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The fundamental solution for a composite anisotropic space (antiplane deformation)

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Introduction. Development of modern technologies in different fields of engineering stimulates the application of the newest composite materials with anisotropic physical properties. Anisotropy influences as the strength of constructive elements so the parameters of their fracture if the defects crackwise. In order to investigate the inertial effect of the anisotropic bimorphs with defects we should use Green's functions of the corresponding dynamic problems of the theory of elasticity. The dynamic problems of the theory elasticity and electroelasticity for piecewise homogeneous bodies were considered, for example, in [1-5]. Below by the method of integral transformations there is constructed the fundamental solution for a composite anisotropic (orthotropic) space at its harmonic with time loading by concentrated shear forces.

1. Statement of the Problem. In Cartesian coordinates x_1, x_2, x_3 consider a composite anisotropic space effected by the action of concentrated on line $x_1 = \xi_1$, $x_2 = \xi_2 > 0$, $-\infty < x_3 < \infty$ harmonically changing with time shear $q(x_1,x_2) = \mathrm{Re}(Q\delta(x_1 - \delta_1, x_2 - \delta_2)e^{-i\omega t})$ of constant along axis x_3 intensity (t is the time, ω is the circular frequency, $\delta(x,y)$ is Dirac-delta function). It is assumed that the materials of the composite space with respect to elastic properties are orthotropic.

Under the given conditions in a composite space there occur a steady wave process corresponding to the state of antiplane deformation. The system of equations of the problem includes the following relations [6]

$$\partial_1\sigma_{13}^{(r)} + \partial_2\sigma_{23}^{(r)} = \rho_r \frac{\partial^2 u_3^{(r)}}{\partial t^2} - \delta_r^{\,1} q(x_1,x_2), \quad \partial_j = \partial/\partial x_j, \tag{1}$$

$$\sigma_{23}^{(r)} = c_{44}^{(r)}\partial_2 u_3^{(r)} \qquad \sigma_{13}^{(r)} = c_{55}^{(r)}\partial_1 u_3^{(r)} \qquad (r = 1, 2). \tag{2}$$

Here (1) is equation of motion, (2) are the equation of the medium state, c_{ij} are the moduli of the material elastisity; ρ is the material density, $\delta_r^{\,j}$ is the Kronecker delta. Index "r" (r = 1,2) for all the quantities referring to r-th half-space; $x_2 > 0$ if r = 1 and $x_2 \le 0$ if r = 2.

The boundary of interphase $x_2 = 0$ should satisfy the conditions of an ideal mechanical contact

$$u_3^{(1)} = u_3^{(2)}, \quad \sigma_{23}^{(1)} = \sigma_{23}^{(2)}. \tag{3}$$

From (1), (2) we obtain Helmholtz equations for the displacement amplitude in a composite space

$$\nabla^2 U_3^{(1)} + \gamma_1^2 U_3^{(1)} = -\frac{Q}{\sqrt{\Delta_1}} \delta(x_1 - \xi_1, \eta_1 - \zeta)(x_2 > 0),$$

$$\nabla_2^2 U_3^{(2)} + \gamma_2^2 U_3^{(2)} = 0 \quad (x_2 \le 0), \; \nabla_r^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial \eta_r^2}, \tag{4}$$

$$u_3^{(r)} = \mathrm{Re}(U_3^{(r)} e^{-i\omega t}), \quad \gamma_r = \frac{\omega}{c_r}, \quad c_r = \sqrt{\frac{\Delta_r}{\rho_r c_{44}^{(r)}}}, \quad \Delta_r = c_{44}^{(r)} c_{55}^{(r)},$$

$$\eta_r = x_2 \mathrm{Im}\mu_r, \quad \zeta = \xi_2 \mathrm{Im}\mu_1 > 0, \quad \mu_r = i\frac{\sqrt{\Delta_r}}{c_{44}^{(r)}} \; (r = 1,2).$$

Thus, the problem is reduced to the definition of the function from differential equations (4), conjugation equations (3) and also from the conditions at infinity.

2. Construction of the fundamental solution for a composite anisotropic space To solve the problem let us apply integral Fourier transform to equations (4)

$$F(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x_1) e^{-ipx_1} dx_1, \quad f(x_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(p) e^{ipx_1} dp. \tag{5}$$

As a result we come to ordinary differential equation with respect to the spectral displacement functions

$$\frac{d^2 \hat{U}_3^{(1)}}{d\eta_1^2} + (\gamma_1^2 - p^2)\hat{U}_3^{(1)} = -\frac{Q}{\sqrt{2\pi}\sqrt{\Delta_1}} e^{-ip\xi_1} \delta(\eta_1 - \zeta) \quad (x_2 > 0), \tag{6}$$

$$\frac{d^2 \hat{U}_3^{(2)}}{d\eta_2^2} + (\gamma_2^2 - p^2)\hat{U}_3^{(2)} = 0 \quad (x_2 \le 0).$$

The general solutions of equations (6) providing the performance of radiation conditions [7] at infinity may be represented as follows

$$\hat{U}_3^{(1)} = Ae^{-\lambda_1\eta_1} + \frac{Qe^{-ip\xi_1}}{2\sqrt{2\pi}\sqrt{\Delta_1\lambda_1}} e^{-\lambda_1|\eta_1-\zeta|} \quad (x_2 > 0), \tag{7}$$

$$\hat{U}_3^{(2)} = Be^{\lambda_2\eta_2} \quad (x_2 \le 0),$$

$$\lambda_r = \begin{cases} -i\sqrt{\gamma_r{}^2 - p^2}, & \gamma_r > |p| \\ \sqrt{p^2 - \gamma^2{}_r}, & \gamma_r < |p| \end{cases} \qquad (r = 1,2).$$

Constants A and B are determineted from conditions (3) on the boundary of conjugation of media $x_2 = 0$, which taking into account (2) in Fourier transformants have following form

$$\hat{U}_3^{(1)} = \hat{U}_3^{(2)},$$

$$c_{44}^{(1)} \operatorname{Im} \mu_1 \frac{d\hat{U}_3^{(1)}}{d\eta_1} = c_{44}^{(2)} \operatorname{Im} \mu_2 \frac{d\hat{U}_3^{(2)}}{d\eta_2}. \tag{8}$$

Proceeding from (7), (8) we obtain the expressions for the spectral function of the displacement amplitude

$$\hat{U}_3^{(1)} = \frac{Qe^{-ip\xi_1}}{2\sqrt{2\pi}\sqrt{\Delta_1\lambda_1}} e^{-\lambda_1|\eta_1-\zeta|} + \frac{Q\alpha(p)}{2\sqrt{2\pi}\sqrt{\Delta_1\lambda_1}} e^{-ip\xi_1} e^{-\lambda_1(\eta_1+\zeta)} \quad (x_2 > 0),$$

$$\hat{U}_3^{(2)} = \frac{Q[1+\alpha(p)]}{2\sqrt{2\pi}\sqrt{\Delta_1\lambda_1}} e^{-ip\xi_1} e^{-\lambda_1\zeta} e^{\lambda_2\eta_2} \quad (x_2 \le 0), \tag{9}$$

$$\alpha(p) = \frac{\lambda_1\sqrt{\Delta_1} - \lambda_2\sqrt{\Delta_2}}{\lambda_1\sqrt{\Delta_1} + \lambda_2\sqrt{\Delta_2}}.$$

Moving to originals according to (5) we find

$$\hat{U}_3^{(1)}(x_1, x_2; \xi_1, \xi_2) = \frac{Q}{2\pi\sqrt{\Delta_1}} \int_0^\infty \frac{\cos p(x_1 - \xi_1)}{\lambda_1} e^{-\lambda_1|x_2-\xi_2|\operatorname{Im}\mu_1} dp +$$

(10)

$$+\frac{Q}{4\pi\sqrt{\Delta_1}}\int_{-\infty}^{\infty}\frac{\alpha(p)}{\lambda_1}e^{ip(x_1-\xi_1)}e^{-\lambda_1(x_2+\xi_2)\mathrm{Im}\mu_1}dp \quad (x_2>0),$$

$$\hat{U}_3^{(2)}(x_1,x_2;\xi_1,\xi_2)=\frac{Q}{4\pi\sqrt{\Delta_1}}\int_{0}^{\infty}\frac{1+\alpha(p)}{\lambda_1}e^{ip(x_1-\xi_1)}e^{-\lambda_1\xi_2\mathrm{Im}\mu_1}e^{\lambda_2x_2\mathrm{Im}\mu_2}dp \quad (x_2\le 0).$$

The integral (10) prescribed on a semi-infinite interval according to the radiation condition will be understood in the general meaning [7]

$$I=\frac{Q}{2\pi\sqrt{\Delta_1}}\int_0^{\infty}\frac{\cos p(x_1-\xi_1)}{\lambda_1}e^{-\lambda_1|x_2-\xi_2|\mathrm{Im}\mu_1}dp=$$

$$=\frac{Q}{2\pi\sqrt{\Delta_1}}\lim_{\varepsilon\to+0}\int_0^{\infty}\frac{e^{-\sqrt{p^2+\lambda_*^2}|x_2-\xi_2|\mathrm{Im}\mu_1}}{\sqrt{p^2+\lambda_*^2}}\cos p(x_1-\xi_1)dp,$$

$$\gamma_*=-i\sqrt{\gamma_1^2+i\varepsilon},\quad \varepsilon>0,\quad \mathrm{Re}\gamma_*>0.$$

Using the value of the integral [8]

$$\int_0^{\infty}\frac{e^{-c\sqrt{x^2+z^2}}}{\sqrt{x^2+z^2}}\cos bx\,dx=K_0\left(z\sqrt{b^2+c^2}\right)\quad (b,\mathrm{Re}c,\mathrm{Re}z>0)$$

and the connection between functions of MacDonald and Hankel [9]

$$K_0(-ix)=\frac{\pi i}{2}H_0^{(1)}(x),$$

we find

$$I=\frac{iQ}{4\sqrt{\Delta_1}}H_0^{(1)}(\gamma_1\rho_*)\quad \rho_*=|z-z_*|,\ z=x_1+\mu_1x_2,\ z_*=\xi_1+\mu_1\xi_2.$$

Thus, the expression for the displacement amplitude when $x_2=0$ may be represented in following form

$$Q_3^{(1)}(x_1,x_2;\xi_1,\xi_2) = I + \frac{Q}{4\pi\sqrt{\Delta_1}} \int_{-\infty}^{\infty} \frac{\alpha(p)}{\lambda_1} e^{ip(x_1-\xi_1)} e^{-\lambda_1(x_2+\xi_2)\mathrm{Im}\mu_1} dp, \qquad (11)$$

where quantity I is fundamental solution for a homogeneous anisotropic space.

Using equations of state (2) and formulas (10), (11) we determine the expressions for the amplitudes of stresses in a composite anisotropic space. Taking into account the formulas of differentation

$$\frac{\partial}{\partial z^n} H_0^{(1)}(\gamma\rho_*) = \left(-\frac{\gamma}{2}\right)^n e^{-in\beta} H_n^{(1)}(\gamma\rho_*), \quad \frac{\partial}{\partial z^n} H_0^{(1)}(\gamma\rho_*) = \left(-\frac{\gamma}{2}\right)^n e^{in\beta} H_n^{(1)}(\gamma\rho_*),$$

$$\frac{\partial^2}{\partial z \partial \bar{z}} H_0^{(1)}(\gamma\rho_*) = -\frac{\gamma^2}{4} H_0^{(1)}(\gamma\rho_*), \quad \gamma = \mathrm{const} > 0,$$

$$\beta = \arg(z - z_*), \quad n = 1,2,\ldots, \quad \partial_1 = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \qquad \partial_2 = \mu_1 \frac{\partial}{\partial z} + \bar{\mu}_1 \frac{\partial}{\partial \bar{\bar{z}}},$$

we will have

$$\sigma_{kj}^{(r)} = \mathrm{Re}(S_{kj}^{(r)} e^{-i\omega t}) \quad (r = 1,2), \qquad (12)$$

$$S_{12}^{(1)}(x_1,x_2;\xi_1,\xi_2) = -\frac{iQ}{4} \gamma_1 H_1^{(1)}(\gamma_1\rho_*)\sin\beta - \frac{Q}{4\pi} \int_{-\infty}^{\infty} \alpha(p) e^{ip(x_1-\xi_1)} e^{-\lambda(x_2+\xi_2)\mathrm{Im}\mu_1} dp,$$

$$S_{12}^{(2)}(x_1,x_2;\xi_1,\xi_2) = -\frac{Q}{4\pi}\sqrt{\frac{\Delta_2}{\Delta_1}} \int_{-\infty}^{\infty} \frac{\lambda_2}{\lambda_1} [1 + \alpha(p)] e^{ip(x_1-\xi_1)} e^{-\lambda_1 \xi_2 \mathrm{Im}\mu_1} e^{\lambda_2 x_2 \mathrm{Im}\mu_2} dp,$$

$$S_{13}^{(1)}(x_1,x_2;\xi_1,\xi_2) = -\frac{iQc_{55}^{(1)}}{4\sqrt{\Delta_1}} \gamma_1 H_1^{(1)}(\gamma_1\rho_*)\cos\beta + \frac{iQc_{55}^{(1)}}{4\sqrt{\Delta_1}} \int_{-\infty}^{\infty} \frac{p\alpha(p)}{\lambda_1} e^{ip(x_1-\xi_1)} e^{-\lambda(x_2+\xi_2)\mathrm{Im}\mu_1} dp,$$

$$S_{13}^{(2)}(x_1,x_2;\xi_1,\xi_2) = \frac{iQc_{55}^{(1)}}{4\sqrt{\Delta_1}} \int_{-\infty}^{\infty} \frac{p}{\lambda_1} [1 + \alpha(p)] e^{ip(x_1-\xi_1)} e^{-\lambda_1 \xi_2 \mathrm{Im}\mu_1} e^{\lambda_2 x_2 \mathrm{Im}\mu_2} dp.$$

It should be noted here that for determination of quantities $S_{kj}^{(r)}$ in (12) we used the procedure of differentiation over the parameter under the sign of an improper integral which is possible in the given case.

3. Numerical results. Let us investigate the distribution of elastic displacements and stresses in a composite anisotropic space under the influence of concentrated shear forces depending on the character of the material anisotropy and the frequency of the harmonic loading. The contour lines of the absolute values of displacement in the area covering point (ξ_1,ξ_2) for various relations of the material elastic moduli are represented in Figs 1-2.. In calculations it was assumed that the beginning of the system of coordinates is in the center of the considered square area and $\xi_1 = 0$, $\xi_2/a = 0.1$, $p_1 = p_2$ (a is the length of the square side). The lighter zones correspond to the maximum values of the investigated quantity.

4. Concluding remarks. The represented results clearly illustrate the influence of the anisotropy of the elastic properties of the materials of a composite orthotropic space and the frequency of harmonic loading on the behaviour of the components of an elastic field at anisotropic deformation in dynamics. The constructed fundamental solution may effectively be used for the calculation of the boundary problems of the theory of elasticity for a composite anisotropic space, weakened by heterogenities (cracks, openings, inclusions), by the method of boundary integral equations.

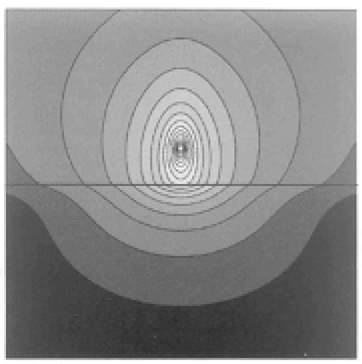

Fig. 1.

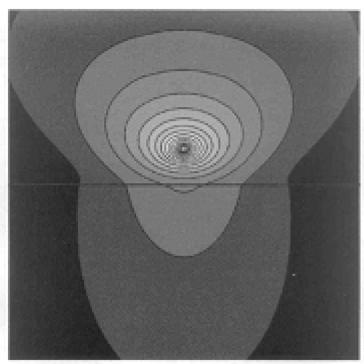

Fig. 2.

Fig. 1. The contour lines of the displacement amplitude modulus in composite space

$(\gamma_2 = 4,\ c_{44}^{(2)} = 4,\ c_{44}^{(2)} = 1,\ c_{55}^{(1)} = 1,\ c_{55}^{(2)} = 2)$.

Fig. 2. The contour lines of the displacement amplitude modulus in composite space

$(\gamma_2 = 4,\ c_{44}^{(1)} = 4,\ c_{44}^{(2)} = 4,\ c_{55}^{(1)} = 2,\ c_{55}^{(2)} = 1)$.

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Литература

1. *Achenbach J. D.* Wave propagation in elastic solids. Amsterdam. North-Holland Publ Co, 1973.
2. *Parton V. Z., Kudryavtsev B. A.* Electromagnetoelasticity. New York: Gordon & Breach. 1988.
3. *Sih G. C. (ed.)* Elastodyanamic crack problems. Leyden. Noordhoff. 1977.
4. *Kosmodamiansky A. S., Storozev V. I.* Dynamic problem of the theory of elasticity for anisotropic media. Kiev. Naukova dumka. 1985. 175 p.
5. *Bardzokas D. I., Filshtinsky M. L.* Electroelasticity of piecewise-uniform bodies. - Sumy (Ukraine): University Book Publ. 2000. 308 p. [in Russian].
6. *Nowacki V.* The theory of elasticity. M. Nauka. 1975. [in Russian].
7. *Vladimirov V. S.* Equations of the mathematical physics. M. Nauka. 1981. 512 p. [in Russian].
8. *Prudnikov A. O., Brychkov Y. A., Marychev O.* I. Integrals and series. M. Nauka. 1981. 799 p. [in Russian].
9. *Liuk U.* Special mathematical functions and their approximation. M. Mir. 1980. 608 p. (in Russian)

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Բաղադրյալ անիզոտրոպ տարածության համար ֆունդամենտալ լուծումը (հակահարթ դեֆորմացիա)

Ինտեգրալ ձևափոխությունների մեթոդով կառուցված է ֆունդամենտալ լուծում բաղադրյալ անիզոտրոպ տարածության համար, որը ենթարկված է եզրային գծերի վրա ժամանակի ընթացքում ներդաշնակ փոփոխվող կենտրոնացված շոշափող ուժերի ազդեցությանը: Տեղափոխությունների և լարումների արտահայտությունները լծորդված կիսատարածություններում ստացված են Ֆուրյեի ինտեգրալների տեսքով: Ստացված են թվային արդյունքներ և բերված են տեղափոխությունների կոնտուրային գծերը:

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Фундаментальное решение для составного анизотропного пространства (антиплоская деформация)

В данной статье методом интегральных преобразований построено фундаментальное решение для составного анизотропного при антиплоской деформации пространства, подверженного воздействию гармонически изменяющейся во времени касательных сосредоточенных на граничных линиях сил. Выражения для перемещений и напряжений в сопряженных полупространствах получены в форме интегралов Фурье. Получены численные результаты и приведены контурное линии перемещений.