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On Design Problem of a Space Elevator Cable

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The concept of the space elevator comprises a very long cable/ribbon (twenty times the Earth radius) with one end fixed to the Earth and the other end free out in the space [1]. Based on this concept we present here the mathematical study and formulation of mechanics problems of strength and elastic stability arising in a simple elastic rod, with a view of a better understanding of this physical phenomenon that could lead to a new knowledge in the field of space elevator studies.

Let us consider the equilibrium state of a very long elastic cable anchored on the Earth equatorial point. The cable subject to both the action of Earth gravity inward force F_1, defined by Newton gravity law

$$F_1(\gamma) = \frac{\rho g_0 R_0^2}{(R_0 + \gamma)^2} \tag{1}$$

and centrifugal outward force F_2, due to Earth daily spinning

$$F_2(\gamma) = -\rho\omega^2(R_0 + \gamma). \tag{2}$$

In (1), (2) γ is a coordinate along cable length counted from Earth surface, ρ is the bulk density of the cable material, $R_0 = 6378$ km is the Earth equatorial radius, $g_0 = 980$ sm·sec^{-2} is the gravity force acceleration on the Earth surface, $\omega = 2\pi/T$ is the circular spinning frequency of the Earth, $T = 86146$ sec is the period of the Earth spinning.

For cables anchored at the points of the Earth interstitial latitudes centrifugal outward force is equal to

$$F_2(\gamma) = -\rho\omega^2(R + \gamma),$$

where R is the radius of the corresponding latitude circumference, which changes from zero at the pole to R_0 (at the equator).

When the cross-section area S(x) of the cable is the function of cable length (tapered cable), using the dimensionless notations the equation determining the cable elastic stress $\sigma(x)$ can be written as,

$$\frac{d[S(x)\sigma(x)]}{dx} + S(x)F(x) = 0, \tag{3}$$

$$F(x) = \rho g_0 R_0 g(x), \qquad g(x) = \left[\alpha(1+x) - \frac{1}{(1+x)^2} \right]. \tag{4}$$

Here the following dimensionless notations are used:

$x = \gamma/R_0$, $L = l/R_0$, $\alpha = \omega^2 R_0/g_0 \approx 1/288$, l is the cable length.

Coefficient α characterizes the ratio of the gravity and centrifugal accelerations on the Earth surface. For the Mars and the Moon this coefficient equals to

$$\alpha_1 \approx 1/218, \quad \alpha_2 \approx 1/176.$$

Equation (3) is to be considered with the following boundary condition at free end

$$\sigma(L) = 0. \tag{5}$$

When S(x)=const, the elastic stress is defined by the following function, satisfying to (3-5)

$$\sigma(x) = \rho g_0 R_0 \tilde{\sigma}_0(x), \tag{6}$$

$$\tilde{\sigma}(x) = \frac{(L-x)[(1+L)(1+x)(2+L+x) - 576]}{576(1+L)(1+x)}.$$

Since function $\tilde{\sigma}_0(x)$ may have only one zero in the interval $x \in [0,L]$, then from condition $\tilde{\sigma}_0(0) = 0$, $(1+L)(2+L) = 576$, follows that for all $L \geq L_0 = 22.5$

$$\tilde{\sigma}_0(x) \geq 0, \quad x \in [0,L_0]. \tag{7}$$

Therefore, when the length of the cable is more than l_0 = 143.325 km the cable is in a pure tension state. For cables with $L = L_0$, $\sigma(0) = 0$ and mechanical tension stress reaches its maximum value $\sigma_0 \approx 0.78\rho g_0 R_0$, at point $x_0 \approx 5.602$ (where $g(x_0) = 0$), which corresponds to the Earth geosynchronous orbit γ_0 = 35.785 km. For cables with $L \geq L_0$, we have the following condition $\sup_x[\sigma(x)] \geq \sigma_0$.

Let us note that the "limit" lengths for space elevator cable located on the Mars or the Moon equators are the following:

$$L_{01} = 19.38 \quad (l_{01} = 65698) \text{ km}$$

$$L_{02} = 17.21 \quad (l_{02} = 29893)\ \text{km}$$

To minimize the cable maximum tension stress and its critical length we need to consider a tapered cable, cross-section area of which is the function of cable length S = S(x).
Based on solution of equation (3) the stress function can be written as

$$\sigma(x) = \frac{\int_x^L F(\alpha)S(\alpha)da}{S(x)}. \tag{8}$$

The solution (8) satisfies the boundary condition (5) at a free end.
Let us consider the cable with the following cross section area function

$$S_*(x)S_0\exp\left[\frac{\rho g_0 R_0}{\tilde{\sigma}}\int_x^L g(x)dx\right], \tag{9}$$

where $\tilde{\sigma}$ is a constant one.
Substituting (8) into (9) leads to

$$\sigma(x) = \tilde{\sigma}\left[1-\exp\left(-\frac{\rho g_0 R_0}{\tilde{\sigma}}\int_x^L g(x)dx\right)\right]. \tag{10}$$

From (10) follows that analogous to the case of the non tapered cable, $\sigma(x \geq 0)$, when $L \geq L_0 \approx 22.4$; for $L = L_0$, $\sigma(0) = 0$ and mechanical tension stress reaches to its maximum value at $x_0 \approx 5.60$

$$\max\ \sigma(x) = \tilde{\sigma}\left[1-\exp\left(-\frac{0.76\rho g_0 R_0}{\tilde{\sigma}}\right)\right]. \tag{11}$$

On the other hand we have

$$\max\ S_* = S_0\exp\left(\frac{0.76\rho g_0 R_0}{\tilde{\sigma}}\right). \tag{12}$$

From (12), (13) one can note that

$$\xi = \frac{\max\ \sigma(x)}{\sigma_0} = \frac{\eta - 1}{\eta\ln\eta},$$

$$\eta = \max S_*/S_0 \tag{13}$$

where η is a tapering ratio of the cable, $\sigma_0 = 0.76\rho g_0 R_0$ is the maximum stress of the of non tapered cable.

On the cable ends we have

$$S_*(0) = S_*(L) = S_0$$

Based on formula (13) the dependence of the maximum stress value from tapering ratio is given in Table 1. As it follows from Table 1. data, the thickening of the cable at the point of the Earth geosynchronous orbit leads to the decreasing of the maximum tension stress. E.g. for a continuous cable of a circular cross section the two times increase of the cable radius at the geosynchronous orbit point leads to the two times decrease of the maximum stress.

Table 1

Dependence of Maximum Stress upon Tapered Ratio

η	1.1	1.6	2.5	4	8	10	20
ξ	0.95	0.79	0.65	0.54	0.42	0.39	0.31

The above-mentioned results determine the projects of the cable design when the "limit" length does not change, while the cable form permits decrease of maximum stress.

We can reach the decreasing both the "limit" length and the maximum stress by appropriate choice of the cross section area function. Let us consider the cross section area functions as the quadratic polynomial functions of the following type

$$S(x) = S_0\left[1 + \frac{ax}{L} + \frac{bx^2}{L^2}\right], \tag{14}$$

where a,b are arbitrary constants.

Substituting (14) into (8), we obtain the stress function depending upon parameters a,b. Since the stress function expression is very cumbrous we do not present it here.

By means of numerical analysis of this expression the cable forms, namely, parameters a,b are determined, providing the implementation of the following conditions

$$\sigma(x) \geq 0, \qquad \max_x \sigma(x) \to \min_S$$

These numerical results for two considered cases are the following:

cable "limit" length $L_0 = 18$, $(b = -1.04,\ a = 2.0)$, $\max_x \sigma(x) = 0.61\rho g_0 R_0$,
cable "limit" length $L_0 = 16.0$, $(b = -2.2,\ a = 4.23)$, $\max_x \sigma(x) = 0.52\rho g_0 R_0$.
Let us now consider the cable with a "counterweight" of mass M_0 attached to its outward end. In this case, we have the following boundary condition

$$\sigma(L) = \frac{M_0 g_0 g(L)}{S_0},$$

$$g(L) = \left[\frac{(1+L)}{288} - \frac{1}{(1+L)^2} \right]. \tag{15}$$

The stress function satisfying the boundary condition (15) is the following

$$\sigma(x) = \rho g_0 R_0 \tilde{\sigma}(x)$$

$$\tilde{\sigma}(x) = \frac{(L-x)[(1+L)(1+x)(2+L+x) - 576]}{576(1+L)(1+x)} + \frac{M_0 L}{M} \left[\frac{(1+L)}{288} - \frac{1}{(1+L)^2} \right], \tag{16}$$

where $M_0 = \rho L R_0 S_0$ is the cable mass.
When the "counterweight" of mass M_0 is attached to the cable outward end, it is possible to consider a design of uniformly tensioned cable when mechanical stress is constant along the cable length. Such a design for a ponderable rod was considered first by S. Timoshenko [2].
Seeking in equation (1) solution $\sigma(x) = \text{const}$ we come to the following solution for the cable cross-section area function.

$$S_*(x) = S_0 \exp\left[\frac{\rho g_0 R_0}{\sigma_*} \int_x^L g(x)dx \right], \tag{17}$$

where

$$\sigma_* = \frac{M_0 g_0 g(L)}{S_0},$$

$g(x)$ is the function defined by (4), S_0 is the cable cross-section area at the outward end, where the mass M_0 is attached. For a cable with "limit" length $L_0 = 22.4$ function $S(x)$ reaches to its maximum

value S^* at point $x \approx 5.60$.
The maximum value of the (17) can be written as

$$S_* = S_0 \exp(\beta), \tag{18}$$

$$\beta \equiv \frac{\rho_0 R_0 S_0 \int_{x_0}^{L_0} g(x)dx}{M_0 g(L_0)}.$$

On the other hand we have

$$\sigma_* = \sigma_0 \beta^{-1}, \tag{19}$$

where $\sigma_0 = 0.76\rho g_0 R_0$ is the maximum stress of the of non tapered cable. From (18), (19) follows that decrease of the maximum stress leads to exponential increase of S_*. E.g. when $\beta = 2$, we have $S_* = 7.39S_0$. For a continuous cable of a circular cross section, it means that the cable radius increase 2.7 times.

Table 2

"Limit" lengths and stress maximum values in depend of the "counterweight" mass

M/M_0	100	50	20	10	5	2	1
L_0	22.3	22.07	21.0	20.48	19.2	16.2	13.2
σ_0	0.76	0.74	0.66	0.63	0.53	13.2	0.22

The "counterweight" decreases slightly the "limit" length and stress maximum values, when $M >> M_0$. In the contrary, the "counterweight" decreases essentially the "limit" length and stress maximum values, when the cable mass is compared with "counterweight" mass.
Let us now consider the elastic stability problem in the non tapered cable when its length is less than "limit" length, $L < L_0$. In this case the compression stresses arise localized near cable base and due to it, the cable may become unstable.
The cable elastic stability equation and appropriate boundary conditions can be written as

$$\frac{d^4W}{dx^4} - \lambda \frac{d}{dx}\left[\tilde{\sigma}_0(x) \frac{dW}{dx} \right] = 0; \tag{20}$$

$$W|_{x=0} = 0, \quad \left.\frac{dW}{dx}\right|_{x=0} = 0,$$

$$\left.\frac{d^2W}{dx^2}\right|_{x=L} = 0, \quad \left.\frac{d^3W}{dx^3}\right|_{x=L} = 0, \tag{21}$$

where the dimensionless function $\tilde{\sigma}_0$ (x) is defined as in (6), EI is the bending rigidity of the cable material, W(x) transverse displacement of the elastic cable, $\lambda \equiv \rho g_0 R_0{}^3 S_0 (EI)^{-1}$. Using notation y(x) = [dW/dx], integrating equation (20) and taking into consideration the boundary conditions we come to the following equation and boundary conditions

$$\frac{d^2y}{dx^2} 0 - \lambda\, \tilde{\sigma}_0\,(x) y = 0, \tag{22}$$

$$y|_{x=0} = 0, \quad \left.\frac{dy}{dx}\right|_{x=L} = 0. \tag{23}$$

Equation (22) with conditions (23) is the self-adjoint boundary value problem determining eigenvalues λ. Since the function $\tilde{\sigma}_0$ (x) changes its sign in the interval $x \in [0,L]$, the eigenvalues can be both positive and negative ones. The minimum positive eigenvalue λ_0 corresponds to the critical load behind which the cable becomes unstable.

Let function $\tilde{\sigma}_0$ (x) change its sign from a negative value to a positive one at point x_0.
Then for minimum positive eigenvalue, λ_0 the following inequality is valid

$$\lambda_0 < \frac{\int_0^{x_0} (y'_0)^2 dx}{\int_0^{x_0} y_0{}^2 |g(x)| dx},$$

where function $y_0(x)$ is any admissible function satisfying to the conditions $y_0(0) = y_0(x_0) = 0$.
Let us consider the cable with L = 22.0. For this cable $x_0 = 0.04$; and taking $y_0(x) = x(x - x_0)$ we come to the inequality $\lambda_0 < 6326$. Based on this inequality we can postulate that for a rod with circular cross section of radius r when $r < r_0$ the elastic rod under consideration is unstable, where

$$r_0 = 0.025 \sqrt{\frac{\rho g_0 R_0^3}{E}}.$$

Therefore, the solution of the self-adjoint boundary value problem (22-23) even in this case (L = 22.0) results in the instability of the cable for any values of elastic and geometrical parameters having practical and science forecasting meanings.

Conclusions.

Based on the solution of the one dimension equation of the elasticity theory qualitative and quantitative results are obtained related to strength and stability problems of a space elevator cable/ribbon. It is assumed that the cable is subjected to action of the Earth gravity inward force, defined by Newton gravity law and the centrifugal outward force, due to the Earth's daily spinning. The numerical data related to the strength characteristics of new modern materials such as carbon nanotubs are omitted, since these data are in [1]. We confined ourselves to results related to the tapered cable design strength problems. Based on the solution of a one dimension elasticity equation solution of the tensile strength does not determine an absolute value of cable cross-section area. This cross section value parameter may be determined while considering the two dimensional problems or dynamics problems, especially elastic wave propagation along a tensioned cable.
On the other hand, there is no doubt that the future objects of the investigation should be constructions of elastic closed shells (pipes) type, which are of interest from an applied point of view. For such constructions, the new mechanical problems should be considered, which take into account circular, transversal stresses and displacements arising in the shells. Among these problems we can list the dynamic interaction of closed shells subject to external media, including electromagnetic, temperature and atmosphere fields actions.

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Ակադեմիկոս Ս. Ա. Համբարձումյան, Մ. Վ. Բելուբեկյան, Կ. Բ. Ղազարյան,
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Տիեզերական վերելակի ճոպանի նախագծի խնդրի մասին

Տիեզերական վերելակի հասկացությունը իր մեջ պարունանում է շատ երկար ճոպան (ժապավեն), որի մի ծայրն ազատ է, իսկ մյուս ծայրն ամրակցված է Երկրին:

Այդ հասկացությունից ելնելով ներկայացված են պարզ, առաձգական ձողի ամրության և կայունության մեխանիկայի խնդիրների ձևակերպումը և մաթեմատիկական հետազոտումը:

Այն նպատակ ունի ավելի լավ հասկանալ բուն ֆիզիկական երևույթը, որն էլ կբերի տիեզերական ճոպանի ուսումնասիրման բնագավառում նոր գիտելիքի:

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К задаче проектирования троса космического лифта

Концепция космического лифта содержит в себе длинный трос (стержень длиной порядка двадцати радиусов Земли), один конец которого закреплен на Земле, другой конец свободен в космосе [1]. С целью лучшего понимания этой концепции в работе представлены математическая постановка и формулировка проблем прочности и устойчивости упругого стержня, находящегося под действием силы гравитации Земли и центростремительной силы, обусловленной суточным вращением Земли.