ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱ
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On Rejection of GM_d Conjecture

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1. Introduction. Let us start with some notation. Denote

$$\bar{\mathbf{x}} = (x_1, x_2, ..., x_d) \in \mathbb{R}^d, \alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in Z_+^d,$$

$$\bar{\mathbf{x}}^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot ... \cdot x_d^{\alpha_d}, |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d.$$

Denote also by $\Pi_n = \Pi_n(\mathbb{R}^d)$ the space of polynomials in d variables of total degree not exceeding n:

$$\Pi_n^d = \left\{ p(\bar{\mathbf{x}}) = \sum_{|\alpha| \le n} a_\alpha \bar{\mathbf{x}}^\alpha, a_\alpha \in \mathbb{R}, \bar{\mathbf{x}} \in \mathbb{R}^d \right\}.$$

We have that

$$N := N(n, d) := dim\Pi_n^d = \binom{n+d}{d}.$$

The interpolation problem with a set of knots

$$\mathcal{X}_s := \{\bar{\mathbf{x}}^{(1)}, \bar{\mathbf{x}}^{(2)}, ..., \bar{\mathbf{x}}^{(s)}\} \subset \mathbb{R}^d$$

and Π_n^d is called poised if for any data $\{c_1, c_2, ..., c_s\}$ there is a unique polynomial $p \in \Pi_n^d$, called interpolation polynomial, such that

$$p(\bar{\mathbf{x}}^{(k)}) = c_k, k = 1, 2, ..., s. \qquad (1)$$

These conditions give a system of s linear equations with N unknowns (the coefficients of the polynomial). The poisedness means that this system has a unique solution for any right side values. This implies the following necessary condition for the poisedness:

$$s = N,$$

i.e., the number of equations equals the number of unknowns. From now on we will assume that this condition holds.

A polynomial $p_A^* \in \Pi_n^d$ is called fundamental for the knot $A = (\bar{\mathbf{x}}^{(k)}) \in \mathcal{X}$, with $\mathcal{X} := \mathcal{X}_N$, if

$$p_A^*(\bar{\mathbf{x}}^{(k)}) = \delta_{jk}, \quad 1 \le j \le N,$$

where δ_{jk} is the symbol of Kronecker. Note that the interpolation problem is poised if and only if all interpolation knots have fundamental polynomials. Let us also mention that in the case of poisedness all fundamental polynomials are unique, since they are interpolation polynomials. It is worth mentioning that in this case the fundamental polynomials are of exact degree n. Indeed, otherwise, if the degree of a fundamental polynomial p_A^* is less than n, then multiplication of p_A^* by a linear polynomial vanishing at A would give a nontrivial polynomial in Π_n^d vanishing on $\mathcal{X}$, contradicting the poisedness of $\mathcal{X}$.

We denote by the same letter, say h, the hyperplane, and the polynomial from Π_1^d which gives rise to the hyperplane.

Definition 1.1. *A shift of linear space of dimension k in $\mathbb{R}^d$ is called k-dimensional flat or k-flat.*

For example a point, line and hyperplane in $\mathbb{R}^d$ are a 0, 1 and $d-1$ flats, respectively. We accept that the empty set is (-1)-flat.

A k-flat h we denote by $h\{k\}$. In the case of hyperplane, i.e., $k = d-1$, we also use the notation $h := h\{d-1\}$. As we will see, each k-flat can contain no more than $N(n,k)$ knots of a GC_n set.

Definition 1.2. *Let $h\{k\}$ be a k-flat. A set of knots $\mathcal{X} \subset h\{k\}$ is said to satisfy geometric characterization for Π_n^k (GC_n for short), if*

1. $\#\mathcal{X} = N(n,k)$

2. For each fixed knot $A \in \mathcal{X}$ there are no more than n $(k-1)$-flats $h_1^A, h_2^A, \ldots, h_m^A$ $(m = m_A \le n)$ in $h\{k\}$ whose union contains all the knots of X but A.

In the case of 2 we say that the knot A uses the (k-1)-flats $h_1^A, h_2^A, \ldots, h_k^A$.

In particular, in the case of $h\{k\} = \mathbb{R}^d$, i.e., $k = d$, $\mathcal{X} \subset \mathbb{R}^d$ is a GC_n set if $\#\mathcal{X} = N$ and for each knot $A \in \mathcal{X}$ there are no more than n hyperplanes in $\mathbb{R}^d$ whose union contains all the knots of X but A. Let us note that the condition 2 in this case means that the fundamental polynomial for the knot A is a product of linear factors:

$$p_A^* = \gamma_A \cdot h_1^A \cdot h_2^A \cdot \ldots \cdot h_k^A,$$

where $h_1^A, h_2^A, ..., h_k^A$ are the hyperplanes used by A and γ_A is a constant. Thus, each GC_n set is Π_n^d-poised. Therefore, the number of hyperplanes used by any knot in GC_n sets is exactly n, i.e., $m_A = n$ for each $A \in \mathcal{X}$.

M. Gasca and J.I. Maeztu made the following conjecture on GC_n sets in $\mathbb{R}^2$:

GM - **conjecture.** *(See [1]) If $\mathcal{X} \subset \mathbb{R}^2$ is a GC_n set, then there exists a line, which passes through $n+1$ knots of $\mathcal{X}$.*

C. de Boor generalized this for $\mathbb{R}^d$:

GM_d - **conjecture.** *(See [2]) If $\mathcal{X} \subset \mathbb{R}^d$ is a GC_n set, then there exists a hyperplane which passes through $N(n, d-1)$ knots of $\mathcal{X}$.*

Above mentioned line and hyperplane are called maximal.

In this paper we provide an example of GC_n set in $\mathbb{R}^6$ with no maximal hyperplane thus rejecting this conjecture. Let us start with generalization of the concept of maximal.

According to [3] Lemma 2.1 for each k-flat $h\{k\}$ $\#(h\{k\} \cap \mathcal{X}) \leq N(n, k)$.

Definition 1.3. *A k-flat $h\{k\}$ is called maximal for GC_n set $\mathcal{X}$, if $h\{k\}$ contains $N(n, k)$ knots of $\mathcal{X}$.*

Thus, each line passing through $n+1$ knots of $\mathcal{X}$ is a maximal line, i.e. maximal 1-flat for $\mathcal{X}$ in $\mathbb{R}^d$.

Next we bring the definition of natural lattices of Chung and Yao [4].

Definition 1.4. *Assume that the set of $n+d$ hyperplanes $H = \{h_1, h_2, ..., h_{n+d}\}$ is in general position. The set of all $N(n, d)$ intersection knots of each d hyperplanes from H, is called a natural lattice of degree n in $\mathbb{R}^d$ or briefly NL_n.*

It is easily seen that every NL_n is GC_n and each hyperplane $h_i, i = 1, 2, ..., n+d$, is maximal for NL_n. Furthermore, $n+d$ is the maximal number of maximal hyperplanes any GC_n set can have.

Definition 1.5. *Let $\mathcal{X} \subset \mathbb{R}^d$ be a GC_n set. We say that $\mathcal{X}$ has default r or that $\mathcal{X}$ is an r-lattice, if the number of maximal hyperplanes of $\mathcal{X}$ equals $n+d-r$.*

Thus, NL_n is $0-lattice$, more NL_n lattices are characterized by the fact that they are $0-lattices$.

In [3] we give the characterization of 1-lattices in $\mathbb{R}^d$. Next we are going to describe it. We start with natural lattice of degree $n-1$, i.e., intersection knots of $n+d-1$ maximal hyperplanes. We call these knots *black*. Then we add $\binom{n+d-1}{d-1}$ arbitrary knots one by one on each intersection line of the maximal hyperplanes and call them *white* knots. We put a restriction on white knots. Namely we require that they are not lying on a hyperplane (otherwise we will get a natural $0-lattice$).

2. An example of GC_2 set in $\mathbb{R}^6$ without maximal hyperplanes.

First we present some preliminaries. Let us start with the following result of Carnicer and Gasca in [5].

Theorem 2.1. *If every planar GC_n set for $n \leq \nu$ has a maximal line, then every such set has at least three maximal lines.*

In [2] C. de Boor made a natural conjecture concerning this result.

Conjecture 2.1. *If every GC_n set in $\mathbb{R}^d$ has a maximal hyperplane, then every such GC_n set has at least $d+1$ maximal hyperplanes.*

In the same paper C. de Boor provides a counterexample GC_2 set in $\mathbb{R}^3$ with just 3 maximal planes thus rejecting the above conjecture in the case $d = 3$. He starts the construction of the example with an 1-lattice of degree 2 in $\mathbb{R}^3$ with 4 maximal planes. Then he shifts one of the white points from a maximal hyperplane so that the resulting set is again GC_2, and in this way making it non-maximal (see Figure 1).

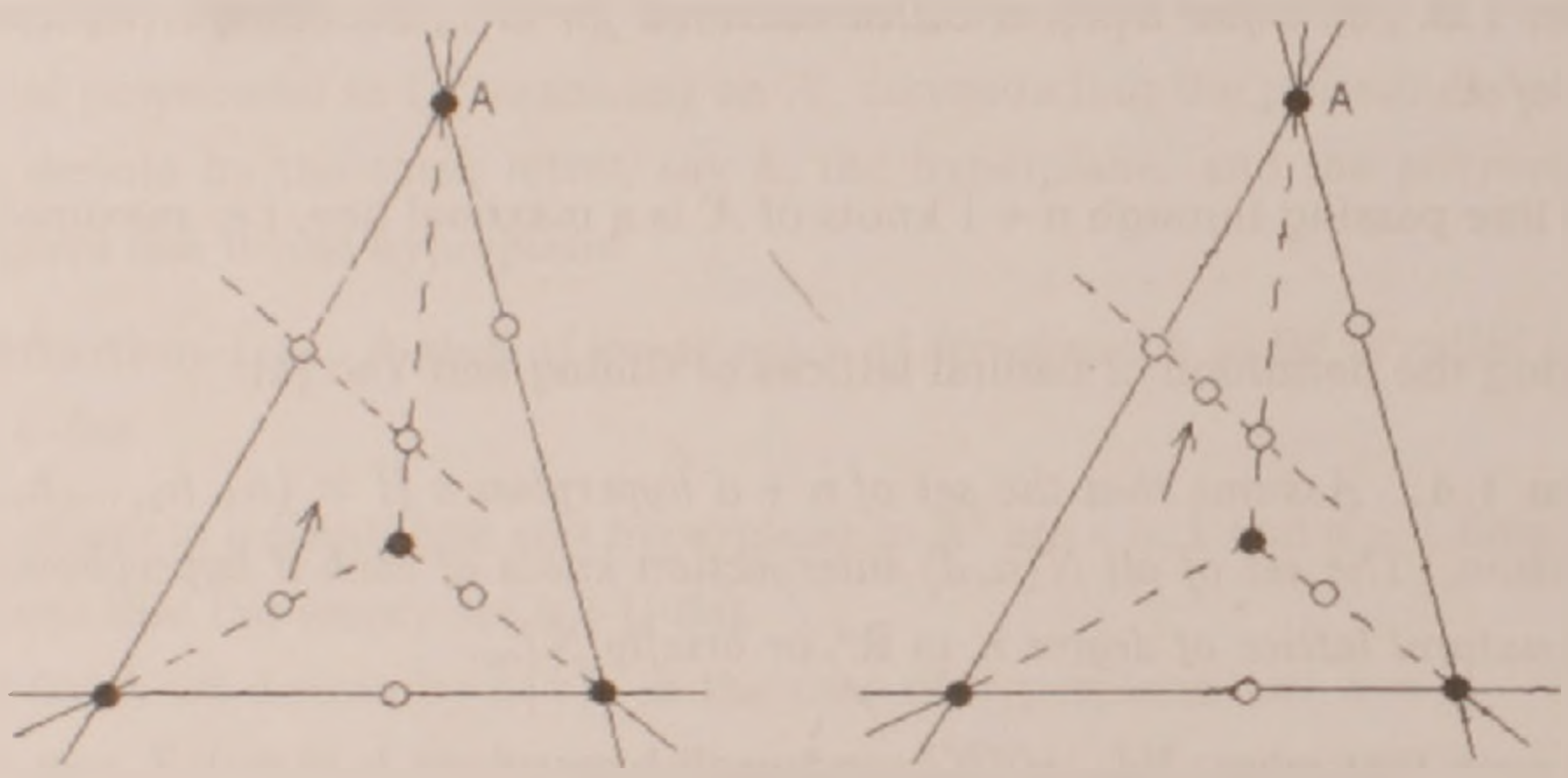

Figure 1: GC_2 sets with 4 and 3 maximal planes.

Next by exploiting the idea of C. de Boor, we are going to construct GC_2 set without maximal hyperplanes in $\mathbb{R}^6$.

We start with the definition of Δ_2-structure:

Definition 2.6. *(See [6]) Six nodes in $\mathbb{R}^2$ are said to have Δ_2-structure, if three nodes are the vertices of a triangle, and the remaining three lie one by one on the lines containing the sides of the triangle.*

Definition 2.7. *Let x be a black knot of a Δ_2-structure. We say that we do the movement* **toward** *the knot x if we move the white knot lying on the line passing through other two black knots of the Δ_2-structure to the line passing through other two white knots.*

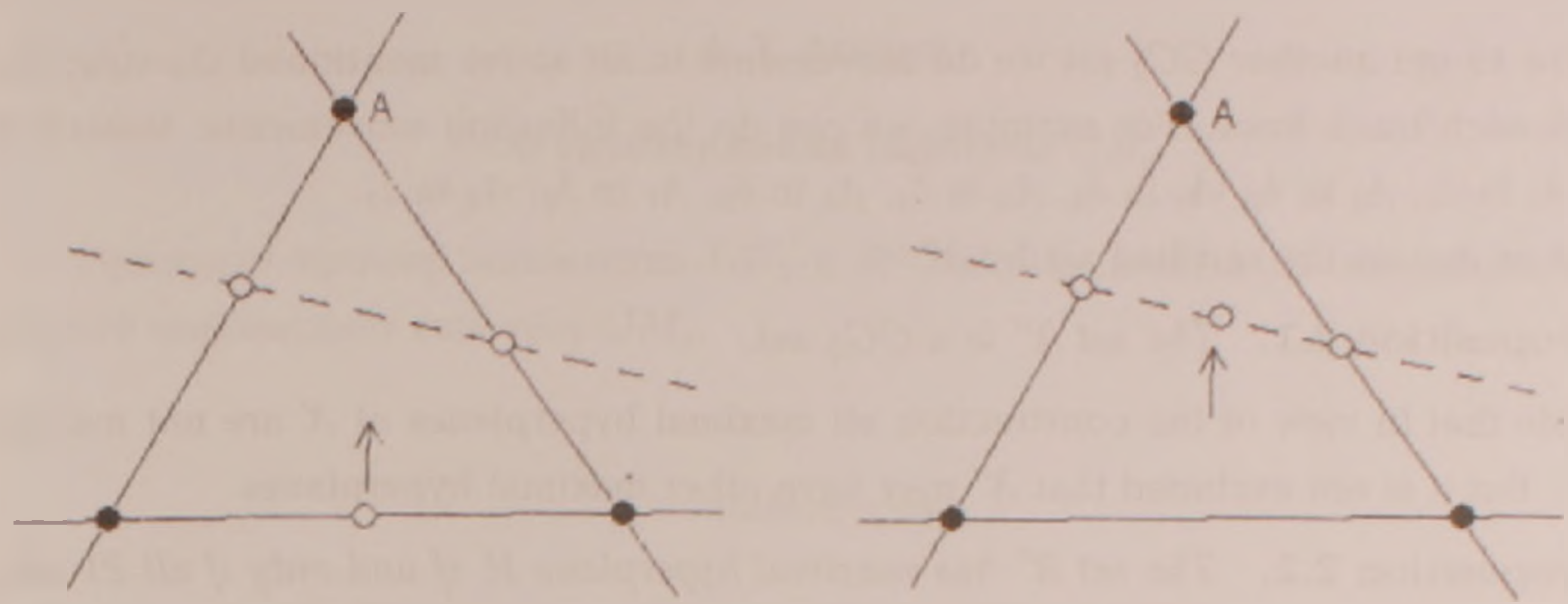

Figure 2: The movement toward the knot A

In the Figure 2 the movement is done toward the knot A.

Assume that $\mathcal{X}$ is 1-lattice in $\mathbb{R}^6$, i.e., 7 black knots and 21 white knots one by one in lines connecting each two black knots. Notice that each black knot uses only one maximal hyperplane which is passing through the remaining black knots. In view of the Definition 2.7, by the movement toward some black knot, in a certain plane containing a Δ_2-structure we turn the maximal hyperplane used by that knot into non-maximal.

Definition 2.8. *We say that two Δ_2-structures are independent from each other or just independent, if they have no more than one common knot, which is black for both.*

Let us denote black knots of $\mathcal{X}$ by $A_1, A_2, A_3, A_4, A_5, A_6, A_7$. We can mention 7 independent Δ_2-structures. These are: $\delta_1 := (A_1, A_2, A_3)$, $\delta_2 := (A_1, A_4, A_5)$, $\delta_3 := (A_1, A_6, A_7)$, $\delta_4 := (A_2, A_4, A_6)$, $\delta_5 := (A_2, A_5, A_7)$, $\delta_6 := (A_3, A_4, A_7)$, $\delta_7 := (A_3, A_5, A_6)$. As we see every black knot belongs to just three Δ_2-structures (see Figure 3).

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
δ_1	○	●	●				
δ_2	●			○	●		
δ_3	●					○	●
δ_4		○		●		●	
δ_5		●			○		●
δ_6			●	●			○
δ_7			○		●	●	

Figure 3: The movement toward the knot A

Now to get another GC_2 set we do movements in all above mentioned Δ_2-structures toward each black knot. For example, we can do the following movements: toward the knot A_1 in δ_1, A_4 in δ_2, A_6 in δ_3, A_2 in δ_4, A_5 in δ_5, A_7 in δ_6, A_3 in δ_7.

Let us denote the resulted set by $\mathcal{X}'$.

Proposition 2.1. *The set $\mathcal{X}'$ is a CG_2 set.*

Note that in view of the construction all maximal hyperplanes of $\mathcal{X}$ are not maximal for $\mathcal{X}'$. But it is not excluded that $\mathcal{X}''$ may have other maximal hyperplanes.

Proposition 2.2. *The set $\mathcal{X}'$ has maximal hyperplane H if and only if all 21 white knots lie on H.*

Proposition 2.3. *There is an 1-lattice of degree 2 in $\mathbb{R}^6$ $\mathcal{X}_0$ such that ${\mathcal{X}'}_0$ has no maximal hyperplane. Moreover $\mathcal{X}_0$ can be obtained from any 1-lattice $\mathcal{X}$ by moving some white knot along the intersection line containing it.*

In such way we reject GM_d - conjecture for $\mathbb{R}^6$. Note that for $\mathbb{R}^d$ $(d \geq 6)$ there are at least $d+1$ pairwise independent Δ_2-structures. Hence we can construct similar counterexample for $\mathbb{R}^d$ $(d \geq 6)$.

In [7] we provide

Conjecture 2.2. *Each GC_n set in $\mathbb{R}^d$ has at least $\binom{d+1}{d-1}$ maximal lines.*

Note that it also holds for the set $\mathcal{X}'$ constructed in this section. Indeed, each movement makes a non-maximal line from maximal one and makes another maximal line from one non-maximal.

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On Rejection of GM_d Conjecture

An example of GC_2 set in $\mathbb{R}^6$ without maximal hyperplanes rejecting the conjecture GM_d is provided.

Ա. Տ. Ափոզյան

GM_d վարկածի հերքման վերաբերյալ

Բերված է GC_2 բազմության օրինակ $\mathbb{R}^6$ տարածությունում, որը չունի մաքսիմալ հիպերհարթություն՝ այդպիսով հերքելով GM_d վարկածը:

А. Т. Апозян

Об опровержении гипотезы GM_d

Приведен пример множества GC_2 в $\mathbb{R}^6$ без максимальных гиперплоскостей, который опровергает гипотезу GM_d.

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