ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱ
НАЦИОНАЛЬНАЯ АКАДЕМИЯ НАУК АРМЕНИИ
NATIONAL ACADEMY OF SCIENCES OF ARMENIA
ДОКЛАДЫ ԶԵԿՈՒՅՑՆԵՐ REPORTS

Հատոր Том Volume 110 2010 № 2

MATHEMATICS

УДК 517.984

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On the Minimal Norm of a Linear Operator Pencil

(Submitted by academician V.S. Zakaryan 30/XI 2009)

Key words: *linear operator pencil, minimal norm, numerical radius, reverse inequalities*

0. Let A and B be linear bounded operators, acting in a Hilbert space $(\mathcal{H}, \langle \bullet, \bullet \rangle)$ and t be a complex number. Denote by $SpA, W(A)$ and $w(A)$ respectively the spectrum, the numerical range and the numerical radius of A. The vector-function $A + tB$ is known as the pencil, generated by A and B. In some problems the least value of $\|A + tB\|$ plays very important role. Evidently there is at least one complex number t_0 such that $\inf\limits_{t\in\mathbb{C}} \|A + tB\| = \|A - t_0 B\|$. In what follows we investigate this problem for a special operator B, find the best value t_0, consider the problem in greater details for three particular cases and prove some inequalities.

1. Proposition 1. Let the operator B be one-to-one. Then

$$\inf_{t\in\mathbb{C}} \|A - tB\| = \sup_{\|x\|=1} \sqrt{\|Ax\|^2 - \frac{|\langle Ax, Bx\rangle|^2}{\|Bx\|}}. \tag{1}$$

Proof . Let $a, b \in \mathcal{H}$ be two non-zero elements. From elementary properties of the Hilbert space

$$\inf_{t\in\mathbb{C}} \|a - tb\|^2 = \|a\|^2 - \frac{|\langle a, b\rangle|^2}{\|b\|^2}.$$

Putting $a = Ax$, $b = Bx$, where x is arbitrary non-zero element from $\mathcal{H}$, we get

$$\inf_{t\in\mathbb{C}} \|(A - tB)x\|^2 = \|Ax\|^2 - \frac{|\langle Ax, Bx\rangle|^2}{\|Bx\|^2}.$$

According to a theorem of Asplund and Ptak [1]

$$\sup_{\|x\|=1} \inf_{t\in\mathbb{C}} \|(A - tB)x\| = \inf_{t\in\mathbb{C}} \sup_{\|x\|=1} \|(A - tB)x\|,$$

and it completes the proof.

To find the best value t_0 we impose an additional restriction on B. Suppose that B is bounded from below, i.e. $\|Bx\| \geq \delta\|x\|$, $\delta > 0$. Let $\{x_n\}$ be a sequence of unit vectors, approximating the supremum in (1). Then

$$\left|\frac{\langle Ax_n, Bx_n\rangle}{\|Bx_n\|} - t_0\,\|Bx_n\|\right|^2 = \frac{|\langle Ax_n, Bx_n\rangle|^2}{\|Bx_n\|^2} - 2\,\mathrm{Re}\,\langle Ax_n, t_0 Bx_n\rangle + |t_0|^2\,\|Bx_n\|^2 =$$

$$= \|(A - t_0 B)\,x_n\|^2 - \|Ax_n\|^2 + \frac{|\langle Ax_n, Bx_n\rangle|^2}{\|Bx_n\|^2}\ \mathbf{6}$$

$$\mathbf{6}\ \|A - t_0 B\|^2 - \|Ax_n\|^2 + \frac{|\langle Ax_n, Bx_n\rangle|^2}{\|Bx_n\|^2} \to 0.$$

As the operator B is bounded from below, the inequality

$$\left|\frac{\langle Ax_n, Bx_n\rangle}{\|Bx_n\|^2} - t_0\right| \leq \frac{1}{\delta}\left|\frac{\langle Ax_n, Bx_n\rangle}{\|Bx_n\|} - t_0\|Bx_n\|\right|$$

is satisfied. Thus,

$$t_0 = \lim_{n\to\infty}\frac{\langle Ax_n, Bx_n\rangle}{\|Bx_n\|^2}. \tag{2}$$

For $B = A^*$ we get $\inf\limits_{t\in\mathrm{C}}\|A - t_0A^*\| = \sup\limits_{\|x\|=1}\sqrt{\|Ax\|^2 - \dfrac{|\langle A^2x, x\rangle|^2}{\|A^*x\|^2}}$. If the operator A is normal, then for any $x \in \mathcal{H}$ the equality $\|A^*x\| = \|Ax\|$ is satisfied. In this case the condition, imposed on B may be dropped. Indeed, the norm in the left hand side may be calculated, taking vectors, belonging to the orthogonal complement of the null-space of A. Hence, A may be assumed to be one-to-one. If $\inf\limits_{t\in\mathrm{C}}\|A - tA^*\| = 0$, then $A = t_0A^*$, otherwise the sequence $\{\|Ax_n\|\}$ is bounded from below. Anyway, we get the following result (cf. [2], (2.10)).

Proposition 2. For any normal operator A

$$\inf_{t\in\mathrm{C}}\|A - t_0A^*\| = \sup_{\|x\|=1}\sqrt{\|Ax\|^2 - \frac{|\langle A^2x, x\rangle|^2}{\|Ax\|^2}}.$$

In this case

$$t_0 = \lim_{n\to\infty}\frac{\langle A^2x_n, x_n\rangle}{\|Ax_n\|^2}.$$

As $|\langle A^2x_n, x_n\rangle| = |\langle Ax_n, A^*x_n\rangle| \leq \|Ax_n\|\cdot\|A^*x_n\| = \|Ax_n\|^2$, we may deduce the inequality $|t_0| \leq 1$. Evidently, for a Hermitian (or skew-Hermitian) operator A $\inf\limits_{t\in\mathrm{C}}\|A - tA^*\| = 0$ and $|t_0| = 1$.

Example 1. Let

$$A = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix},\ \lambda_1, \lambda_2 \in \mathrm{C}.$$

The vector, maximizing the right hand side in (1) has the coordinates

$$x = \left\{ \frac{\sqrt{|\lambda_2|}}{\sqrt{|\lambda_1| + |\lambda_2|}}; \frac{\sqrt{|\lambda_1|}}{\sqrt{|\lambda_1| + |\lambda_2|}} \right\}. \tag{3}$$

We have

$$\inf_{t \in \mathrm{C}} \|A - tA^*\| = 2\frac{|\operatorname{Im}(\overline{\lambda_1}\lambda_2)|}{|\lambda_1| + |\lambda_2|}.$$

If $\lambda_1 = 1$, $\lambda_2 = i$, we get $\inf\limits_{t \in \mathrm{C}} \|A - tA^*\| = 1$ and $t_0 = 0$.

Further two particular cases of the special interest are considered. For the first one we put $B = I$, where I is the identity operator.

Proposition 3. For any operator A

$$\inf_{t \in \mathrm{C}} \|A - tI\| = \sup_{\|x\|=1} \sqrt{\|Ax\|^2 - |\langle Ax, x\rangle|^2}. \tag{4}$$

The proof will be omitted. For this case

$$t_0 = \lim_{n \to \infty} \langle Ax_n, x_n\rangle. \tag{5}$$

Evidently $t_0 \in \overline{W}(A)$, where the upper bar denotes the closure of the set.

Remark 1. The right hand side of (4) is known as the square root of the Bjork-Thomee constants of A (see [3]), where it has been shown that it coincides with Mirsky's constant

$$M(A) = \sup\{|\langle Au, \nu\rangle| : \|u\| = \|\nu\| = 1,\ \langle u, \nu\rangle = 0\}.$$

From formula (4) follows Dragomir's inequality ([4], 3.11)

$$\|A\|^2 - w^2(A) \le \inf_{t \in \mathrm{C}} \|A - tI\|^2. \tag{6}$$

Let A be the operator from Example 1. The supremum in the right hand side of (4) is attained on the element $x = \left\{ \frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2} \right\}$. Thus,

$$\|Ax\|^2 - |\langle Ax, x\rangle|^2 = \frac{|\lambda_1|^2 + |\lambda_2|^2}{2} - \frac{|\lambda_1 + \lambda_2|^2}{4}.$$

According to the parallelogram law this difference is equal to $\frac{|\lambda_1 - \lambda_2|^2}{4}$, so $\inf\limits_{t \in \mathrm{C}} \|A - tI\| = \frac{|\lambda_1 - \lambda_2|^2}{2}$. The best value is $t_0 = \frac{\lambda_1 + \lambda_2}{2}$.

Remark 2. For the operator defined by the matrix

$$A = \begin{pmatrix} \lambda_1 & \lambda_3 \\ 0 & \lambda_2 \end{pmatrix},\ \lambda_1, \lambda_2, \lambda_3 \in \mathrm{C}$$

in [5] is proved that

$$\inf_{t\in \mathrm{C}} \|A - tI\| = \frac{|\lambda_3| + \sqrt{|\lambda_1 - \lambda_2|^2 + |\lambda_3|^2}}{2}.$$

According to Schur decomposition $A = UTU^{-1}$, where U is unitary and T is triangular. As the operator norm is unitary invariant, the last formula settles the general case in two dimensional space.

The second problem is related with $m(A) = \inf_{t\in \mathrm{C}} \|I - tA\|$. This expression occurs when a Hilbert space operator equation $Ax = b$ is solved by Richardson stationary iterations ([6]). It defines the rate of convergence of iterations to the solution of the equation. Evidently $m(A) \le 1$ and a non trivial situation arises when $m(A) < 1$. Formula $A^{-1} = t \sum_{n=0}^{\infty} (I - tA)^n$ shows that in this case A is invertible.

Proposition 4. For any operator A

$$\inf_{t\in \mathrm{C}} \|I - tA\| = \sup_{ax\neq\theta} \sqrt{1 - \frac{|\langle Ax, x\rangle|^2}{\|Ax\|^2 \cdot \|x\|^2}}. \tag{7}$$

The proof may be found in [6]. Denote

$$p(A) = \inf_{Ax\neq\theta} \frac{|\langle Ax, x\rangle|}{\|Ax\| \cdot \|x\|}. \tag{8}$$

According to [6] for any operator A

$$m^2(A) + p^2(A) = 1. \tag{9}$$

We get

$$t_0 = \lim_{n\to\infty} \frac{\langle x_n, Ax_n\rangle}{\|Ax_n\|^2}, \tag{10}$$

where $\{x_n\}$ is a sequence of unit vectors, approximating the infimum in (8). Let $y_n = Ax_n$. Then $\lim_{n\to\infty} \frac{\langle x_n, Ax_n\rangle}{\|Ax_n\|^2} = \lim_{n\to\infty} \frac{\langle A^{-1}y_n, y_n\rangle}{\|y_n\|^2}$, meaning that $t_0 \in \overline{W}(A^{-1})$.

Proposition 5. For any operator A

$$\inf_{t\in \mathrm{C}} \|A - tI\| \le \|A\| \cdot \inf_{t\in \mathrm{C}} \|I - tA\|. \tag{11}$$

Proof. Let x have unit norm. Then $|\langle Ax, x\rangle| \ge p(A)\|Ax\|$ and

$$\|Ax\|^2 - |\langle Ax, x\rangle|^2 \le (1 - p^2(A))\|Ax\|^2.$$

Calculating the supremum of the both sides, we get (11).

Corollary. If $\inf_{t\in \mathrm{C}} \|A - tI\| = \|A\|$, then $\inf_{t\in \mathrm{C}} \|I - tA\| = 1$.

Example 3. Let $\lambda_1 = 2$, $\lambda_2 = -1$ in Example 1. Then $\inf\limits_{t\in C} \|I - tA\| = 1$ and $\inf\limits_{t\in C} \|A - tI\| = 1.5 < \|A\|$, so conditions $\inf\limits_{t\in C} \|I - tA\| = 1$ and $\inf\limits_{t\in C} \|A - tI\| = \|A\|$ are not equivalent.

Recalling (6),(11), we arrive at Dragomir's another inequality

$$\|A\|^2 - w^2(A) \le \|A\|^2 \inf_{t\in C} \|I - tA\|^2.$$

For the above considered two dimensional normal operator the vector, maximizing the right hand side of (7) is defined [7] by formula (3),

$$p = \frac{\sqrt{|\lambda_1\lambda_2|}}{|\lambda_1| + |\lambda_2|} |\operatorname{sgn} \overline{\lambda_1} + \operatorname{sgn} \overline{\lambda_2}|,$$

$$t_0 = \frac{\operatorname{sgn} \overline{\lambda}_1 + \operatorname{sgn} \overline{\lambda}_2}{|\lambda_1| + |\lambda_2|}$$

and

$$M = \frac{|\lambda_1 - \lambda_2|}{|\lambda_1| + |\lambda_2|}.$$

As $\|A\| = \max\{|\lambda_1|, |\lambda_2|\}$, the equality $|\lambda_1| = |\lambda_2|$ implies that the both sides of inequality (11) have the same value.

Now we suppose that $\inf\limits_{t\in C} \|I - tA\| < 1$. Then

$$\inf_{t\in C} \|A - tI\| = \inf_{\alpha\neq 0} \frac{\|I - \alpha A\|}{|\alpha|} \le \frac{1}{|t_0|} \inf_{t\in C} \|I - tA\|, \tag{12}$$

where t_0 is defined by (10).

When $\lambda_1, \lambda_2 > 0$, inequality (12) is reduced to an equality.

By the same way

$$\inf_{t\in C} \|I - tA\| = \inf_{\alpha\neq 0} \frac{\|A - \alpha I\|}{\|\alpha\|} \le \frac{1}{|t_0|} \inf_{t\in C} \|A - tI\|, \tag{13}$$

where t_0 is defined by (5).

2. It is interesting to note that considered above minimal norms for some operators have interesting geometrical meanings. Let for any complex t the norm of the operator $A - tI$ and its spectral radius $r(A - tI)$ coincide. Then

$$\inf_{t\in C} \|A - tI\| = \inf_{t\in C} \sup_{z\in SpA} |z - t|.$$

The expression in the right hand side is the radius of the smallest circle $C(z_0, R)$ containing the spectrum SpA of A. Hence the minimal norm of $A - tI$ equals the radius R of this circle and the optimal parameter t_0 is the affix z_0 of the circle's center. It is known [7] that for any compact subset $F \subset C$ the smallest circle exists, is unique and contains on its boundary at least two points, belonging to F.

For the second problem we have

$$\inf_{t\in\mathbb{C}}\|I-tA\|=\inf_{t\in\mathbb{C}}\sup_{z\in SpA}|tz-1|=\inf_{z_0\in\mathbb{C}}\frac{1}{|z_0|}\sup_{z\in SpA}|z-z_0|=\inf_{z_0\in\mathbb{C}}\frac{R}{|z_0|},$$

so we look for a circle, containing SpA and having the least $\dfrac{R}{|z_0|}$ ratio among all circles, satisfying this condition. This circle exists [8] if and only if the coordinate system's origin does not belong to the convex hull of SpA.

Using formulas (8) and (4), we can establish Cauchy-Bunyakovsky-Schwarz type reverse inequalities.

Example 4. Let $A=diag\left\{\lambda_1,\lambda_2,\ldots,\lambda_n\right\},\ \xi=\{\xi_1,\xi_2,\ldots,\xi_n\}\in\mathbb{C}^n$. Then

$$\|A\xi\|^2=\sum_k|\lambda_k\xi_k|^2\,,\ \langle A\xi,\xi\rangle=\sum_k\lambda_k\,|\xi_k|^2.$$

By (9)

$$\left|\sum_k\lambda_k\,|\xi_k|^2\right|^2\geqslant\left(1-m^2\,(A)\right)\sum_k|\lambda_k\xi_k|^2\sum_k|\xi_k|^2.$$

Denoting $|\xi_k|^2/\sum\limits_k|\xi_k|^2=p_k$, we get $p_k\geqslant 0,\sum p_k=1$ and

$$\left|\sum_k\lambda_k p_k\right|^2\geqslant\left(1-m^2\,(A)\right)\sum_k|\lambda_k|^2 p_k. \tag{14}$$

For arbitrary set of complex numbers $\{\lambda_k\}$ an algorithm of definition of $m(A)$ is described in [9].

For a particular case the last inequality is reduced to Kantorovich inequality. If $0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_n$, then $1-m^2\,(A)=\dfrac{4\lambda_1\lambda_n}{(\lambda_1+\lambda_n)^2}$ and

$$\left(\sum_k\lambda_k p_k\right)^2\geqslant\frac{4\lambda_1\lambda_n}{(\lambda_1+\lambda_n)^2}\sum_k\lambda_k^2 p_k.$$

The same inequality remains true, if for example, $\{\lambda_k\}$ is a subset of the closed circle with the center at $\dfrac{\lambda_1+\lambda_n}{2}$ and of radius $\dfrac{\lambda_n-\lambda_1}{2}$.

According to (4)

$$\sum p_k\lambda_k^2-\left(\sum p_k\lambda_k\right)^2\leqslant\frac{(\lambda_n-\lambda_1)^2}{4},$$

which may be compared with Shisha-Mond inequality ([10], 5.54)

$$\sqrt{\sum p_k\lambda_k^2}-\sum p_k\lambda_k\leqslant\frac{\lambda_n-\lambda_1}{4\,(\lambda_n+\lambda_1)}.$$

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On the Minimal Norm of a Linear Operator Pencil

The following problem is studying: how close one of the two given Hilbert space operators may be approximated by the multiples of another. Some particular cases are studied: the first, when the operator is approximated by its adjoint, and in more detailed manner; the second, when the operator is approximated by scalar operator and the identity operator is approximated by multiples of a fixed one. Some extremal geometrical problems are investigated and the generalization of known inequalities are established.

Л. З. Геворгян

О минимальной норме линейного операторного пучка

Исследуется следующая задача: насколько тесно один из двух данных операторов, действующих в гильбертовом пространстве, может быть аппроксимирован кратными другого. Рассмотрены частные случаи: аппроксимация оператора кратными сопряженного оператора; аппроксимация оператора скалярными операторами; аппроксимация единичного оператора кратными данного оператора, а также некоторые экстремальные геометрические задачи. Обобщены известные неравенства.

Լ. Զ. Գևորգյան

Օպերատորային գծային փնջի նվազագույն նորմի մասին

Ուսումնասիրվում է հետեւյալ խնդիրը: Հիլբերթյան փարածությունում գործող երկու օպերատորներից մեկը որքան սերտորեն կարող է մոտարկվել մյուսի պատիկներով: Քննարկվել են նաեւ մասնավոր դեպքեր, երբ օպերատորը մոտարկվում է իր համալուծով, օպերատորը մոտարկվում է սկալյար օպերատորով եւ միավոր օպերատորը մոտարկվում է տրված օպերատորի պատիկներով: Դիտարկվել են որոշ էքստրեմալ երկրաչափական խնդիրներ, եւ ընդհանրացվել են հայտնի անհավասարություններ:

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