ and Ω_- lying respectively on the right and left sides of the parabola $\Omega = \omega_0 + 3\alpha a^2/8\omega_0$, see Fig 1. A typical frequency-response curve for $\alpha > 0$ is presented in the same figure. Note that the cases $\alpha > 0$ and $\alpha < 0$ are symmetric with respect to the axis $\Omega = \omega_0$. Such multivaluable behavior $a(\Omega)$ is typical for nonlinear systems. For some values Ω it is possible to have three different regimes a and γ of the periodic solution (2). Some of them might be stable, and some unstable.

3. The stability study of periodic solutions. To study the stability of periodic solution (2) we take small variation

$$u(t) = u_0(t) + v(t), \tag{6}$$

substitute this expression into (1), and then linearize the obtained equation with respect to $v(t)$. Thus we get a linear equation for the variation

$$\ddot{v} + 2\mu\dot{v} + [\omega_0^2 + 3\alpha u_0^2(t)]v = 0. \tag{7}$$

According to Lyapunov's theorem [3] the stability of periodic solution (2) is governed by equation (7). If the solution $v(t)$ to this equation is stable, the periodic motion (2) is stable, and vice versa.

Substituting (2) into (7) and omitting in the square brackets the terms of higher order than $O(\alpha)$ we find

$$\ddot{v} + 2\mu\dot{v} + [\omega_0^2 + 3\alpha a^2\cos^2(\Omega t - \gamma)]v = 0. \tag{8}$$

Introducing notation $\tau = \Omega t - \gamma$ we obtain damped Hill's equation in the form

$$\frac{d^2v}{d\tau^2} + \frac{2\mu}{\omega_0}\frac{dv}{d\tau} + \left[1 - \frac{2\sigma}{\omega_0} + \frac{3\alpha a^2}{\omega_0^2}\cos^2\tau\right]v = 0. \tag{9}$$

Comparison with the standard Hill's equation with damping [3,10] implies

$$\beta = \frac{2\mu}{\omega_0},\ \omega^2 = 1 - \frac{2\sigma}{\omega_0},\ \varepsilon = \frac{3\alpha a^2}{\omega_0^2},\ \varphi(\tau) = \cos^2\tau,\ k = 2. \tag{10}$$

Calculating the Fourier coefficients we find the instability region in the first approximation as a semi-cone [3,10]

$$\left(\Omega - \omega_0 - \frac{3\alpha a^2}{4\omega_0}\right)^2 + \mu^2 < \left(\frac{3\alpha a^2}{8\omega_0}\right)^2. \tag{11}$$

The excitation frequency Ω and the amplitude a are related by formula (5). Condition (11) can be transformed to the inequality

$$\left(\Omega - \omega_0 - \frac{3\alpha a^2}{8\omega_0}\right)\left(\Omega - \omega_0 - \frac{9\alpha a^2}{8\omega_0}\right) + \mu^2 < 0. \tag{12}$$

Verifying condition (12) for the branch Ω_- from (5) for $\alpha > 0$ we obtain the inequality of the opposite sign implying the stability of the periodic solution (2). Substituting in (12) the branch Ω_+ from (5) we obtain that for $\alpha > 0$ it is satisfied only when

$$a_* < a < a_{**}. \tag{13}$$

where a_* and a_{**} are the two roots of the equation

$$\frac{f^4}{9\alpha^2\omega_0^2a^8} = \frac{f^2}{4\omega_0^2a^2} - \mu^2. \tag{14}$$

We note that equation (14) is equivalent to the condition $d\Omega_+/da = 0$ indicating that the unstable regime (13) lies between the points A and B shown in Fig. 1 by dashed line. Other parts of the branch Ω_+, shown in Fig. 1 by solid line, correspond to the stable periodic solutions (2). The points A and B are the catastrophe points due to a jump of the amplitude a. The case $\alpha < 0$ can be treated similarly.

Simple analysis of equation (14) shows that for nonzero ω_0, μ, α, f it possesses two distinct roots $0 < a_* < a_{**}$ if the following inequality holds

$$\frac{\mu^6\omega_0^6}{f^4\alpha^2} < \frac{3^5}{4^8}. \tag{15}$$

For the opposite sign of this inequality equation (14) has no real roots, and in the case of equality in (15) it possesses a double real root. If condition (15) is violated then periodic solution (2) is stable on both branches Ω_-, Ω_+ and there is no jump in the amplitude a on the frequency-response curve.

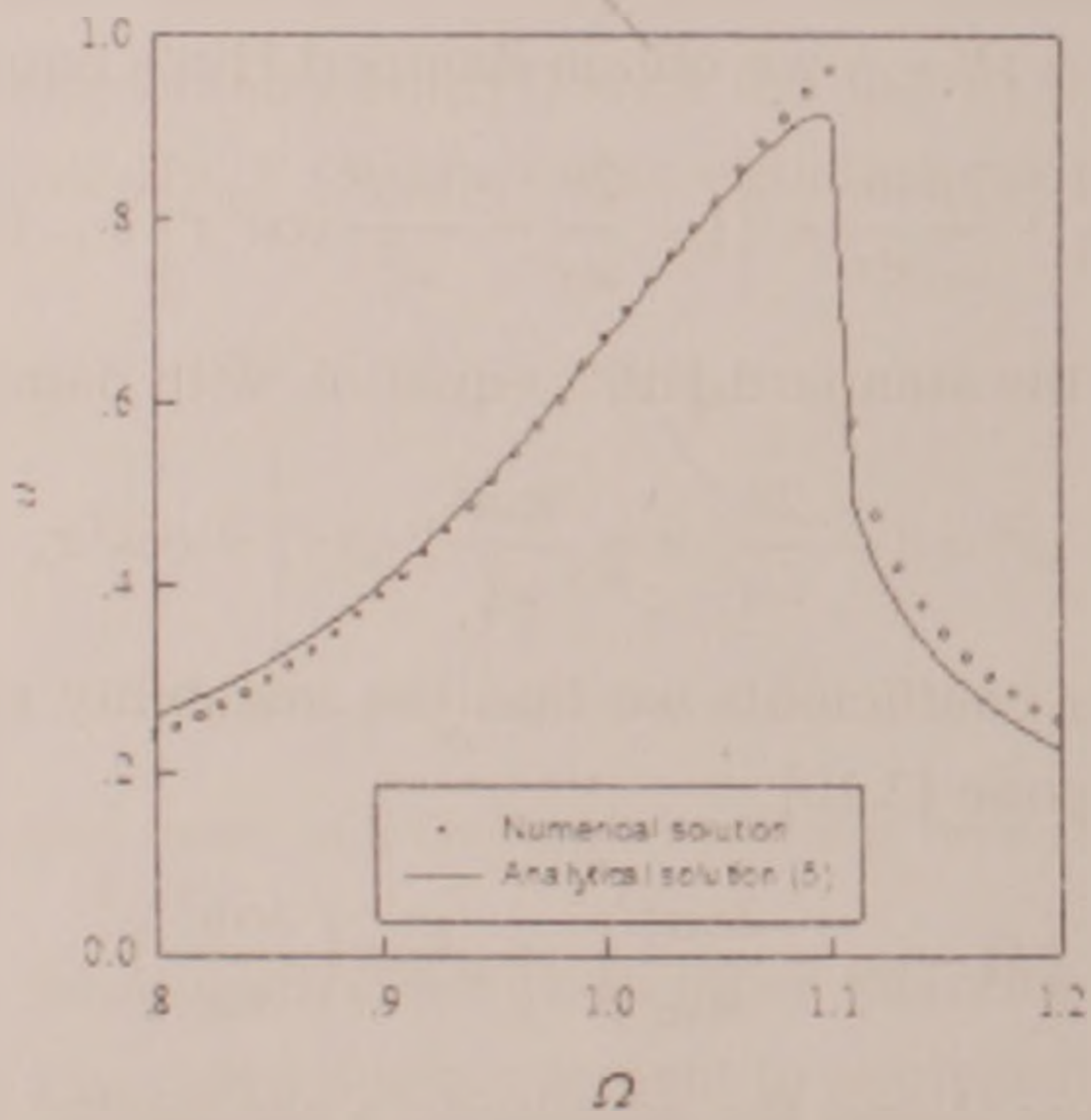

Fig. 2. Frequency-response curves with $f = 0.1$, $\mu = 0.05$, $\omega_0 = 1$ and $\alpha = 1/3$. Solid line: analytical solution (5). Circles: numerical solution.

A real frequency-response curve (5) for Duffing's equation at the specific values of parameters is presented and compared with numerical simulation results in Fig. 2. It is seen that in this case the part AB corresponding to the unstable regime of the curve is almost vertical since inequality (15) is satisfied but the roots a_* and a_{**} are very close to each other.

It is interesting to observe that the frequency-response curve changes if we introduce another detuning parameter $\Omega^2 - \omega_0^2 = \sigma$ where σ is the small parameter of the same order as α. Then instead of (3) the analytical expression for the curve takes the form [11]

$$\left(1-\left(\frac{\Omega}{\omega_0}\right)^2+\frac{3\alpha}{4\omega_0^2}a^2\right)^2+\left(\frac{2\mu\Omega}{\omega_0^2}\right)^2=\left(\frac{f}{a\omega_0^2}\right)^2 \qquad (16)$$

In spite of expressions (3) and (16) are asymptotically equivalent near the primary resonance, in this case we have better agreement between analytical and numerical simulation results, Fig. 3.

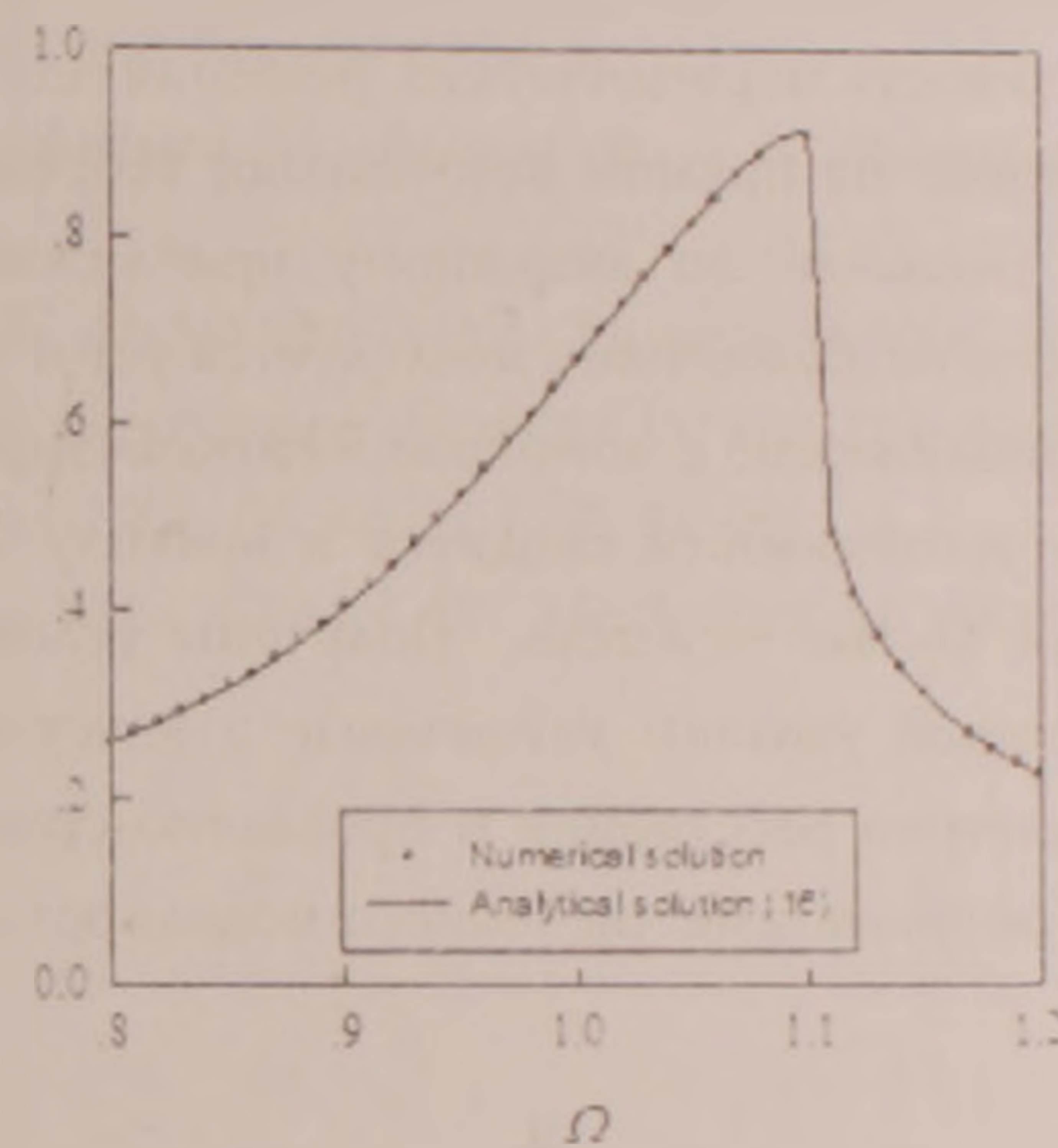

Fig. 3. Frequency-response curve with $f = 0.1$, $\mu = 0.05$, $\omega_0 = 1$ and $\alpha = 1/3$. Solid line: analytical solution with the modified detuned parameter (16). Circles: numerical solution

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[1]Institute of Mechanics, Lomonosov Moscow State University, Russia
[2]Dalian University of Technology, China

Foreign member of NAS RA A. P. Seyranian, Y. Wang

On Stability of Periodic Solutions for Duffing's Equation

The stability study of periodic solutions for Duffing's equation is investigated. The method of the investigation is related to direct application of the Lyapunov theorem for the stability of periodic motions based on a linear approximation. For the harmonically excited Duffing's equation we examine the stability of periodic solutions obtained approximately with the use of the perturbation technique. For primary resonance we reduce the stability problem to the stability study of solutions of a linear Mathieu-Hill equation. The stability

conditions are derived on the frequency-response curve, both stable and unstable regimes are found. Two types of detuning parameter are considered and corresponding frequency-response curves are compared with the results obtained by numerical simulation.

Иностранный член НАН РА А. П. Сейранян, У. Ванг

Об устойчивости периодических решений для уравнения Дуффинга

Исследована устойчивость периодических решений для уравнения Дуффинга. Метод исследования основан на прямом применении теоремы Ляпунова об устойчивости периодических решений по линейному приближению. Для уравнения Дуффинга с гармоническим возбуждением исследуется устойчивость периодических решений, полученных приближенно с помощью метода возмущений. Для основного резонанса исследование устойчивости сводится к анализу областей устойчивости для линейного уравнения Матье — Хилла. Получены условия устойчивости и на амплитудно-частотной кривой указаны устойчивые и неустойчивые режимы. Рассмотрены два вида параметров расстройки и проведено сравнение амплитудночастотных характеристик с результатами численного моделирования.

ՀՀ ԳԱԱ արտասահմանյան անդամ Ա. Պ. Սեյրանյան, Ու. Վանգ

Դուֆինգի հավասարման պարբերական լուծումների կայունության մասին

Հետազոտված է Դուֆինգի հավասարման պարբերական լուծումների կայունությունը: Հետազոտության մեթոդը հիմնված է գծային մոտավորությամբ կայունության պարբերական լուծումների վրա Լյապունովի թեորեմի ուղղակի կիրառմամբ: Հետազոտված է հարմոնիկ գրգռված Դուֆինգի հավասարման համար պարբերական լուծումների կայունությունը, որոնք ստացվում են մոտավոր ձևով գրգռման մեթոդի օգնությամբ: Կայունության հետազոտությունը հիմնական ռեզոնանսի համար բերվում է Մաթյե - Հիլլի գծային հավասարման կայունության տիրույթների անալիզի: Ստացված են կայունության պայմանները, եւ ցույց են տրված լայնույթահաճախային կորագծի վրա կայուն եւ անկայուն ռեժիմները: Դիտարկված են խանգարման երկու տեսակ պարամետրեր, եւ արված է լայնույթահաճախային բնութագրիչների համեմատությունը թվային մոդելավորման արդյունքների հետ:

References

1. *Nayfeh A.H., Mook D.T.* - Nonlinear Oscillations. Wiley. 1979.

2. *Thomsen J.J.* Vibrations and Stability. Advanced Theory, Analysis and Tools. Springer. 2003.

3. *Seyranian A.P., Mailybaev A.A.* Multiparameter Stability Theory with Mechanical Applications. World Scientific. 2003.

4. *Njoku F.I., Omari P.* - Applied Mathematics and Computation. 2003. V. 135. P. 471-490.

5. *Chen H., Li Y.* - Nonlinearity. 2008. V. 21. P. 2485-2503.

6. *Huang J.L., Su R.K.L., Chen S.H.* - Computers and Structures. 2009. V. 87. P. 1624-1630.

7. *Bogolyubov N.N., Mitropolsky Yu.A.* Asymptotic Methods in the Theory of Nonlinear Oscillations. Gordon and Breach. 1961.

8. *Guennoun K., Houssni M., Belhaq M.* - Nonlinear Dynamics. 2002. V. 27. P. 211-236.

9. *El-Bassiouny A.F., Abdel-Khalik A.* - Physica Scripta. 2010. V. 81. 01 5008 (8 P.)

10. *Seyranian A.P.* Resonance domains for the Hill equation with allowance for damping. Doklady Physics. 2001. V. 46. N 1. P. 41-44.

11. *Liu Y.Z., Chen W.L., Chen L.Q.* Mechanics of Vibrations. Higher Education Press, Beijing. 1998.