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Invariant Imbedding in Stochastic Geometry

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INTRODUCTION

V. A. Ambartsumian has often pointed out ((1), Epilogue) that versions of the "invariance principle" he has used in the study of scattering of light may be effective in other mathematical problems as well.

R. Bellman and his followers (2) have developed and systematically applied similar ideas. Completely recognizing the priority of V. A. Ambartsumian, they coined the term of "Invariant Imbedding" to designate the corresponding mathematical approach, presumably to become useful in mathematical physics at large. Outside mathematical physics, an analytical procedure which can be attributed to invariant imbedding, has been applied in integral geometry in (3), where it helped to discover basic combinatorics governing the relation between measures in the space of lines and metrics.

The present article applies invariant imbedding in the related field of stochastic geometry. We derive differential equations for the probability distribution of the number of hits of a test segment by the lines of a random line process, valid under certain factorization assumptions. The imbedding parameters are the direction and the length of the test segment. The results are valid for random line processes that are translation invariant in distribution and possess first and second moment measures. There are also some smoothness assumptions.

Section 1 contains necessary prerequisites from the theory of translation invariant random line processes. The proofs of the properties listed in this section can be easily worked out within the standard framework of the "method of fixed realizations" as presented in (4).

In §2, invariant imbedding is applied towards derivation of differential relations involving Palm type probability distributions of the line process. In §3 we consider the marked point processes of hits induced on test lines by the lines of the line process. The marks are the angles at which the hits occur. We show that certain degree of independence between the point process of hits and the sequence of hit angles transforms these relations to differential equations of rather conventional nature. As stated by the concluding Theorem, under another additional assumption of "sufficient mixing" and absence of correlation between the cotangents of the angles, these equations can be resolved yielding Poisson distribution for random numbers of hits on test segments. Among earlier attempts to consider similar questions we mention (5) (this paper was the first to study the general random line processes), Chapter 10 in (4) and (6). They all used approaches different from the present one.

§1. TRANSLATION INVARIANT LINE PROCESSES

We consider random line processes in the Euclidean plane $\mathbb{R}^2$. A line process is defined to be (4) a random point process in the space of lines. Our notation will be g for a line in $\mathbb{R}^2$ and $\{g_i\}$ for a random line process. The latter notation stresses the fact that a line process is a countable random set of lines. Occasionally we use the letter $\mathbf{M}$ to denote the space of *realizations* of line processes, $\mathbf{m} \in \mathbf{M}$. By P we denote the probability distribution of $\{g_i\}$ (a probability measure on $\mathbf{M}$). We say that a line g "hits" a segment γ if $\gamma \cap g$ reduces to a point in the *relative interior* of γ.

Given a "test segment" γ, we will consider the event

$$\binom{\gamma}{k} = \{\gamma \text{ is hit by exactly } k \text{ lines from } \{g_i\}\}.$$

Given two test segments γ_1 and γ_2 and two nonnegative integers k_1, k_2, we write $\begin{pmatrix}\gamma_1 & \gamma_2\\ k_1 & k_2\end{pmatrix}$ for the intersection of $\binom{\gamma_1}{k_1}$ and $\binom{\gamma_2}{k_2}$. This notation extends to any number of test segments. For the probabilities of the events we use notation like $P\binom{\gamma}{k}$. In the definition that follows and elsewhere we write dg for the unique (up to constant factor) measure in the space of lines which is invariant with respect to Euclidean motions of $\mathbb{R}^2$.

Definition 1. A line process $\{g_i\}$ belongs to the class TICD2 if its probability distribution P is invariant with respect to the group of translations of the plane (Translation Invariant) and the first and second moment measures of $\{g_i\}$ are of the form $f_1 dg$ and (outside $g_1 = g_2$) $f_2 dg_1 dg_2$ respectively, with Continuous Densities f_1 and f_2.

This section contains a list of some properties of the processes from the class TICD2 that we will need for Invariant Imbedding in §2. Their proofs can be easily obtained by the method of "fixed realizations" presented in ([4]), and therefore are omitted.

Let γ be an arbitrary segment in the plane. In this and the next section γ will remain fixed. We construct the rectangle R shown on Fig. 1. by attaching the lateral sides v_1 and v_2 to γ, otherwise called *vertical windows*. Two segments h_1 and h_2 attached to γ to make continuations of γ we call *horizontal windows*. The length of the windows, vertical or horizontal, let be l, and we will assume that l *tends to zero*.

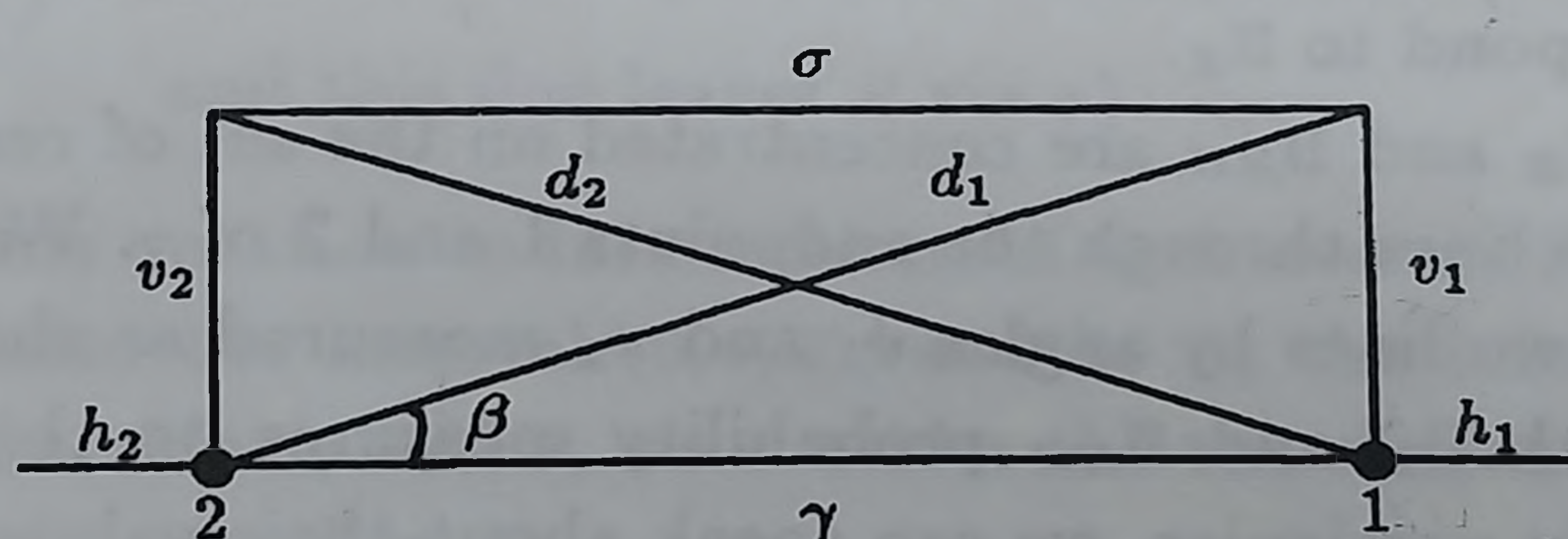

Fig. 1

Property P1 : for any window w, vertical or horizontal,

$$P\binom{w}{1} = O(l), \quad P\binom{w}{2} = O(l^2) \text{ and } P\binom{w}{k} = o(l^2) \text{ for } k > 2.$$

We will use special short notation

$$H = \binom{h_1}{1}, \quad \text{and } V = \binom{v_1}{1}.$$

The *intensities* of the process of intersection points induced by $\{g_i\}$ on a horizontal (λ_H) or vertical (λ_V) test lines are well defined :

$$\lambda_H = \lim_{l \to 0} l^{-1} P(H) \text{ and } \lambda_V = \lim_{l \to 0} l^{-1} P(V).$$

Property P2 : the following limits exist

$$\lim_{l\to 0} l^{-2} P(HH) = c_{HH}, \quad \lim_{l\to 0} l^{-2} P(A) = c_A \text{ and } \lim_{l\to 0} l^{-2} P(B) = c_B,$$

where $HH = \begin{pmatrix} h_1 & h_2 \\ 1 & 1 \end{pmatrix}$, while $A \subset \begin{pmatrix} v_1 & v_2 \\ 1 & 1 \end{pmatrix}$ is the subevent that occurs whenever v_1 and v_2 are intersected by the *same line* from $\{g_i\}$, while B is the relative complement of A within $\begin{pmatrix} v_1 & v_2 \\ 1 & 1 \end{pmatrix}$ i.e. B stands for the case where the intersections are caused by *two different lines* from $\{g_i\}$.

Property P3 :

$$\sum_{j_1+j_2\geq 3} P\begin{pmatrix} v_1 & v_2 \\ j_1 & j_2 \end{pmatrix} = o(l^2).$$

We will need the concepts of Palm type distributions Π_V, Π_H, Π_B and Π_{HH} of a process $\{g_i\}$ (for a rigorous geometrical theory of Palm distributions see ([4])).

Roughly, each of these Palm type distributions Π_Z is the limiting, as l tends to 0, conditional distributions of $\{g_i\}$, *conditional upon the event* Z. For each $Z \in \{H, V, B, HH\}$ we can speak about the line process that correspond to Π_Z.

Both Π_B and Π_{HH} are concentrated on the set of realizations that possess two lines through the endpoints 1 and 2 of γ. We parameterize the latter two lines by angles ψ_1 and ψ_2 measured as shown on Fig. 2, making both Π_B and Π_{HH} probability measures on the space $(0,\pi) \times (0,\pi) \times \mathrm{M}$. In particular, we can speak about their values on the events of the type $\begin{pmatrix} \gamma \\ k \end{pmatrix} \cap \{\Theta_1\} \cap \{\Theta_2\}$ where Θ_1, Θ_2 are two subintervals of $(0,\pi)$, $\{\Theta_i\}$ stands for the event $\psi_i \in \Theta_i$ *that occurs at the endpoint i.* Both probability distributions Π_H and Π_V are concentrated on the set of realizations that possess a line through the endpoint 1 of γ, i.e. both live on the $(0,\pi) \times \mathrm{M}$. In particular, the values of Π_H and Π_V on the events of the type $\begin{pmatrix} \gamma \\ k \end{pmatrix} \cap \{\Theta_1\}$ are well defined.

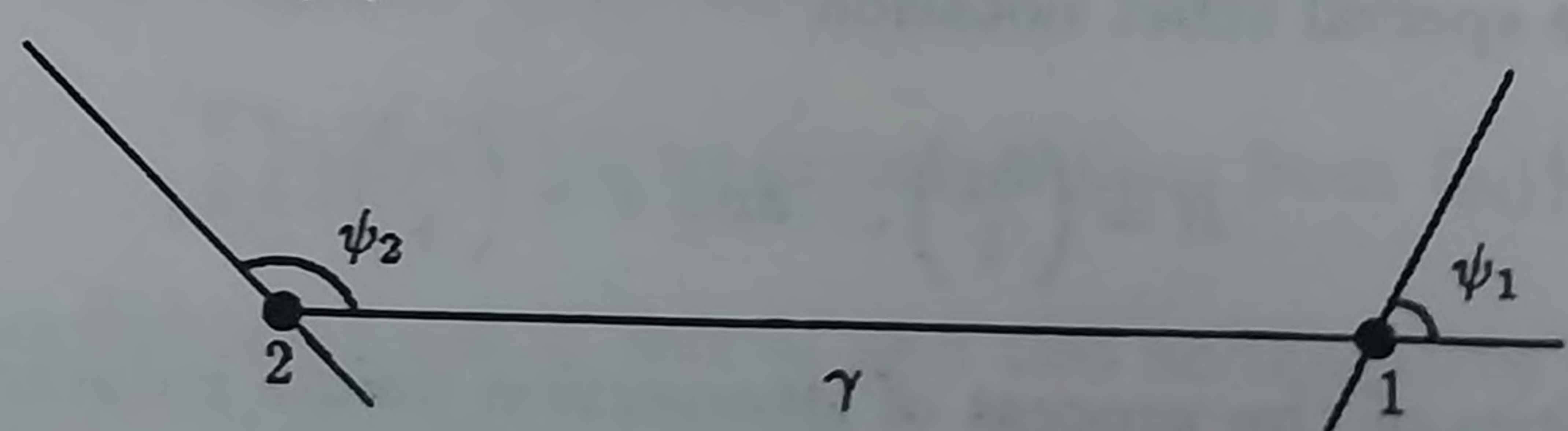

Fig. 2

We write E_Z for the *expectation* with respect to the probability measure Π_Z.

Property P4. Let $F(\Psi_1, \mathrm{m})$ be a bounded function defined on $(0,\pi) \times \mathrm{M}$. If $\{g_i\} \in TICD2$, then for every choice of γ

$$\lambda_H E_H [F(\psi_1, \mathrm{m}) \mid \cot \psi|] = \lambda_V E_V F(\psi_1, \mathrm{m}).$$

Property P5. Let $F(\Psi_1, \Psi_2, \mathrm{m})$ be a bounded function defined on $(0,\pi) \times (0,\pi) \times \mathrm{M}$. If $\{g_i\} \in TICD2$, then for every choice of γ

$$c_{HH} E_{HH} [F(\psi_1, \psi_2, \mathrm{m}) | \cot \psi_1 \cot \psi_2|] = c_B E_B F(\psi_1, \psi_2, \mathrm{m}).$$

Palm type probability of arbitrary event in M can not in general be calculated as a limit of the corresponding conditional probabilities of the same event. The following Proposition 4 illustrates this instability.

For a side u of the rectangle R, $u \neq v_1$, we define the event $\binom{v_1}{u} \subset \binom{v_1}{1}$ as

$$\binom{v_1}{u} = \{\text{there is exactly one line in } \{g_i\} \text{ that hits } v_1 \text{ and this line leaves } R \text{ via } u\}$$

and extend this notation to intersections of such events. From now on by S_1 we denote the interval $(0, \pi/2)$, and by S_2 the interval $(\pi/2, \pi)$. Along with the sides γ and σ, we consider the two diagonals d_1 and d_2 of the rectangle R on Fig. 1.

Property P6. If $\{g_i\} \in TICD2$, then four limit relations of the form

$$\lim_{l \to 0} l^{-2} P \begin{pmatrix} u & v_1 \\ k & u_1 \end{pmatrix} = c_V \Pi_V \left[\begin{pmatrix} \gamma \\ k - r \end{pmatrix} \cap \{S_i\} \right]$$

hold. The map

$$(u, u_1) \mapsto (r, i)$$

on which they depend is given by

$$(\gamma, \gamma) \mapsto (1,1) \qquad (d_1, \gamma) \mapsto (0,1) \qquad (\gamma, d_1) \mapsto (0,2) \qquad (d_1, d_1) \mapsto (1,2)$$

Property P7. If $\{g_i\} \in TICD2$, then sixteen limit relations of the form

$$\lim_{l \to 0} l^{-2} P \begin{pmatrix} u & v_1 & v_2 \\ k & u_1 & u_2 \end{pmatrix} = c_B \Pi_B \left[\begin{pmatrix} \gamma \\ k - r \end{pmatrix} \cap \{S_i\} \cap \{S_j\} \right]$$

hold. The map

$$(u, u_1, u_2) \mapsto (r, i, j)$$

on which they depend is given by the table

$(\gamma,\gamma,\gamma) \mapsto (2,1,2)$ $(\gamma,\gamma,\sigma) \mapsto (1,1,1)$ $(\gamma,\sigma,\gamma) \mapsto (1,2,2)$ $(\gamma,\sigma,\sigma) \mapsto (0,2,1)$

$(\sigma,\gamma,\gamma) \mapsto (0,1,2)$ $(\sigma,\gamma,\sigma) \mapsto (1,1,1)$ $(\sigma,\sigma,\gamma) \mapsto (1,2,2)$ $(\sigma,\sigma,\sigma) \mapsto (2,2,1)$

$(d_1,\gamma,\gamma) \mapsto (1,1,2)$ $(d_1,\gamma,\sigma) \mapsto (0,1,1)$ $(d_1,\sigma,\gamma) \mapsto (2,2,2)$ $(d_1,\sigma,\sigma) \mapsto (1,2,1)$

$(d_2,\gamma,\gamma) \mapsto (1,1,2)$ $(d_2,\gamma,\sigma) \mapsto (2,1,1)$ $(d_2,\sigma,\gamma) \mapsto (0,2,2)$ $(d_2,\sigma,\sigma) \mapsto (1,2,1)$

Limit conditioning by the event A has a special status. In fact, the method of fixed realizations yields ([4]) the existence of Palm distribution Π_A only for line processes that are invariant with respect to the group of Euclidean motions (translations *and* rotations). For $\{g_i\} \in TICD2$ a condition of existence of the limit

$$x_k = \lim_{l\to 0}[P(A)]^{-1}P(\binom{\chi}{k} \cap A) = c_A^{-1} \lim_{l\to 0} l^{-2}P(\binom{\chi}{k} \cap A). \tag{1}$$

is contained in Proposition 2 of the next section. In (1), the segment χ is defined as follows : whenever the event A occurs, $\{g_i\}$ contains a line which hits both v_1 and v_2, and we take χ to be the segment cut from that unique line by the vertical windows.

§2. INVARIANT IMBEDDING

We formulate the two propositions of the present section for the line processes from the class TICD2, although application of the invariant imbedding approach requires only the properties P 1-P7 listed in §1, rather than proper translation invariance of $\{g_i\}$. We use the notation

$\Delta x_k = x_k - x_{k-1}$ for the first and

$\Delta^2 y_k = y_k - 2y_{k-1} + y_{k-2}$ for the second difference with respect to k.

Proposition 1. If $\{g_i\} \in TICD2$, then the following limit exists

$$\begin{aligned}\lim_{l\to 0}(\lambda_V l)^{-1}\left[P\binom{d_1}{k} - P\binom{\gamma}{k}\right] &= \Pi_V\left(\binom{\gamma}{k-1} \cap \{S_2\}\right) + \\ + \Pi_V\left(\binom{\gamma}{k} \cap \{S_1\}\right) - \Pi_V\left(\binom{\gamma}{k} \cap \{S_2\}\right) &- \Pi_V\left(\binom{\gamma}{k-1} \cap \{S_1\}\right),\end{aligned} \tag{2}$$

where S_1 is the interval $(0, \pi/2)$, S_2 is the interval $(\pi/2, \pi)$.

The proof of this proposition, based on P1 and P4, we leave to the reader because it is a simplified ("first order") version of the proof of Proposition 2, which we give in complete detail.

Proposition 2. If $\{g_i\} \in TICD2$ and the limit

$$L_k(\gamma) = \lim_{l \to 0} l^{-2} \left[P\begin{pmatrix} d_1 \\ k \end{pmatrix} - P\begin{pmatrix} \gamma \\ k \end{pmatrix} - P\begin{pmatrix} \sigma \\ k \end{pmatrix} + P\begin{pmatrix} d_2 \\ k \end{pmatrix} \right]$$

exists, then the limits x_k in (1) exist and

$$L_k(\gamma) = -2c_A \Delta x_k + c_B \Delta^2 y_k, \tag{3}$$

where

$$\begin{aligned} y_k = \Pi_B\left(\begin{pmatrix} \gamma \\ k \end{pmatrix} \cap \{S_1\} \cap \{S_1\}\right) + \Pi_B\left(\begin{pmatrix} \gamma \\ k \end{pmatrix} \cap \{S_2\} \cap \{S_2\}\right) - \\ - \Pi_B\left(\begin{pmatrix} \gamma \\ k \end{pmatrix} \cap \{S_1\} \cap \{S_2\}\right) - \Pi_B\left(\begin{pmatrix} \gamma \\ k \end{pmatrix} \cap \{S_2\} \cap \{S_1\}\right) \end{aligned} \tag{4}$$

with S_1 and S_2 same as in Proposition 1.

Proof : For convenience in writing, we occasionally use the notation (see Fig. 1) $\gamma = \sigma_1$ and $\sigma = \sigma_2$. For each choice of τ from the collection $\{\sigma_1, \sigma_2, d_1, d_2\}$ we represent $\begin{pmatrix} \tau \\ k \end{pmatrix}$ as a union of mutually exclusive events

$$\begin{pmatrix} \tau \\ k \end{pmatrix} = \bigcup_{j_1, j_2 \geq 0} \begin{pmatrix} \tau & v_1 & v_2 \\ k & j_1 & j_2 \end{pmatrix}.$$

By P3, when $l \to 0$

$$\sum_{j_1 + j_2 \geq 3} P\begin{pmatrix} \tau & v_1 & v_2 \\ k & j_1 & j_2 \end{pmatrix} = o(l^2),$$

and therefore

$$P\begin{pmatrix} \tau \\ k \end{pmatrix} = \sum_{0 \leq j_1 + j_2 \leq 2} P\begin{pmatrix} \tau & v_1 & v_2 \\ k & j_1 & j_2 \end{pmatrix} + o(l^2). \tag{5}$$

A line which enters a triangle crossing one of its sides leaves the triangle crossing one of the remaining sides. Therefore we have the set identities

$$\begin{pmatrix} \sigma_1 & v_1 & v_2 \\ k & 0 & j_2 \end{pmatrix} = \begin{pmatrix} d_1 & v_1 & v_2 \\ k & 0 & j_2 \end{pmatrix}, \quad \begin{pmatrix} \sigma_1 & v_1 & v_2 \\ k & j_1 & 0 \end{pmatrix} = \begin{pmatrix} d_2 & v_1 & v_2 \\ k & j_1 & 0 \end{pmatrix}, \tag{6}$$

as well as similar identities for σ_2.

In the expression

$$D = P\begin{pmatrix} d_1 \\ k \end{pmatrix} - P\begin{pmatrix} \gamma \\ k \end{pmatrix} - P\begin{pmatrix} \sigma \\ k \end{pmatrix} + P\begin{pmatrix} d_2 \\ k \end{pmatrix}.$$

we replace the individual probabilities by their decompositions (5). Further, the probability of each event $\begin{pmatrix} \sigma_i & v_1 & v_2 \\ k & j_1 & j_2 \end{pmatrix}$, where either j_1 or

j_2 or both equal zero we replace according to (6) (or the analog of (6) for σ_2). In the resulting sum the probabilities of each event $\begin{pmatrix} d_i & v_1 & v_2 \\ k & j_1 & j_2 \end{pmatrix}$ where at least one of the indices j_1 or j_2 equals zero, will enter twice with opposite signs. So these terms cancel out and we get

$$D = -\sum_{i=1,2} P\begin{pmatrix} \sigma_i & v_1 & v_2 \\ k & 1 & 1 \end{pmatrix} + \sum_{i=1,2} P\begin{pmatrix} d_i & v_1 & v_2 \\ k & 1 & 1 \end{pmatrix} + o(l^2). \tag{7}$$

We have

$$\begin{pmatrix} \tau & v_1 & v_2 \\ k & 1 & 1 \end{pmatrix} = \bigcup_{(u_1,u_2)\in U} \begin{pmatrix} \tau & v_1 & v_2 \\ k & u_1 & u_2 \end{pmatrix}, \tag{8}$$

where $U = \{v_2, \gamma, \sigma\} \times \{v_1, \gamma, \sigma\}$, and the events under the union are mutually exclusive.
Therefore the probabilities in (7) can be replaced by sums of probabilities of the events according to (8). By P3, for pairs $(u_1, u_2) = (v_2, \gamma)$, (v_2, σ), (γ, v_1) and (σ, v_1) we have

$$P\begin{pmatrix} \gamma & v_1 & v_2 \\ k & u_1 & u_2 \end{pmatrix} = o(l^2),$$

and (7) takes the form

$$D = -\sum_{i=1,2} P\begin{pmatrix} \sigma_i & v_1 & v_2 \\ k & v_2 & v_1 \end{pmatrix} + \sum_{i=1,2} P\begin{pmatrix} d_i & v_1 & v_2 \\ k & v_2 & v_1 \end{pmatrix} +$$

$$+\sum_{i=1,2}\left[\sum_{(u_1,u_2)\in U_1}\left[-P\begin{pmatrix} \sigma_i & v_1 & v_2 \\ k & u_1 & u_2 \end{pmatrix} + P\begin{pmatrix} d_i & v_1 & v_2 \\ k & u_1 & u_2 \end{pmatrix}\right]\right] + o(l^2), \tag{9}$$

where in the last sum $U_1 = \{\gamma, \sigma\} \times \{\gamma, \sigma\}$. Note that $\begin{pmatrix} v_1 & v_2 \\ v_2 & v_1 \end{pmatrix}$ coincides with A as defined in P2.
We divide (9) by l^2 and let $l \to 0$. Using P7, one can check that

$$\lim_{l\to 0} l^{-2} \sum_{i=1,2}\left[\sum_{(u_1,u_2)\in U_1}\left[-P\begin{pmatrix} \sigma_i & v_1 & v_2 \\ k & u_1 & u_2 \end{pmatrix} + P\begin{pmatrix} d_i & v_1 & v_2 \\ k & u_1 & u_2 \end{pmatrix}\right]\right] = c_B \Delta^2 y_k.$$

By the definition of the segment χ

$$\begin{pmatrix} \sigma_i \\ k \end{pmatrix} \cap A = \begin{pmatrix} \chi \\ k \end{pmatrix} \cap A \quad \text{and} \quad \begin{pmatrix} d_i \\ k \end{pmatrix} \cap A = \begin{pmatrix} \chi \\ k-1 \end{pmatrix} \cap A.$$

The existence of the limits, $i = 1, 2$

$$\lim_{l\to 0} l^{-2} P\begin{pmatrix} \sigma_i & v_1 & v_2 \\ k & v_2 & v_1 \end{pmatrix} = c_A x_k \quad \text{and} \quad \lim_{l\to 0} l^{-2} P\begin{pmatrix} d_i & v_1 & v_2 \\ k & v_2 & v_1 \end{pmatrix} = c_A x_{k-1}$$

follows from the existence of the limits $L_k(t, \alpha)$ (first for x_0 because $x_{-1} = 0$ and then in succession, for all x_k). Proposition 7 is completely proved.

§3. FACTORIZATION AND SUFFICIENT MIXING CONDITIONS

By translation invariance, the probabilities $P\binom{\gamma}{k}$ depend solely on the length t and direction α of the segment γ. So we can use "functional" notation

$$P\binom{\gamma}{k} = p_k(t,\alpha)$$

and the left-hand side of (2) reduces to

$$\lim_{l\to 0} \frac{p_k(\sqrt{t^2+l^2},\alpha+\beta) - p_k(t,\alpha)}{l} = t^{-1}\frac{\partial p_k(t,\alpha)}{\partial \alpha},$$

while for (3) applying Taylor expansion we find

$$L_k(\gamma) = \lim_{l\to 0} \frac{p_k(\sqrt{t^2+l^2},\alpha+\beta) - 2p_k(t,\alpha) + p_k(\sqrt{t^2+l^2},\alpha-\beta)}{l^2} =$$

$$= t^{-1}\cdot\frac{\partial p_k(t,\alpha)}{\partial t} + t^{-2}\cdot\frac{\partial^2 p_k(t,\alpha)}{\partial \alpha^2}. \tag{10}$$

Validity of (10) is essentially the *smoothness assumption* to which we refer in the Theorem below. Turning to the right-hand sides of (2) and (3), we first transform them using P4 and P5. By a remarkable interplay of signs

$$\lambda_V \Pi_V\left(\binom{\gamma}{k}\cap\{S_1\}\right) - \lambda_V \Pi_V\left(\binom{\gamma}{k}\cap\{S_2\}\right) = \lambda_H E_H I\binom{\gamma}{k}\cot\psi_1, \tag{11}$$

where I stands for the indicator function of the corresponding event (dependence on $\mathbf{m}\in\mathbf{M}$ suppressed). Similarly, for the quantities y_k in (8) we have

$$y_k = E_{HH} I\binom{\gamma}{k}\cot\psi_1\cot\psi_2.$$

We are ready to consider the consequences of certain *factorization* assumptions F1,F2 and F3, expressed in terms of the probability distribution of random *marked point process* $\{P_i,\psi_i\}_g$ of intersections induced by $\{g_i\}$ on a test line g. In this notation, $P_i = g\cap g_i$, while the mark ψ_i is the angle at which the intersection at P_i occurs. We note in advance, that jointly, the three assumptions F1,F2 and F3 are essentially less restrictive than the *Cox independence* well known in stochastic geometry ([5]). We say, that $\{P_i,\psi_i\}_g$ has Cox independence property, if for test line g of any direction α, the sequence of angles $\{\psi_i\}$ is independent of the point process $\{P_i\}$, and $\{\psi_i\}$ is a sequence of independent angles. Doubly stochastic Poisson line process $\{g_i\}$

governed by random measure of the form $\xi \cdot f_1(\phi)dg$, where the factor ξ is random while $f_1(\phi)$ is nonrandom (Cox line processes), all have this property.

Assumption F1 : for any direction α, any t and k the random variables $\cot\psi_1$ and $I\binom{\gamma}{k}$ are uncorrelated, i.e.

$$E_H I\binom{\gamma}{k}\cot\psi_1 = \Pi_H\binom{\gamma}{k} E_H\cot\psi_1.$$

We have

$$\lambda(\alpha) = \int \sin\psi f_1(\phi)d\psi,$$

where f_1 is the density of the first moment measure of $\{g_i\}$ (because of translation invariance, f_1 depends solely on the direction ϕ of the line g), ψ is the angle between the directions ϕ and α. Therefore the probability density of random angle ψ_1 is $(\lambda(\alpha))^{-1}\sin\psi f_1(\phi)d\psi$. Hence

$$E_H\cot\psi_1 = (\lambda(\alpha))^{-1}\int\cos\psi f_1(\phi)d\psi = -(\lambda(\alpha))^{-1}\lambda'(\alpha),$$

with $\lambda'(\alpha)$ denoting the first derivative in α. By so called Palm formulae for point processes in one dimention, see (4)

$$\lambda(\alpha)\left[\Pi_H\binom{\gamma}{k-1} - \Pi_H\binom{\gamma}{k}\right] = \frac{\partial p_k(t,\alpha)}{\partial t}. \tag{12}$$

We come to the conclusion that under the assumption F1, the relation (2) transforms to the differential equation

$$\frac{\partial p_k(t,\alpha)}{\partial\alpha} = t\cdot(\lambda(\alpha))^{-1}\lambda'(\alpha)\frac{\partial p_k(t,\alpha)}{\partial t}. \tag{13}$$

The equation (13) can be easily solved by standard method of characteristics. Its general solution has the form

$$p_k(t,\alpha) = q_k(\lambda(\alpha)t), \tag{14}$$

where $q_k(\cdot)$ is some function of one argument.

Assumption F2 : for any direction α, any t and k the random variables $\cot\psi_1\,\cot\psi_2$ and $I\binom{\gamma}{k}$ are uncorrelated, i.e.

$$E_{HH} I\binom{\gamma}{k}\cot\psi_1\cot\psi_2 = \Pi_{HH}\binom{\gamma}{k} E_{HH}\cot\psi_1\cot\psi_2.$$

One can easily derive the second order analog of (12) :

$$c_{HH}\Delta^2\Pi_{HH}\binom{\gamma}{k} = \frac{\partial^2 p_k(t,\alpha)}{\partial t^2}.$$

We conclude that under assumption F2, (3) reduces to

$$t\cdot\frac{\partial p_k(t,\alpha)}{\partial t} + \frac{\partial^2 p_k(t,\alpha)}{\partial\alpha^2} = -2\,c_A t^2\,\Delta x_k + t^2 a(t,\alpha)\frac{\partial^2 p_k(t,\alpha)}{\partial t^2}, \tag{15}$$

where $a(t,\alpha) = E_{HH}\cot\psi_1\cot\psi_2$.

We observe ([4]) that $c_A(\gamma) = t^{-1} f_1(\alpha)$ and $2f_1(\alpha) = \lambda(\alpha) + \lambda''(\alpha)$. By a direct substitution of (14) into (15) we get the following corollary of F1 and F2 acting jointly :

$$(\lambda + \lambda'')\, q_k' + t[(\lambda')^2 - \lambda^2 \cdot a(t,\alpha)]\, q_k'' = -(\lambda + \lambda'')\, \Delta x_k. \tag{16}$$

Assumption F3 : for any direction α and any t

$$E_{HH} \cot \psi_1 \cot \psi_2 = E_H \cot \psi_1 E_H \cot \psi_2 = [\lambda'(\alpha)]^2 [\lambda(\alpha)]^{-2}$$

Under F3 the equation (16) transforms to

$$q_k' = -\Delta x_k. \tag{17}$$

This infinite system of equations is easily solved if we make one more additional assumption that

$$x_k = P\binom{\gamma}{k} = p_k(t,\alpha) \tag{18}$$

meaning limiting independence of the events A and $\binom{\chi}{k}$. We call (18) the assumption of *sufficient mixing*. Roughly, (18) means that the circumstance that χ is chosen to lie on one of the lines from the random collection $\{g_i\}$ can be ignored, as far as the distribution of the number of hits on that segment is considered. In the limit, as $l \to 0$, χ receives length t and direction α.

Under (18), the solution of (17) satisfying natural initial conditions $q_0(0) = 1$ and $q_k(0) = 0$ for $k > 0$ yields Poisson probabilities with unit parameter $q_k(t) = \frac{t^k}{k!} e^{-t}$. This result we formulate as a theorem.

THEOREM. Let $\{g_i\} \in TICD2$ *possesses smooth hitting probabilities* $p_k(t,\alpha)$. *If for any direction* α *and length* t, $\{g_i\}$ *possesses the three factorization properties F1, F2 and F3, as well as the property of sufficient mixing, then* $p_k(t,\alpha)$ *are Poisson probabilities with parameter* $\lambda(\alpha)t$, *where* $\lambda(\alpha)$ *is the* sin-*transform of the density of the first moment measure.*

We note in conclusion, that if the condition of sufficient mixing is removed, then the Theorem becomes invalid, as demonstrated by any Cox line processes $\{g_i\} \in TICD2$ for which the factor ξ is essentially random. For them the probabilities $p_k(t,\alpha)$ become mixtures of Poisson probabilities. The latter reduce to Poisson probabilities whenever ξ is nonrandom. But in that case the line process $\{g_i\}$ becomes Poisson. Clearly, for Poisson $\{g_i\}$ the sufficient mixing condition is satisfied.

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Հայաստանի ԳԱԱ ակադեմիկոս **Ռ. Վ. ՀԱՄԲԱՐՁՈՒՄՅԱՆ**

Հաստատագիր ներդրման եղանակը ստոխաստիկ երկրաչափության մեջ

Հոդվածում հաստատագիր ներդրման եղանակը կիրառվում է ստոխաստիկ երկրաչափության մի խնդրում: Հարթության վրա դիտարկվում են տեղաշարժների նկատմամբ հաստատագիր ուղիղների երկրորդ կարգի պատահական պրոցեսներ:

Ցույց է տրված, որ եթե բավարարված են ֆակտորիզացիայի և խառնելիության որոշ պայմաններ, ապա փորձնական հատվածը հատող պրոցեսին պատկանող ուղիղների քանակի բաշխումը միշտ Պուասոնյան է:

Академик НАН Армении **Р. В. АМБАРЦУМЯН**

Инвариантное вложение в стохастической геометрии

В статье принцип инвариантного вложения применяется в одной задаче стохастической геометрии. При некоторых предположениях факторизации выводятся дифференциальные уравнения для вероятностей, описывающих распределение числа пересечений тестового сегмента прямыми, принадлежащими трансляционно-инвариантному случайному процессу прямых на плоскости.

Показано, что при дополнительном условии т.н. "достаточного перемешивания" полученные уравнения допускают лишь пуассоновские решения.

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