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The Moore - Penrose Inverse for a Certain Class of Block Matrices

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1. The term *semi-magic square* is attributed to an $n \times n$ real matrix having the sum of elements in each row and each column equal to an identical constant (see [1], for instance). As was shown in [2] , the Moore - Penrose inverse of a semi-magic square is semi-magic as well. In this paper we introduce more general classes of matrices (see Definitions 1.1 and 2.1 below) and establish a similar property of their Moore - Penrose inverse.

Let $\mathbf{R}^n$ be the space of real n-dimensional column vectors and $\mathbf{R}^{m\times n}$ be the space of real $m\times n$ matrices.

Definition 1.1. A matrix $A = [a_{ij}] \in \mathbf{R}^{m\times n}$ is referred to as *magic rectangle* if there exist constants r and c such that the sum of elements in each row and each column is equal to r and c , respectively, i.e.

$$\sum_{j=1}^{n} a_{ij} = r , \quad i = 1,2,\dots,m , \tag{1.1}$$

$$\sum_{i=1}^{m} a_{ij} = c , \quad j = 1,2,\dots,n . \tag{1.2}$$

We will call the constants r and c *row sum* and *column sum*, respectively.

Example 1.1. The matrix

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 3 & 4 & -1 \end{bmatrix}$$

is a magic rectangle with $r = 6$ and $c = 4$. ◃

Let us denote by MR(m, n) the set of $m \times n$ magic rectangles. It can be readily shown that MR (m,n) is a subspace of $\mathbf{R}^{m\times n}$. Next, MR(m, n; r, c) will denote the set of $m\times n$ magic rectangles with row sum r and column sum c. Let $e_n = [11\dots1]^T$ be the n-dimensional column vectors of ones. Then the properties (1.1) and (1.2) can be written, respectively, as follows:

$$Ae_n = re_m, \quad A^T e_m = c\, e_n. \tag{1.3}$$

It can be easily found the following relation between row sum, column sum and the size of the matrix:

$$mr = nc. \tag{1.4}$$

Basing on the relation (1.4), we can use another notation for the set of magic rectangles with fixed row sum and column sum. Namely, simultaneously with the notation MR(m, n; r, c) we will use a notation MR[m, n : γ] which implies that $r = n\gamma$ and $c = m\gamma$.

Obviously, for m = n a magic rectangle becomes a semi-magic square.

2. Let us define a class of block matrices composed of magic rectangles.

Definition 2.1 A matrix $A = [a_{ij}] \in \mathbf{R}^{m\times n}$ represented in the block form

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1q} \\ A_{21} & A_{22} & \dots & A_{2q} \\ \dots & \dots & \dots & \dots \\ A_{p1} & A_{p2} & \dots & A_{pq} \end{bmatrix} \tag{2.1}$$

with submatrices $A_{ij} \in MR(m_i, n_j; r_{ij}, c_{ij})$ (or $A_{ij} \in MR[m_i, n_j : \gamma_{ij}]$, in another notation), where $i = 1,2,\dots,p$, $j = 1,2,\dots,q$ and $\sum_{i=1}^{p} m_i = m$, $\sum_{j=1}^{q} n_j = n$, will be referred to as *block magic rectangle*.

Insert diagonal matrices

$$M = \begin{bmatrix} m_1 & & & \\ & m_2 & 0 & \\ & 0 & \ddots & \\ & & & m_p \end{bmatrix} \in \mathbf{R}^{p\times p}, \quad N = \begin{bmatrix} n_1 & & & \\ & n_2 & 0 & \\ & 0 & \ddots & \\ & & & n_q \end{bmatrix} \in \mathbf{R}^{q\times q} \tag{2.2}$$

and $p \times q$ matrices

$$R = \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1q} \\ r_{21} & r_{22} & \dots & r_{2q} \\ \dots & \dots & \dots & \dots \\ r_{p1} & r_{p2} & \dots & r_{pq} \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1q} \\ c_{21} & c_{22} & \dots & c_{2q} \\ \dots & \dots & \dots & \dots \\ c_{p1} & c_{p2} & \dots & c_{pq} \end{bmatrix} \tag{2.3}$$

into our consideration. By

$$\mathrm{BMR}(M, N\,; R\,, C) \tag{2.4}$$

we will denote the set of block magic rectangles partitioned into blocks correspondingly to (2.2) with row sums and column sums defined by the matrices (2.3).

In accordance with relation (1.4) and notation accepted in the previous section, we have

$$r_{ij} = \gamma_{ij} n_j\,, \quad c_{ij} = \gamma_{ij} m_i \tag{2.5}$$

for $i = 1,2,\dots,p$ and $j = 1,2,\dots,q$. Consider a matrix

$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \dots & \gamma_{1q} \\ \gamma_{21} & \gamma_{22} & \dots & \gamma_{2q} \\ \dots & \dots & \dots & \dots \\ \gamma_{p1} & \gamma_{p2} & \dots & \gamma_{pq} \end{bmatrix}. \tag{2.6}$$

Then the relations (2.5) can be written in the matrix form:

$$R = \Gamma N\,, \quad C = M\Gamma\,. \tag{2.7}$$

Therefore, we will also use another notation for the set of block magic rectangles (2.4), that is

$$\mathrm{BMR}[M,N : \Gamma]\,. \tag{2.8}$$

The properties (1.3) for the blocks of matrix (2.1) look as

$$A_{ij} e_{n_j} = r_{ij} e_{m_i}\,, \quad A_{ij}^T e_{m_i} = c_{ij} e_{n_j}\,, \tag{2.9}$$

where $i = 1,2,\dots,p$ and $j = 1,2,\dots,q$. Let us define block matrices

$$E_{mp} = \begin{bmatrix} e_{m_1} & 0 & \dots & 0 \\ 0 & e_{m_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e_{m_p} \end{bmatrix} \in \mathbf{R}^{m\times p}, \; E_{nq} = \begin{bmatrix} e_{n_1} & 0 & \dots & 0 \\ 0 & e_{n_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e_{n_q} \end{bmatrix} \in \mathbf{R}^{n\times q}$$

with blocks of the size $m_i \times 1$, $i = 1,2,\dots,p$ (in the matrix E_{mp}) and $n_j \times 1$, $j = 1,2,\dots,q$ (in the matrix E_{nq}). By a straightforward verification it can be easily shown that relations (2.9) are equivalent to the following ones:

$$AE_{nq} = E_{mp}R\,, \quad A^TE_{mp} = E_{nq}C^T\,.$$

3. Remind, that the Moore-Penrose inverse $A^+ \in \mathbf{R}^{n\times m}$ of a matrix $A \in \mathbf{R}^{m\times n}$ is uniquely determined by the properties

$$\text{a)}\ AA^+A = A\,, \qquad \text{b)}\ A^+AA^+ = A^+\,,$$

$$\text{c)}\ (A^+A)^T = A^+A\,, \qquad \text{d)}\ (AA^+)^T = AA^+$$

(see [3] , for instance).

The main result of the paper is formulated as follows.

Theorem 3.1 *If* $A \in \mathrm{BMR}(M, N\,;\, R\,,\, C)$ *then* $A^+ \in \mathrm{BMR}(N, M\,;\, \tilde{R}, \tilde{C})$, *where*

$$\tilde{R} = N^{-1/2}(M^{1/2}RN^{-1/2})^+M^{1/2}\,, \tag{3.1}$$

$$\tilde{C} = N^{1/2}(M^{-1/2}CN^{1/2})^+M^{-1/2}\,. \tag{3.2}$$

Let us give an equivalent formulation of Theorem 3.1 , connected with the notation of type (2.8) for a set of block magic rectangles.

Theorem 3.1A *If* $A \in \mathrm{BMR}[M, N : \Gamma]$ *then* $A^+ \in \mathrm{BMR}[N, M : \tilde{\Gamma}]$, *where*

$$\tilde{\Gamma} = N^{-1/2}(M^{1/2}\Gamma N^{1/2})^+M^{-1/2}\,. \tag{3.3}$$

Example 3.1. Consider a block magic rectangle

$$A = \left[\begin{array}{ccc|cc|c} 1 & 0 & 5 & 1 & 1 & 0 \\ 3 & 4 & -1 & 1 & 1 & 0 \\ \hline 1 & 0 & 2 & 4 & 2 & 2 \\ 1 & 2 & 0 & 1 & 5 & 2 \\ 1 & 0 & 2 & 3 & 3 & 2 \\ 1 & 2 & 0 & 4 & 2 & 2 \end{array}\right] \in \mathbf{R}^{6\times 6} \qquad (p = 2\,,\, q = 3)\,.$$

According to (2.2),(2.3),(2.5),(2.6), we have

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad N = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$R = \begin{bmatrix} 6 & 2 & 0 \\ 3 & 6 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 2 & 0 \\ 4 & 12 & 8 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix}.$$

The Moore-Penrose inverse of the matrix A calculated using MATLAB is

$$A^{+} = \left[\begin{array}{cc|cccc} -0.2411 & 0.4256 & 0.3212 & -0.1788 & 0.4879 & -0.6788 \\ 0.2589 & -0.0744 & -0.2788 & 0.1212 & -0.4121 & 0.5212 \\ 0.2589 & -0.0744 & -0.0788 & 0.0212 & -0.1121 & 0.1212 \\ \hline -0.0267 & -0.0267 & 0.0864 & -0.1136 & -0.0136 & 0.1864 \\ -0.0267 & -0.0267 & -0.0136 & 0.1864 & 0.0864 & -0.1136 \\ \hline -0.0583 & -0.0583 & 0.0340 & 0.0340 & 0.0340 & 0.0340 \end{array}\right].$$

For this matrix

$$\tilde{R} = \begin{bmatrix} 0.1845 & -0.0485 \\ -0.0534 & 0.1456 \\ -0.1165 & 0.1359 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 0.2767 & -0.0364 \\ -0.0534 & 0.0728 \\ -0.0583 & 0.0340 \end{bmatrix},$$

$$\tilde{\Gamma} = \begin{bmatrix} 0.0922 & -0.0121 \\ -0.0267 & 0.0364 \\ -0.0583 & 0.0340 \end{bmatrix}.$$

It is of interest to examine some particular cases of block magic rectangles for which the general formula, describing the Moore-Penrose inverse, is considerably simplified.

Case 1. Suppose that in block representation (2.1) $p = q = 1$, i.e. let $A \in MR(m, n\,; r, c)$. Then, by formulae (3.1) and (3.2), we get

$$\tilde{r} = n^{-1/2}(m^{1/2} r n^{-1/2})^{+} m^{1/2} = n^{-1/2} n^{1/2} r^{+} m^{-1/2} m^{1/2} = r^{+},$$

$$\tilde{c} = n^{1/2}(m^{-1/2} c\, n^{1/2})^{+} m^{-1/2} = n^{1/2} n^{-1/2} c^{+} m^{1/2} m^{-1/2} = c^{+}.$$

Recall that for a scalar a, which can be considered as 1×1 matrix, $a^{+} = 1/a$, if $a \neq 0$, and $a^{+} = 0$, if $a = 0$. So, we arrive at the following statement.

Theorem 3.2 *If* $A \in MR(m, n\,; r, c)$ *then* $A^{+} \in MR(n, m\,; r^{+}, c^{+})$.

Note, that the last proposition has been obtained recently in [4].

We can also give an equivalent formulation of Theorem 3.2 . Let $A \in MR[m, n : \gamma]$. According to (3.3), we find

$$\tilde{\gamma} = n^{-1/2}(m^{1/2}\gamma n^{1/2})^{+} m^{-1/2} = n^{-1/2} n^{-1/2} \gamma^{+} m^{-1/2} m^{-1/2} = \frac{1}{mn}\gamma^{+}.$$

Thus, the result can be stated as follows.

Theorem 3.2A *If* $A \in MR[m, n : \gamma]$ *then* $A^{+} \in MR[n, m : (mn)^{-1}\gamma^{+}]$.

Case 2. Let in block form (2.1) $m_1 = m_2 = \ldots = m_p \equiv k$ and $n_1 = n_2 = \ldots = n_q \equiv l$. It is clear, that $k = m/p$ and $l = n/q$. Thereby, according to (2.2), we have $M = kI_p$ and $N = lI_q$ where I_p and I_q are identity matrices of pth and qth order, respectively. In this case our formulae (3.1) and (3.2) become extremely simple, i.e.

$$\tilde{R} = \sqrt{\frac{k}{l}}\left(\sqrt{\frac{k}{l}}R\right)^{+} = R^{+}, \qquad \tilde{C} = \sqrt{\frac{l}{k}}\left(\sqrt{\frac{l}{k}}C\right)^{+} = C^{+}.$$

Thus, we get

Theorem 3.3 *If* $A \in BMR(kI_p, lI_q; R, C)$ *then* $A^{+} \in BMR(lI_q, kI_p; R^{+}, C^{+})$.

Next, from (3.3) we obtain

$$\tilde{\Gamma} = \frac{1}{\sqrt{kl}}(\sqrt{kl}\Gamma)^{+} = \frac{1}{kl}\Gamma^{+}.$$

Consequently, the result may be formulated as follows.

Theorem 3.3A *If* $A \in BMR[kI_p, lI_q : \Gamma]$ *then* $A^{+} \in BMR[lI_q, kI_p : (kl)^{-1}\Gamma^{+}]$.

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Յու. Ռ. Հակոբյան, Ա. Ն. Էլոյան

Մուր - Պենրոուզի հակադարձը բլոկային մատրիցների մեկ դասի համար

Կիսակախարդական քառակուսիներ անվանում են բոլոր տողերում և սյուներում տարրերի հավասար գումարներ ունեցող $n \times n$ մատրիցները: Հոդվածում այս հասկացությունը տարածվում է հատուկ տեսքի բլոկային մատրիցների դասի վրա, որն անվանել են *բլոկային կախարդական ուղղանկյուններ:* Ապացուցվում է, որ բլոկային կախարդական ուղղանկյան Մուր-Պենրոուզի հակադարձումը դուրս չի բերում մատրիցը այդ դասի շրջանակներից:

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Обратная матрица Мура - Пенроуза для одного класса блочных матриц

Полумагическими квадратами называются $n \times n$ матрицы, у которых сумма элементов во всех строках и столбцах одинакова. В настоящей статье это понятие обобщается на специальный класс блочных матриц, которые были названы блочно-магическими прямоугольниками. Доказывается, что обращение Мура - Пенроуза блочно-магического прямоугольника не выводит матрицу за пределы этого класса.