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A description of a truncated operator's spectrum and intermediate problems

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Let A be a selfadjoint (bounded) operator, acting in a Hilbert space H, L be a subspace in H and P_L be the orthogonal projector onto L. Any operator in H generates a sesquilinear form (Af, g) on $H \times H$. If we consider this form only for elements f, g belonging to L, we arrive at the operator $A_0 = P_L A \mid L$ said to be truncated. First we describe the spectrum of a truncated operator in a particular case, then apply it to the oscillation's investigation of constrained mechanical systems.

Let μ be a nonnegative locally finite measure with supp $\mu \subset \mathbb{R}$, M be the multiplication operator by the independent variable in the (complex) space L^2_μ, $\mathcal{P}_{n-1}$ $(n \in \mathbb{N})$ $-$ the set of polynomials Q on one real variable, satisfying the condition deg $Q \leq n-1$ (we suppose that any $\mathcal{P}_{n-1}$ is a subspace of L^2_μ, e.g. μ has compact support and an infinite number of growth points). The subspace $\mathcal{P}_{n-1}$, evidently, is not invariant under M and we consider the restriction of the form (Mf, g) on $\mathcal{P}_{n-1}$.

Proposition 1. *The spectrum of* $P_{\mathcal{P}_{n-1}} M \mid \mathcal{P}_{n-1}$ *coincides with the roots* $\{x_k\}$ *of the* $n-$ *th* $\mu-$ *orthogonal polynomial* P_n *and the corresponding eigenelements are the Lagrange fundamental polynomials constructed by these points.*

Proof. Let

$$l_m(x) = \prod_{\substack{i=1\\ i\neq m}}^{n} \frac{x - x_i}{x_m - x_i},\ m = 1, 2, \ldots, n$$

be the Lagrange fundamental polynomials, constructed by the points $\{x_k\}$. Denoting

$$\omega(x) = \prod_{i=1}^{n} (x - s_i).$$

one arrives to the formula

$$l_m(x) = \frac{\omega(x)}{(x - x_m)\,\omega'(x_m)}.$$

We prove that $\{l_m\}$ are $\mu -$ orthogonal. Indeed,

$$(l_m, l_k) = \int \omega(x) \cdot \frac{\omega(x)}{\omega'(x_m)\,\omega'(x_k)\,(x - x_k)\,(x - x_m)} \mu(dx) = 0,$$

as the first term under the integral sign coincides up to a scalar multiplier with P_n and second term at $m \neq k$ is a polynomial of degree $n-2$. If $m = k$, one gets

$$\int l_m^2(x)\,\mu(dx) = \lambda_m > 0,$$

where the last numbers are known as weights or Christoffel numbers. Expanding polynomials $p, q \in \mathcal{P}_{n-1}$ according to the Lagrange interpolation formula

$$p(x) = \sum_{k=1}^{n} p(x_k)\, l_k(x), \quad q(x) = \sum_{m=1}^{n} q(x_m)\, l_m(x)$$

we arrive to the formula

$$(p, q) = \sum_{k=1}^{n} \lambda_k p(x_k)\, \overline{q(x_k)}$$

established in the theory of numerical calculations ([1], formula 3.4.1) as an approximative eguality

$$\int f(x)\,\mu(dx) \simeq \sum_{k=1}^{n} \lambda_k f(x_k).$$

Note that this equality is exact for any polynomial R with degi $R \leq 2n-1$.

Then

$$(xq(x), l_m(x)) = ((x - x_m)\, q(x), l_m(x)) + x_m (q(x), l_m(x)) = x_m (q(x), l_m(x)),$$

implying

$$(xl_k(x), l_m(x)) = \lambda_k x_k \delta_{km}$$

(δ_{mk} is Kronecker's delta) and

$$P_{\mathcal{P}_{n-1}} M l_k = x_k l_k. \square$$

For any set S of different real numbers $\{s_1, s_2, \cdots, s_n\}$ and a set Λ of positive numbers $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ one can construct a space of $\Lambda -$ orthonormal on S polynomials. Let introduce in the space $\mathcal{F}(S)$ of all functions defined at least on S a sesquilinear form by the formula

$$(f, g) = \sum_{k=1}^{n} \lambda_k f(s_k)\, \overline{g(s_k)}. \tag{1}$$

This expression defines a scalar product on $\mathcal{P}_{n-1}$. Note that for any two functions $f, g \in \mathcal{F}(S)$

$$(f, g) = \left(\tilde{f}, \tilde{g}\right),$$

where

$$\tilde{f}(x) = \sum_{k=1}^{n} f(s_k)\, l_k(x)$$

is the Lagrange interpolation polynomial for f. Applying to the monomials $\{x^m\}_0^{n-1}$ the Gram-Schmidt orthogonalization process we arrive to the set $\{p_k(x)\}_0^{n-1}$ of orthonormal polynomials (with positive coefficient of the leading term).

We seek an explicit formula for$p_{n-1}(x)$. To this end note that the function

$$r_m(x) = p_{n-1}(x)\frac{\omega(x)}{x - s_m},$$

where $\omega(x)$ is the product, corresponding to the points $\{s_1, s_2, \cdots, s_n\}$, is a polynomial of degree $2n - 2$, so

$$\int r_m(x)\,\mu(dx) = \lambda_m p_{n-1}(s_m)\,\omega'(s_m).$$

The integral may be calculated, decomposing the fraction

$$\frac{\omega(x)}{x - s_m}$$

as follows

$$\frac{\omega(x)}{x - x_m} = \frac{p_{n-1}(x)}{\mu_{n-1}} + q_{n-2}(x),$$

where μ_{n-1} is the coefficient of x^{n-1} in $p_{n-1}(x)$ and deg $q_{n-2} \leq n - 2$. Then

$$\int r_m(x)\,\mu(dx) = \int\left(\frac{p_{n-1}(x)}{\mu_{n-1}} + q_{n-2}(x)\right)p_{n-1}(x)\,\mu(dx) = \frac{1}{\mu_{n-1}},$$

and

$$p_{n-1}(s_m) = \frac{1}{\mu_{n-1}\lambda_m\omega'(s_m)}. \tag{2}$$

The last equality implies

$$p_{n-1}(x) = \frac{1}{\mu_{n-1}}\sum_{k=1}^{n}\frac{1}{\lambda_k\omega'(s_k)}l_k(x). \tag{3}$$

The equation

$$p_{n-1}(x) = 0 \tag{4}$$

is equivalent to

$$\frac{p_{n-1}(x)}{p_n(x)} = 0.$$

As

$$\frac{p_{n-1}(x)}{p_n(x)} = \sum_{m=1}^{n}\underset{x=s_k}{Res}\left(\frac{p_{n-1}(x)}{p_n(x)}\right)\cdot\frac{1}{x - s_m}$$

we arrive at

$$\sum_{m=1}^{n}\frac{p_{n-1}(s_m)}{p_n'(s_m)}\cdot\frac{1}{x - s_m} = 0. \tag{5}$$

To investigate this equation, we note that the coefficients $p_{n-1}(s_m)/p'_n(s_m)$ according to formula (2) are positive, therefore

$$\left(\frac{p_{n-1}(x)}{p_n(x)}\right)' = -\sum_{m=1}^{n} \frac{p_{n-1}(s_m)}{p'_n(s_m)} \cdot \frac{1}{(x-s_m)^2} < 0. \qquad (6)$$

As

$$\lim_{x\to\pm\infty} \frac{p_{n-1}(x)}{p_n(x)} = 0$$

and

$$\lim_{x\to s_k\mp 0} \frac{p_{n-1}(x)}{p_n(x)} = \mp\infty,$$

equation (4) has $n-1$ simple roots $\{\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\}$, alternating $\{s_1, s_2, \cdots, s_n\}$ (details in [2], ch. IV, 4.3).

These formulas (in particular, formula (5)) may be applied to investigate so-called intermediate problems. As it is known (cf. [3], ch. III) an oscillating system with n degrees of freedom is described by a selfadjoint matrix $\mathcal{A}$, acting in a n- dimensional Hilbert space H. The oscillation frequencies are equal to eigenvalues of $\mathcal{A}$ and the eigenelements define the directions of so-called normal oscillations. If the system is constrained, its behavior is described by the truncated operator and oscillation frequencies are shifted.

The eigenvalues of any matrix can be calculated by Courant-Fisher minimax principle. It provides a two-sided estimate for shifted frequencies. We propose below an explicit formula for eigenvalues and eigenelements and resolve the inverse problem.

As any selfadjoint operator is an orthogonal sum of selfadjoint operators with cyclic vectors, we assume that $\mathcal{A}$ has a cyclic vector, i.e. its spectrum $Sp\mathcal{A} = \{s_1, s_2, \cdots, s_n\}$ is simple. The constraint will be taken as a subspace L defined by the equation

$$\overline{a}_1 x_1 + \overline{a}_2 x_2 + \cdots + \overline{a}_n x_n = 0 \qquad (7)$$

in the coordinate system $\{e_k\}_1^n$ consisting of eigenelements of $\mathcal{A}$.

Proposition 2. *The spectrum of* $\mathcal{A}_0 = P_L\mathcal{A}|L$ *coincides with the root* $\{\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\}$, *of equation*

$$\sum_{k=1}^{n} \frac{|a_k|^2}{x - s_k} = 0. \qquad (8)$$

Proof. Without loss of generality we suppose that no coefficient $\{a_k\}_1^n$ is equal to zero. Define the weights $\{\lambda_k\}_1^n$ by the formula

$$\lambda_k = \frac{1}{|a_k \omega'(s_k)|^2}. \qquad (9)$$

Let U be an operator from H into $\mathcal{P}_{n-1}$ putting in correspondence to the element e_k the polynomial $\bar{a}_k\omega'(s_k)\, l_k(x)$ and expanded linearly onto H. As

$$\left(\sum_{k=1}^{n} a_k e_k\right) = \sum_{k=1}^{n} |a_k|^2\, \omega'(s_k)\, l_k(x) = \mu_{n-1} p_{n-1}(x),$$

we get $UL = \mathcal{P}_{n-1} \ominus \{\mathcal{P}_{n-1}\} = \mathcal{P}_{n-2}$. So the spectrum of the matrix $\mathcal{A}, = \mathcal{P}_{\mathcal{L}}\, \mathcal{A}\, |\mathcal{L}$ coincides with the roots of p_{n-1} and we must resolve the equation

$$\sum_{k=1}^{n} \frac{1}{\lambda_k \omega'(s_k)} l_k(x) = \sum_{k=1}^{n} |a_k|^2\, \omega'(s_k)\, l_k(x) = 0.$$

As

$$l_k(x) = \frac{\omega(x)}{(x - s_k)\,\omega'(s_k)}$$

$(z- s_k)u>'(s_k)$

we get finally (8).□

The same equation may be rewritten as

$$\frac{\det(\mathcal{A}_0 - xI)}{\det(\mathcal{A} - xI)} = 0. \tag{10}$$

The eigenelements of truncated matrix $\mathcal{A}$, are equal to

$$f_j = \sum_{k=1}^{n} \frac{e_k}{\bar{a}_k \omega'(s_k)} \prod_{\substack{i=1 \\ i \neq j}} (s_k - \sigma_j). \tag{11}$$

Direct calculations yield another formula

$$f_j = const \sum_{k=1}^{n} \frac{a_k}{\sigma_j - s_k} e_k \tag{12}$$

We suppose now that two sets $\{\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\}$, $\{s_1, s_2, \cdots, s_n\}$, of alternating real numbers are known and seek the subspace L, generating the shift $\{s_1, s_2, \cdots, s_n\} \mapsto \{\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\}$. This problem may be resolved by the following method.

Note that

$$p_{n-1}(x) = \mu_{n-1} \prod_{j=1}^{n-1} (x - \sigma_j)$$

and

$$\mu_{n-1}^2 \sum_{m=1}^{n} \lambda_m \prod_{j=1}^{n-1} (s_m - \sigma_j)^2 = 1,$$

From formula (2) it follows that

$$\sum_{k=1}^{n} \lambda_k \prod_{j=1}^{n-1} (s_k - \sigma_j)^2 - \lambda_m \omega'(s_m) \prod_{j=1}^{n-1} (s_m - \sigma_j) = 0,\ m = 1, 2, \cdots, n. \tag{13}$$

If we define $\{\lambda_m\}_1^n$ from the last equalities, formula (9) permits to recover (up to a common factor and specific phase multiplier) the coefficients $\{a_n\}$, hence the subspace L.

The coefficients $\{a_k\}$ may be found directly, if we consider equation (8) as a system of linear equations

$$\sum_{k=1}^{n} \frac{|a_k|^2}{\sigma_j - s_k} = 0, \ j = 1, 2, \cdots, n-1.$$

Introducing vectors $r_j = \{(\sigma_j - s_1)^{-1}, (\sigma_j - s_2)^{-1}, \cdots, (\sigma_j - s_n)^{-1}\}$, we find $\{a_k\}$ as the outer product of $\{r_j\}_1^{n-1}$.

Comparing (11) and (12) we get

$$|a_k|^2 = const \frac{\prod_{i=1}^{n} (s_k - \sigma_i)}{\prod_{\substack{j=1 \\ j \neq k}}^{n} (s_k - s_j)}.$$

These formulas may be applied also in the case, where A is an infinite- dimensional Hilbert space selfadjoint operator, belonging to the Schatten trace class. As it is well known (cf. [4], ch. IV) for any such operator the sequence of eigenvalues $\{s_k\}$ satisfies the condition

$$\sum_{k=1}^{\infty} |s_k| < \infty,$$

the infinite product

$$\prod_{k=1}^{\infty} (1 - z s_k), \ z \in \mathbb{C}$$

converges to an entire function $D_A(z)$, said to be the characteristic determinant of A. The roots of equation $D_A(z) = 0$ coincide with s_k^{-1} and one has the equality

$$D_A(z) = \lim_{n \to \infty} \det (\delta_{ik} - z (A e_k, e_i)), \ i, k = 1, 2, \cdots, n. \qquad (14)$$

Easily can be seen that if coefficients $\{a_k\}_1^{\infty}$, defining the subspace L, belong to l^2, the series

$$F(x) = \sum_{k=1}^{\infty} \frac{|a_k|^2}{x - s_k}$$

converges for any non-zero $x, x \neq s_k, k \in \mathbb{N}$ to a continuous function. This convergence is uniform on any segment $[a, b]$ which does not contain the points 0 and $\{s_k\}$. The same is true for the (formal) derivative

$$F'(x) = -\sum_{k=1}^{\infty} \frac{|a_k|^2}{(x - s_k)^2},$$

so F is decreasing. Let x_m be a root of the equation $F(x) = 0$ and ϵ- a sufficiently small positive number. As $F(x_m - \epsilon) > 0$ and $F(x_m + \epsilon) < 0$ and

$$F_N(x) = \sum_{k=1}^{N} \frac{|a_k|^2}{x - s_k}$$

converges to $F(x)$, we get $F_N(x_m - \epsilon) > 0$ and $F_N(x_m + \epsilon) < 0$ for sufficiently large N, hence the roots $\{x_m^N\}$ of the equation $F_N(x) = 0$ tend to x_m as $N \to \infty$. According to formulae (10), (14) the non -zero terms of the spectrum of A_0 coincide with the roots of the following equation

$$\sum_{k=1}^{\infty} \frac{|a_k|^2}{x - s_k} = 0.$$

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Լ. Զ. Գևորգյան

Հատված օպերատորի սպեկտրի նկարագրություն և միջանկյալ խնդիրներ

Ինքնահամալուծ օպերատորի՝ ոչ ինվարիանտ ենթատարածության վրա սահմանափակման սպեկտրը նկարագրվում է զուգորդված օրթոգոնալ բազմանդամների միջոցով: Ստացված նկարագրությունն օգտագործվում է տատանողական համակարգը կապերով կաշկանդելու հետևանքով առաջացած հաճախությունների տեղաշարժը որոշելու համար: Լուծված է նաև հակադարձ՝ հաճախությունների տեղաշարժով կապերի բնույթը որոշելու խնդիրը:

Л. З. Геворкян

Описание спектра усеченного оператора и промежуточные проблемы

Спектр ограничения самосопряженного оператора на неинвариантное подпространство описывается при помощи ассоциированных ортогональных полиномов. Полученное описание применяется для определения сдвига частот колебательной системы при наложении связей. Решена также обратная задача определения характера связей при помощи сдвига частот.

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