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Fourier Formulae for Equidistant Hermite Trigonometric Interpolation

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1. For a given smooth function $f \in C^{p-1}[-1,1]$ we consider the sequence $T_{p,N}(f)(x)$, $p \geq 1$, $N \geq 1$, of trigonometric Hermite interpolation polynomials with prescribed values

$$T_{p,N}^{(s)}(f)\left(\frac{2k}{2N+1}\right) = f^{(s)}\left(\frac{2k}{2N+1}\right),\ s = 0, \cdots, p-1;\ |k| \leq N.$$

Trigonometric Hermite interpolation on equidistant nodes were discussed by different authors (see, for example, Kress [1], Nersessian [2], Sahakyan [3] and Berrut, Welscher [4] with references therein). A new idea has recently come up in Hermite trigonometric interpolation: considering the separate discrete Fourier transforms (DFTs) of the various derivatives of f and then writing the Hermite interpolant in terms of the thereby obtained coefficients. Berrut and Welcher [4] developed a formula for the Fourier coefficients in terms of those of the two classical trigonometric polynomials interpolating the values and those of the derivative separately. This formula treats the most customary case, i.e., the classical Hermite interpolant that uses only the first order derivatives at every point for an even number of equidistant points. As showed the authors this formula yields the coefficients with a single FFT. They also gave an aliasing formula for the error in the coefficients which, on its turn, yields error bounds and convergence results for differentiable as well as analytic functions.

It is well known that the resulting error of Hermite trigonometric interpolation is strongly dependent on the smoothness of the interpolated function. Interpolation of a 2-periodic and smooth function is highly effective. When the interpolated function has a point of discontinuity, the interpolation leads to the Gibbs phenomenon. The oscillations caused by this phenomenon are typically propagated into regions away from the singularity and degrade the quality of the approximations. Different ways of treating this deficiency have been suggested in the literature for the case $p = 1$. The idea of increasing the convergence rate by subtracting a polynomial that represents the discontinuities in the function and some of its first derivatives ("jumps") was suggested by Krylov [5] in 1906 and later, in 1964, by Lanczos [6]. The key problem lies in determining the singularity amplitudes that has been realized by Eckhoff [7-9] where the values of the "jumps" are solutions of the corresponding system of linear equations. The Krylov-Lanczos and the Krylov-Lanczos-Eckhoff methods were developed and generalized by a number of authors, see [10-15] with references therein.

In this paper we consider trigonometric Hermite interpolation with equidistant interpolation nodes and uniform multiplicities. Our method of construction of the trigonometric Hermite interpolants may be considered as a continuation of the method of Berrut and Welcher [4]. We derive relatively compact formula for the trigonometric Hermite interpolation that gives the interpolants as functions of the coefficients in the DFTs of the derivative values. We consider the case of an odd number of equidistant points for an arbitrary high number of derivatives of equal order at each of these nodes. Although we are discussing only the case of odd number of points our approach is valid also for even number of nodes. The accelerating convergence of interpolations were achieved by application of the Krylov-Lanczos approach. We also give formulae for the corrections that should be applied in order to soften the effect of the "jumps" at the endpoints when the interpolated function is not periodic. In this paper it will be assumed that the exact values of the "jumps" are known.

2. Let $f \in C^{p-1}[-1,1]$, $p \geq 1$. Let $\check{f}_m^{(j)}$ denote the discrete Fourier coefficients of the j-th derivative of f

$$\check{f}_m^{(j)} := \frac{1}{2N+1} \sum_{k=-N}^{N} f^{(j)}(x_k) e^{-i\pi m x_k}, \; x_k = \frac{2k}{2N+1}, \; j = 0, \cdots, p-1, \; |m| \leq N.$$

We set $\check{f}_m := \check{f}_m^{(0)}$.

Following [4], the sequence $T_{p,N}(f)$, $p \geq 1$, $N \geq 1$, of Hermite trigonometric interpolation polynomials will be defined by the formula

$$T_{p,N}(f)(x) := \sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_{j=0}^{p-1} a_j(m) \check{f}_m^{(j)},$$

where $\sigma = 0$ for odd values of parameter p and $\sigma = 1$ for even values. The unknown functions $\{a_j\}$ will be determined from the condition that $T_{p,N}(f)$ is exact for the set of functions $\{e^{i\pi r x}\}$, $r = -N(1-\sigma) - \left[\frac{p}{2}\right](2N+1), \cdots, N(1+\sigma) + \left[\frac{p-1}{2}\right](2N+1)$, where $[x]$ denotes the greatest integer less than or equal to x. We set $r = n + s(2N+1)$, $n = -N(1-\sigma), \cdots, N(1+\sigma)$; $s = -\left[\frac{p}{2}\right], \cdots, \left[\frac{p-1}{2}\right]$ and obtain the system of linear equations

$$e^{i\pi(m+s(2N+1))x} = \sum_{j=0}^{p-1} \alpha_s^j(m) a_j(m), \tag{1}$$

with Vandermonde matrix, for determining the functions $\{a_j\}$, where

$$\alpha_s(m) := i\pi(m + s(2N+1)).$$

We proceed as in [11] and construct the explicit solution of (1)

$$a_j(m) = \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} c_{k,j}(m) e^{i\pi(m+k(2N+1)))x},$$

where

$$c_{k,j}(m) = \frac{1}{\prod\limits_{\substack{\ell=-[\frac{p}{2}] \\ \ell\neq k}}^{[\frac{p-1}{2}]} (\alpha_k(m) - \alpha_\ell(m))} \sum_{s=j+1}^{p} \gamma_s(m) \alpha_k^{s-j-1}(m), \tag{2}$$

and the γ_s are the coefficients of the polynomial

$$\prod_{s=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} (x - \alpha_s(m)) = \sum_{s=0}^{p} \gamma_s(m)\, x^s.$$

This leads to the explicit form of the trigonometric Hermite interpolants

$$T_{p,N}(f)(x) = \sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_{j=0}^{p-1} \tilde{f}_m^{(j)} \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} c_{k,j}(m) e^{i\pi(m+k(2N+1))x}. \tag{3}$$

Remark. Trigonometric Hermite interpolation $T_{p,N}(f)(x)$ realizes trigonometric Hermite interpolation for odd number of grid points. It is worth to mention that overall idea of this section with further acceleration of convergence by the Krylov-Lanczos method is valid also for even number of nodes.

Theorem 0.1 *Let $f \in C^{p-1}[-1,1]$, $p \geq 1$. Then $T_{p,N}(f)$ is a trigonometric Hermite interpolation of f on equidistant grid $x_k = \frac{2k}{2N+1}$, $|k| \leq N$ with uniform multiplicities*

$$T_{p,N}^{(s)}(f)(x_k) = f^{(s)}(x_k),\ s = 0, \cdots, p-1.$$

Definition of the coefficients γ_s implies

$$\gamma_s(m) = (-i\pi(2N+1))^{p-s} \sum_{-[\frac{p}{2}]\le k_1<\cdots<k_{p-s}\le[\frac{p-1}{2}]} \prod_{\ell=k_1}^{k_{p-s}} \left(\frac{m}{2N+1}+\ell\right).$$

Inserting this into (2) we get for $k = -\left[\frac{p}{2}\right], \cdots, \left[\frac{p-1}{2}\right]$ and $j = 0, \cdots, p-1$

$$c_{k,j}(m) = \frac{1}{(i\pi(2N+1))^j}\beta_{k,j}\left(\frac{m}{2N+1}\right), \tag{4}$$

where

$$\beta_{k,j}(x) := \frac{1}{\prod\limits_{\substack{\ell=-[\frac{p}{2}] \\ \ell\neq k}}^{[\frac{p-1}{2}]} (k-\ell)} \sum_{s=j+1}^{p} (-1)^{p-s}\rho_s(x)(x+k)^{s-j-1}, \tag{5}$$

and the $\rho_j(x)$ are the coefficients of the polynomial

$$\prod_{s=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} (y+(x+s)) = \sum_{s=0}^{p} \rho_s(x) y^s. \tag{6}$$

Integration of the interpolation $T_{p,N}(f)(x)$ over the interval $(-1,1)$ leads to the quadrature formula

$$Q_{p,N}(f) := \int_{-1}^{1} T_{p,N}(f)(x)dx = \frac{2}{\prod\limits_{\substack{\ell=-[p/2] \\ \ell\neq 0}}^{[\frac{p-1}{2}]} \ell} \sum_{j=0}^{p-1} \frac{(-1)^j\rho_{j+1}(0)}{(i\pi)^j(2N+1))^{j+1}} \sum_{k=-N}^{N} f^{(j)}(x_k).$$

3. For $f \in C^q[-1,1]$ denote by $A_k(f)$ the "jump" of the k-th derivative of f

$$A_k(f) := f^{(k)}(1) - f^{(k)}(-1), \ k = 0, \cdots, q.$$

Throughout the paper it will be assumed that the exact values of the "jumps" are known.

By $f_n^{(j)}$ define the Fourier coefficients of the j-th derivative of f

$$f_n^{(j)} := \frac{1}{2}\int_{-1}^{1} f^{(j)}(x)e^{-i\pi nx}dx, \ j \ge 0.$$

We set $f_n := f_n^{(0)}$.

The following lemma is crucial for the Krylov-Lanczos method.

Lemma 0.2 *Suppose $f \in C^q[-1,1]$ for some $q \geq 1$. Then the following formula holds for $n \neq 0$*

$$\check{f}_n = \frac{(-1)^{n+1}}{2} \sum_{k=0}^{q-1} \frac{A_k(f)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^q} \int_{-1}^{1} f^{(q)}(x) e^{-i\pi n x} dx. \tag{7}$$

Equation (7) implies the basic expansion

$$f(x) = F(x) + \sum_{k=0}^{q-1} A_k(f) B_k(x) \tag{8}$$

of the approximated function where the B_k are 2-periodic extensions of the Bernoulli polynomials with the Fourier coefficients

$$B_{k,n} := \begin{cases} 0, n = 0, \\ \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, n \neq 0, \end{cases}$$

and F is a 2-periodic and smooth function ($F \in C^{q-1}(R)$) on the real line with the discrete Fourier coefficients

$$\check{F}_n = \check{f}_n - \sum_{k=0}^{q-1} A_k(f) \check{B}_{k,n}. \tag{9}$$

It is well known that [6]

$$B_0(x) = \frac{x}{2}, \; B_k(x) = \int B_{k-1}(x) dx, \; \int_{-1}^{1} B_k(x) dx = 0.$$

Equation (8) yields ($p \leq q$)

$$F^{(j)}(x) = f^{(j)}(x) - \frac{A_{j-1}(f)}{2} - \sum_{k=j}^{q-1} A_k(f) B_{k-j}(x), \; j = 1, \cdots, p-1. \tag{10}$$

Therefore

$$\check{F}_n^{(j)} = \check{f}_n^{(j)} - \sum_{k=j}^{q-1} A_k(f) \check{B}_{k-j,n}, \; n \neq 0, \; j = 1, \cdots, p-1, \tag{11}$$

and

$$\check{F}_0^{(j)} = \check{f}_0^{(j)} - \frac{A_{j-1}(f)}{2} - \sum_{k=j}^{q-1} A_k(f) \check{B}_{k-j,0}, \; j = 1, \cdots, p-1. \tag{12}$$

Approximation of F in (8) by $T_{p,N}(f)$, for $q \geq p$, leads to the following Hermite interpolation that we will call *Hermite-Krylov-Lanczos interpolation*

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$$\begin{aligned} T_{q,p,N}(f)(x) &:= \sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_{j=0}^{p-1} \check{F}_m^{(j)} \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} c_{k,j}(m) e^{i\pi(m+k(2N+1))x} \\ &+ \sum_{k=0}^{q-1} A_k(f) B_k(x),\ q \geq p, \end{aligned} \tag{13}$$

where the coefficients $\check{F}_m^{(j)}$ are defined by (9), (11), and (12).

Integration of the interpolation (13) over the interval $(-1,1)$ leads to the following quadrature formula that we will call *Hermite-Krylov-Lanczos quadrature*

$$Q_{q,p,N}(f) := \int_{-1}^{1} T_{q,p,N}(f)(x)dx = \frac{2}{\prod_{\substack{\ell=-[p/2] \\ \ell\neq 0}}^{[\frac{p-1}{2}]} \ell} \sum_{j=0}^{p-1} \frac{(-1)^j \rho_{j+1}(0)}{(i\pi)^j (2N+1)^{j+1}} \sum_{k=-N}^{N} F^{(j)}(x_k),\ q \geq p, \tag{14}$$

where for $F^{(j)}(x)$ we have representation (10).

We put

$$R_{q,p,N}(f)(x) := f(x) - T_{q,p,N}(f)(x),\ r_{q,p,N}(f) := \int_{-1}^{1} f(x)dx - Q_{q,p,N}(f),$$

and by $||f||$ we denote the standard norm in the space $L_2(-1,1)$

$$||f|| := \left(\int_{-1}^{1} |f(x)|^2 dx \right)^{1/2}.$$

The next theorem reveals the asymptotic behavior of the trigonometric Hermite interpolation.

Theorem 0.3 *Let $f \in C^q[-1,1]$, $q \geq 1$ be such that $f^{(q)}$ is absolutely continuous on $[-1,1]$. Then the following estimate holds ($q \geq p$)*

$$\lim_{N\to\infty} (2N+1)^{q+\frac{1}{2}} ||R_{q,p,N}(f)|| = |A_q(f)| t(q,p),$$

$$t(q,p) := \frac{1}{\sqrt{2}\,\pi^{q+1}} \left(\int_{-\frac{1-\sigma}{2}}^{\frac{1+\sigma}{2}} \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} \left| \sum_{j=0}^{p-1} \beta_{k,j}(x) \sum_s{}^{*} \frac{(-1)^s}{(x+s)^{q-j+1}} \right|^2 dx + \frac{2^{2q+2}}{(2q+1)p^{2q+1}} \right)^{\frac{1}{2}},$$

where

$$\sum_s{}^{*} := \sum_{s=-\infty}^{-[\frac{p}{2}]-1} + \sum_{s=[\frac{p-1}{2}]+1}^{\infty},$$

and the $\beta_{k,j}$ are defined by (5).

In the next theorem we reveal the asymptotic behavior of the Hermite-Krylov-Lanczos quadrature.

Theorem 0.4 *Let $f \in C^q[-1,1]$, $q \geq 1$ be such that $f^{(q)}$ is absolutely continuous on $[-1,1]$. Then the following estimate holds ($q \geq p$)*

$$\lim_{N\to\infty}(2N+1)^{q+1} r_{q,p,N}(f) = \frac{A_q(f)}{(i\pi)^{q+1}\prod_{\substack{\ell=-[\frac{p}{2}] \\ \ell\neq 0}}^{[\frac{p-1}{2}]}\ell}\sum_{j=0}^{p-1}(-1)^j\rho_{j+1}(0)\sum_s{}^{*}\frac{(-1)^s}{s^{q-j+1}},$$

where

$$\sum_s{}^{*} := \sum_{s=-\infty}^{-[\frac{p}{2}]-1} + \sum_{s=[\frac{p-1}{2}]+1}^{\infty},$$

and the $\rho_s(x)$ are defined by (6).

4. Consider the following simple function

$$f(x) = \sin(x-1). \tag{15}$$

In Table the uniform errors $\max_{|x|\leq 1}|R_{q,p,N}(f)(x)|$ are presented for various values of p, q and $N=1$ while interpolating the function (15) by the Hermite-Krylov-Lanczos interpolation. We see that the errors vary from the value 0.15 ($p=q=1$) to the value $3\cdot 10^{-15}$ ($q=10$ and $p=6$). Hence, using the same number of nodes ($2N+1=3$) we increase the precision of the quadrature by increasing the values of parameters p and q.

The uniform errors while approximating the function (15) by the Hermite-Krylov-Lanczos interpolation for various values of the parameters p and q. Here, $N=1$.

$q\backslash p$	1	2	3	4	5	6
1	0.15	–	–	–	–	–
2	$6.4\cdot10^{-3}$	$2.4\cdot10^{-3}$	–	–	–	–
3	$2.4\cdot10^{-3}$	$6.7\cdot10^{-4}$	$2.6\cdot10^{-4}$	–	–	–
4	$1.8\cdot10^{-4}$	$2.5\cdot10^{-5}$	$4.9\cdot10^{-6}$	$2.3\cdot10^{-6}$	–	–
5	$5.2\cdot10^{-5}$	$5.9\cdot10^{-6}$	$9.9\cdot10^{-7}$	$4.4\cdot10^{-7}$	$1.9\cdot10^{-7}$	–
6	$4.7\cdot10^{-6}$	$2.9\cdot10^{-7}$	$2.4\cdot10^{-8}$	$7.7\cdot10^{-9}$	$2.4\cdot10^{-9}$	$1.2\cdot10^{-9}$
7	$1.2\cdot10^{-6}$	$5.7\cdot10^{-8}$	$3.8\cdot10^{-9}$	$1.1\cdot10^{-9}$	$3.3\cdot10^{-10}$	$1.6\cdot10^{-10}$
8	$1.2\cdot10^{-7}$	$3.2\cdot10^{-9}$	$1.1\cdot10^{-10}$	$2.4\cdot10^{-11}$	$4.7\cdot10^{-12}$	$1.9\cdot10^{-12}$
9	$3.1\cdot10^{-8}$	$5.8\cdot10^{-10}$	$1.5\cdot10^{-11}$	$2.9\cdot10^{-12}$	$5.4\cdot10^{-13}$	$2.1\cdot10^{-13}$
10	$3.1\cdot10^{-9}$	$3.6\cdot10^{-11}$	$4.5\cdot10^{-13}$	$6.9\cdot10^{-14}$	$8.5\cdot10^{-15}$	$2.7\cdot10^{-15}$

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Institute of Mathematics NAS RA

A. V. Poghosyan

Fourier Formulae for Equidistant Hermite Trigonometric Interpolation

A sequence of Hermite trigonometric interpolation polynomials with equidistant interpolation nodes and uniform multiplicities is investigated. We derive relatively compact formula that gives the interpolants as functions of the coefficients in the DFTs of the derivative values. The coefficients can be calculated by the FFT algorithm. Corresponding quadrature formulae are derived and explored. Convergence acceleration based on the Krylov-Lanczos approach for accelerating both the convergence of interpolation and quadrature is considered. Exact constants of the asymptotic errors are obtained and some numerical illustrations are presented.

Ա. Վ. Պողոսյան

Հերմիտի եռանկյունաչափական հավասարահեռ ինտերպոլյացիայի համար Ֆուրիեի բանաձեւ

Ուսումնասիրվում է Հերմիտի եռանկյունաչափական ինտերպոլյացիան հավասարահեռ ցանցի վրա եւ վերջինիս համար ներկայացվում է բացահայտ բանաձեւ ֆունկցիայի եւ նրա ածանցյալների Ֆուրիեի դիսկրետ ձեւափոխությունների տերմիններով: Ուսումնասիրվում է ինտերպոլյացիայի զուգամիտության արագացման խնդիրը Կռիլով-Լանցոշի հայտնի մոտեցման կիրառմամբ: Զուգահեռաբար ուսումնասիրվում են համապատասխան քառակուսելիության բանաձեւերը: Դիտարկվում են զուգամիտության հարցեր, ստացվում են սխալանքի ասիմպտոտորեն ճշգրիտ գնահատականներ: Թվային արդյունքներր ներկայացնում են մեթոդի հնարավորությունները:

А. В. Погосян

Формула Фурье для Эрмитовой тригонометрической равноотстоящей интерполяции

Изучена Эрмитова тригонометрическая интерполяция на равномерной сети и представлена явная формула, которая реализуется посредством дискретного преобразования Фурье значений функции и ее производных. Рассматривается задача

ускорения сходимости интерполяции применением подхода Крылова – Ланцоша. Параллельно исследованы соответствующие квадратурные формулы. Получены асимптотически точные оценки ошибок интерполяций и квадратур. Результаты численных экспериментов подтверждают точность теоретических оценок.

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