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# Classical analog of extended phase space SUSY: spectrum generating algebra

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**Abstract**

We address the classical analog of the *extended phase space* (EPS) (N=2)-supersymmetric quantum mechanics (SUSY QM). We derive the integrals of motion and calculate the parameters that measure the breaking of EPS-SUSY. We analyze the non-perturbative mechanism for its breaking in the instanton picture, which has resulted from tunneling between the classical vacua of the theory. Particular attention is given to the independent group theoretical method and its connection to the factorization procedure in developing a spectrum generating algebra (SGA) for the EPS-SUSY.

***Keywords:*** *Extended phase space quantum mechanics–Supersymmetric quantum mechanics–Symmetry breaking–Spectrum generating algebra*

## 1. Introduction

Much use has been made of the technique of EPS QM Nasiri (2006), Sobouti & Nasiri (1993) for keeping the symmetry between coordinates ($q$) and momenta ($p$) in the process of quantization. It was observed that the concept of an *extended* Lagrangian, $\mathcal{L}(p,\,q,\,\dot{p},\,\dot{q})$ in phase space allows a subsequent extension of Hamilton's principle to actions minimum along the actual trajectories in $(p,\,q)-$, rather than in $q-$space. This extension, in turn, allows a definition of "second" momenta $\pi_p = \delta\,\mathcal{L}/\delta\,\dot{p}$ and $\pi_q = \delta\,\mathcal{L}/\delta\,\dot{q}$, and a subsequent introduction of an *extended* phase space $(p,\,q,\,\pi_p,\,\pi_q)$ and of an *extended* Hamiltonian, $H_{ext}(p,\,q,\,\pi_p,\,\pi_q)$. In particular, the vanishing of $\pi_q$ or $\pi_p$ is the condition for $p$ and $q$ to constitute a canonical pair. In the language of statistical quantum mechanics, this choice picks a pure state (*actual path*). Otherwise, one is dealing with a mixed state (*virtual path*). In Ter-Kazarian & Sobouti (2008) we address the EPS stochastic quantization of Hamiltonian systems with first class holonomic constraints. This, in a natural way, results in the Faddeev-Popov conventional path-integral measure for gage systems. Consequently, in Ter-Kazarian (2009), we construct an (N=2) realization of the EPS-SUSY algebra. A new feature of this formulation over the ordinary SUSY QM formalism E. (1981, 1982), Gamboa & Plyushchay (1998), Inomata (1992), Inomata & Junker (1991), Nicolai (1976), Salomonson & Van Holten (1982), E. (rf:) is, in particular, the fact that even if the SUSY was broken in $q-$ or $p-$ spaces, it can still be good in $(q,p)-$ space, and vise versa. We also explore the shape-invariance Balantekin (1998), Gangopadhyaya (1998), Gendenshtein (1983) of exactly solvable EPS SUSY potentials. These aspects of conventional SUSY QM and its connection to the factorization method have been extensively investigated by several authors. In the present article, we investigate further the above mentioned aspects of EPS-SUSY QM.

We proceed according to the following structure. In section 2 we address the classical analog of the EPS (N=2)-SUSY QM. In section 3 we calculate the parameters that measure the breaking of EPS SUSY. We analyze non-perturbative mechanism for EPS SUSY breaking in the instanton picture. The section 4 deals with the independent group theoretical methods with nonlinear extensions of Lie algebras from the EPS SUSY QM. Unless otherwise stated we take the geometrized units. Also, an implicit summation on repeated indices are assumed throughout this paper.

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## 2. The integrals of motion

The sufficient details of EPS quantization are given below to make the rest of the paper understandable. The extended Lagrangian can be written as

$$\mathcal{L}_{ext}(p,\,q,\,\dot{p},\,\dot{q}) = -\dot{q}_i\,p_i - q_i\,\dot{p}_i + \mathcal{L}^q + \mathcal{L}^p, \tag{1}$$

where a dynamical system with N degrees of freedom described by the 2N independent coordinates $q = (q_1, \dots q_N)$ and momenta $p = (p_1, \dots p_N)$ which are not, in general, canonical pairs. A Lagrangian $\mathcal{L}^q(q,\,\dot{q})$ is given in $q-$representation and the corresponding $\mathcal{L}^p(p,\,\dot{p})$ in $p-$representation. A dot will indicate differentiation with respect to $t$. One may now defines an extended Hamiltonian

$$H_{ext}(p,\,q,\,\pi_p,\,\pi_q) = \pi_{q_i}\,\dot{q}_i + \pi_{p_i}\,\dot{p}_i - \mathcal{L}_{ext}(p,\,q,\,\dot{p},\,\dot{q}) = H(p+\pi_q,\,q) - H(p,\,q+\pi_p), \tag{2}$$

where $H(p,\,q) = p_i\dot{q}_i - \mathcal{L}^q = q_i\dot{p}_i - \mathcal{L}^p$ is the conventional Hamiltonian of the system. Here $p$ and $q$ will be considered as independent c-number operators on the integrable complex function $\chi(q,p)$. For $\pi_p$ and $\pi_q$, however, the differential operators and commutation brackets will be borrowed from the conventional quantum mechanics, $\pi_{q_i} = -i\dfrac{\partial}{\partial\,q_i}$, $\;[\pi_{q_i}, q_j] = -i\delta_{ij}$, and similar relations hold for the pair $\pi_{p_i}$ and $p_i$. Note also the following $[p_i, q_j] = [p_i, p_j] = [q_i, q_j] = \left[\pi_{p_i}, \pi_{q_j}\right] = \left[\pi_{p_i}, \pi_{p_j}\right] = \left[\pi_{q_i}, \pi_{q_j}\right] = 0$. The solutions of Schrödinger-like equation $i\,\dfrac{\partial}{\partial\,t}\chi = H_{ext}\,\chi$, are

$$\chi(q,p,t) = \overline{\chi}(q,p,t)e^{-ipq} = a_{\alpha\beta}\,\psi_\alpha(q,\,t)\,\phi^*_\beta(p,\,t)\,e^{-i\,pq}, \tag{3}$$

where $a = a^\dagger$ positive definite, $\psi_\alpha$ and $\phi^*_\alpha$ are solutions of the conventional Schrödinger equation in $q-$ and $p-$representations, respectively. The normalizable $\chi$ is a physically acceptable solution. The exponential factor is a consequence of the total time derivative, $-d(qp)/dt$, in Eq. (1) which can be eliminated due to $(p+\pi_q)\chi(q,p,t) = (\pi_q\overline{\chi}(q,p,t))\,e^{-ipq}$, and so on. This substitution gives $i\,\dfrac{\partial}{\partial\,t}\,\overline{\chi}(q,p,t) = \overline{H}_{ext}\,\overline{\chi}(q,p,t)$, provided by the reduced Hamiltonian, $\overline{H}_{ext}$ where we put $p+\pi_q \to \pi_q$ and $q+\pi_p \to \pi_p$. So, we may replace $H_{ext}$ by $\overline{H}_{ext}$, and $\chi(q,p,t)$ by $\overline{\chi}(q,p,t)$, respectively, but retain the former notational conventions. The interested reader is referred to Nasiri (2006) for further details.

Following Gamboa & Plyushchay (1998), let us consider a nonrelativistic particle of unit mass with two $(\alpha = 1, 2)$ odd (Grassmann) degrees of freedom. The classical extended Lagrangian Eq. (1) explicitly can be written

$$\mathcal{L}_{ext}(p,\,q,\,\dot{p},\,\dot{q}) = -\dot{q}\,p - q\,\dot{p} + \frac{1}{2}\,\dot{q}^2 - F(q) + \frac{1}{2}\,\dot{p}^2 - G(p) - R(q,\,p)\,N + \frac{1}{2}\,\psi_\alpha\dot{\psi}_\alpha, \tag{4}$$

provided by $N = \psi_1\,\psi_2 = -i\,\psi_+\,\psi_-$, where $\psi_\alpha$ are the two odd (Grassmann) degrees of freedom. Here $F(q) : \mathcal{R} \to \mathcal{R}$, $\;G(p) : \mathcal{R} \to \mathcal{R}$ and $R(q,\,p) : \mathcal{R} \to \mathcal{R}$ are arbitrary piecewise continuously differentiable functions defined over the 1-dimensional Euclidean space $\mathcal{R}$. The nontrivial Poisson-Dirac brackets of the system Eq. (4) are

$$\begin{aligned} &\{q,\,\pi_q\} = 1, \quad \{p,\,\pi_p\} = 1, \quad \{\psi_\alpha,\,\psi_\beta\} = \delta_{\alpha\beta}, \\ &\{\psi_+,\,\psi_-\} = 1, \quad \psi^2_\pm = 0, \quad \psi_\pm = \frac{1}{\sqrt{2}}\,(\psi_1 \pm i\psi_2)\,. \end{aligned} \tag{5}$$

The extended Lagrangian Eq. (4) yields the Hamiltonian $H_{ext}$ which, in turn, gives the following equations of motion:

$$\begin{aligned} &\dot{q} = \pi_q, \quad \dot{p} = \pi_p, \quad \dot{\pi}_q = -F'_q(q) - R'_q(q,\,p)\,N, \\ &\dot{\pi}_p = -G'_p(p) + R'_p(q,\,p)\,N, \quad \dot{\psi}_\pm = \pm iR(q,\,p)\,\psi_\pm, \end{aligned} \tag{6}$$

where $N$ is the integral of motion additional to $H_{ext}$. Along the trajectories $q(t)$ and $p(t)$ in $(p,q)-$ spaces, the solution to the equations of motion for odd variables is

$$\psi_\pm(t) = \psi_\pm(t_0)\,\exp\left[\pm i\int_{t_0}^t R(q(\tau),\,p(\tau))\,d\tau\right]. \tag{7}$$

Hence, the odd quantities

$$\theta_\pm = \psi_\pm(t)\,\exp\left[\mp i\int_{t_0}^t R(q(\tau),\,p(\tau))\,d\tau\right] \tag{8}$$


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are nonlocal in time integrals of motion. In the trivial case $R=0$, we have $\dot{\psi}_\pm=0$, and $\theta_\pm=\psi_\pm$. Suppose the system has even complex conjugate quantities $B_{q,p\pm}$, $(B_{q,p+})^*=B_{q,p-}$, whose evolution looks, up to the term proportional to $N$ like the evolution of odd variables in Eq. (6). Then local odd integrals of motion could be constructed in the form $Q_{X\pm}=B_{X\pm}\,\psi_{\pm,}$, where $X$ denotes concisely either $q-$ or $p-$ representations (no summation on $X$ is assumed). Let us introduce the oscillator-like bosonic variables $B_{X\pm}$ as

$$\begin{array}{ll} B_{q\pm}:\mathcal{L}^2(\mathcal{R})\to\mathcal{L}^2(\mathcal{R}), & B_{q\pm}=\left[p+\pi_q\pm iW(q)\right], \\ B_{p\pm}:\mathcal{L}^2(\mathcal{R})\to\mathcal{L}^2(\mathcal{R}), & B_{p\pm}=\left[q+\pi_p\pm iV(p)\right], \end{array} \tag{9}$$

where $W(q):\mathcal{R}\to\mathcal{R}$ and $V(p):\mathcal{R}\to\mathcal{R}$ are the piecewise continuously differentiable functions called SUSY potentials. In particular, if $R(q,\,p)=R_q(q)-R_p(p)$, we obtain

$$\begin{array}{l} \dot{Q}_{q\pm}=\pm i\left[\left(W_q'-R_{q\pm}'\right)-\sqrt{i}\left(F_q'-WW_q'\right)\,\psi_\mp\right], \\ \dot{Q}_{p\pm}=\pm i\left[\left(V_p'-R_{p\pm}'\right)-\sqrt{i}\left(G_p'-VV_p'\right)\,\psi_\mp\right]. \end{array} \tag{10}$$

This shows that either $\dot{Q}_{q\pm}=0$ or $\dot{Q}_{p\pm}=0$ when $W_q'(q)=R_{q\pm}'(q)$ and $F_q'=\frac{1}{2}(W^2)_q'$ or $V_p'(p)=R_{p\pm}'(p)$ and $G_p'=\frac{1}{2}(V^2)_p'$, respectively. Therefore, when the functions $R_X$, $F(q)$ and $G(p)$ are related as

$$R_{q\pm}'(q)=W_q'(q),\quad F_q=\tfrac{1}{2}W^2+C_q,\quad R_{p\pm}'(p)=V_p'(p),\quad G_p=\tfrac{1}{2}V^2+C_p, \tag{11}$$

where $C_X$ are constants, then odd quantities $Q_{X\pm}$ are integrals of motion in addition to $H_{ext}$ and $N$. Using the explicit form $H_{ext}=H_q-H_p$ :

$$H_q=\frac{1}{2}\,\pi_q^2+F^2(q)+R_q\,N,\quad H_p=\frac{1}{2}\,\pi_p^2+G^2(p)+R_p\,N, \tag{12}$$

it can be seen that $Q_{X\pm}$ and $N$ together with the $H_q$ and $H_p$ form the classical analog of the EPS SUSY algebra

$$\begin{array}{l} \{Q_{X+},\,Q_{X-}\}=2(H_X-C_X),\quad \{H_X,\,Q_{X\pm}\}= \\ \{Q_{X\pm},\,Q_{X\pm}\}=0,\quad \{N,\,Q_{X\pm}\}=\pm iQ_{X\pm},\quad \{N,\,H_X\}=0, \end{array} \tag{13}$$

with constants $C_X$ playing a role of a central charges in $(q,p)-$ spaces, $N$ is classical analog of the grading operator. Putting $C_q=C_p=0$, we arrive at classical analog of the EPS SUSY QM Ter-Kazarian (2009) given by the extended Lagrangian

$$\mathcal{L}_{ext}(p,\,q,\,\dot{p},\,\dot{q})=\frac{1}{2}\,\pi_q^2-\frac{1}{2}\,W^2(q)+\frac{1}{2}\,\pi_p^2-\frac{1}{2}\,V^2(p)+\psi_1\psi_2(W_q'+V_p')+\frac{1}{2}\psi_\alpha\dot{\psi}_\alpha. \tag{14}$$

We conclude that the classical system Eq. (4) is characterized by the presence of two additional local in time odd integrals of motion, $Q_{X\pm}$, being supersymmetry generators. Along the actual trajectories in $q-$space, the Eq. (14) reproduces the results obtained in Ter-Kazarian (2009). In the matrix formulation of EPS (N=2)-SUSY QM, the $\hat{\psi}_\pm$ will be two real fermionic creation and annihilation nilpotent operators describing the fermionic variables. The $\hat{\psi}_\pm$, having anticommuting c-number eigenvalues, imply

$$\hat{\psi}_\pm=\sqrt{\frac{1}{2}}\left(\hat{\psi}_1\pm i\hat{\psi}_2\right),\quad \left\{\hat{\psi}_\alpha,\,\hat{\psi}_\beta\right\}=\delta_{\alpha\beta},\quad \left\{\hat{\psi}_+,\,\hat{\psi}_-\right\}=1,\quad \hat{\psi}_\pm^2=0. \tag{15}$$

They can be represented by finite dimensional matrices $\hat{\psi}_\pm=\sigma^\pm$,

$$\hat{\psi}_+=\sigma^+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \hat{\psi}_-=\sigma^-=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \hat{f}=\frac{1}{2}\sigma_3, \tag{16}$$

where the $\sigma^\pm=\frac{\sigma_1\pm i\sigma_2}{2}$ denote the usual raising and lowering operators for the eigenvalues of $\sigma_3$. The fermionic operator Eq. (46) commutes with the $H_{ext}$ and is diagonal in this representation with conserved eigenvalues $\pm\frac{1}{2}$. Due to it the wave functions become two-component objects:

$$\chi(q,p)=\begin{pmatrix}\chi_{+1/2}(q,p)\\ \chi_{-1/2}(q,p)\end{pmatrix}=\begin{pmatrix}\chi_1(q,p)\\ \chi_2(q,p)\end{pmatrix}=\begin{pmatrix}\psi_1(q)\phi_1(p)\\ \psi_2(q)\phi_2(p)\end{pmatrix}, \tag{17}$$

where the states $\psi_{1,2}(q)$, $\phi_{1,2}(p)$ correspond to fermionic quantum number $f=\pm\dfrac{1}{2}$, respectively, in $q-$ and $p-$ spaces. They belong to Hilbert space $\mathcal{H}=\mathcal{H}_0\otimes\mathcal{C}^2=\left[\mathcal{L}^2(\mathcal{R})\otimes\mathcal{L}^2(\mathcal{R})\right]\otimes\mathcal{C}^2$. Hence the Hamiltonian $H_{ext}$ of EPS (N=2)-SUSY QM system becomes a $2\times2$ matrix Ter-Kazarian (2009).


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# 3. The EPS SUSY breaking

To calculate the parameters that measure the breaking of EPS SUSY, let us first find the approximate ground state solutions to the extended Schrödinger-like equation

$$H_{ext}\,\chi(q,p) = (H_q - H_p)\,\chi(q,p) = \varepsilon\,\chi(q,p), \tag{18}$$

with the supposedly small groundstate energy $\varepsilon$. As we mentioned above the solutions for non-zero $\varepsilon$ come in pairs of the form $\chi_\uparrow(q,p) = \begin{pmatrix} \chi_1(q,p) \\ 0 \end{pmatrix}$ or $\chi_\downarrow(q,p) = \begin{pmatrix} 0 \\ \chi_2(q,p) \end{pmatrix}$, related by supersymmetry, where $\chi_{1,2}(q,p) = \psi_{1,2}(q)\phi_{1,2}(p)$. The state space of the system is defined by all the normalizable solutions of Eq. (18) and the individual states are characterized by the energies $\varepsilon_q$ and $\varepsilon_q$ and the fermionic quantum number $f$. One of these solutions is acceptable only if $W(q)$ and $V(p)$ become infinite at both $q \to \pm\infty$ and $p \to \pm\infty$, respectively, with the same sign. If this condition is not satisfied, neither of the solutions is normalizable, and they cannot represent the groundstate of the system. The Eq. (18) yields the following relations between energy eigenstates with fermionic quantum number $\pm\frac{1}{2}$:

$$\begin{array}{l}\left(-\dfrac{\partial}{\partial q} + W(q)\right)\sqrt{\varepsilon_q}\,\psi_2(q)\phi_1(p) - \left(-\dfrac{\partial}{\partial p} + V(p)\right)\sqrt{\varepsilon_p}\,\psi_1(q)\phi_2(p) = \\ \sqrt{2}\,\varepsilon\psi_1(q)\phi_1(p),\end{array} \tag{19}$$

and

$$\begin{array}{l}\left(\dfrac{\partial}{\partial q} + W(q)\right)\sqrt{\varepsilon_q}\,\psi_1(q)\phi_2(p) - \left(\dfrac{\partial}{\partial p} + V(p)\right)\sqrt{\varepsilon_p}\,\psi_2(q)\phi_1(p) = \\ \sqrt{2}\,\varepsilon\psi_2(q)\phi_2(p),\end{array} \tag{20}$$

where $\varepsilon = \varepsilon_q - \varepsilon_p, \quad \varepsilon_q$ and $\varepsilon_p$ are the eigenvalues of $H_q$ and $H_p$, respectively. The technique now is to devise an iterative approximation scheme, as developed in Salomonson & Van Holten (1982), to solve Eq. (19) and Eq. (20) by taking a trial wave function for $\chi_2(q,p)$, substitute this into the first equation (19) and integrate it to obtain an approximation for $\chi_1(q,p)$. This can be used as an ansatz in the second equation Eq. (20) to find an improved solution for $\chi_2(q,p)$, etc. As it was shown in Salomonson & Van Holten (1982), the procedure converges for well-behaved potentials with a judicious choice of initial trial function. If the $W_q$ and $V_p$ are odd, then $\psi_1(-q) = \psi_2(q), \quad \phi_1(-p) = \phi_2(p)$, since they satisfy the same eigenvalue equation. The independent nature of $q$ and $p$ gives the freedom of taking $q = 0, \quad p = 0$ which yield an expression for energies: $\sqrt{2\varepsilon_q} = W(0) + \psi_1'(0)\,/\psi_1(0)\,, \quad \sqrt{2\varepsilon_p} = V(0) + \phi_1'(0)\,/\phi_1(0)$ . Suppose the potentials $W_q(q)$ and $V_p(p)$ have a maximum, at $q_-$ and $p_-$, and minimum, at $q_+$ and $p_+$, respectively. For the simplicity sake we choose the trial wave function as $\chi^{(0)}_{1,2}(q) = \delta(q - q_\pm)\delta(p - p_\pm)$. After one iteration, we obtain

$$\begin{array}{l}\chi_1^{(1)}(q,p) = \dfrac{1}{N}\theta(q - q_-)\,\theta(p - p_-)\,e^{-\int_0^q dq'\,W(q') - \int_0^p dp'\,V(p')}, \\ \chi_2^{(1)}(q,p) = \dfrac{1}{N}\theta(q_+ - q)\,\theta(p_+ - p)\,e^{\int_0^q dq'\,W(q') + \int_0^p dp'\,V(p')},\end{array} \tag{21}$$

where $N$ is the normalization factor. The next approximation leads to

$$\begin{array}{l}\chi_1^{(2)}(q,p) = \dfrac{1}{N'}\,e^{-\int_0^q dq'\,W(q') + \int_0^p dp'\,V(p'))}\displaystyle\int_{max(-q,q_-)}^{\infty} dq' e^{-2\int_0^{q'} dq''\,W(q'')}\times \\ \displaystyle\int_{max(-p,p_-)}^{\infty} dp' e^{-2\int_0^{p'} dp''\,V(p'')}, \\ \chi_2^{(2)}(q,p) = \dfrac{1}{N'}\,e^{\int_0^q dq'\,W(q') - \int_0^p dp'\,V(p'))}\displaystyle\int_{\infty}^{min(-q,q_+)} dq' e^{2\int_0^{q'} dq''\,W(q'')}\times \\ \displaystyle\int_{\infty}^{min(-p,p_+)} dp' e^{2\int_0^{p'} dp''\,V(p'')}.\end{array} \tag{22}$$

It can be verified that to this level of precision Eq. (21) is self-consistent solution. Actually, for example, for $q' > q_-$ the exponential $e^{-2\int_0^{q'} W(q'')dq''}$ will peak sharply around $q_+$ and may be approximated by a $\delta-$function $c\delta(q_+ - q')$; similarly $e^{2\int_0^{q'} W(q'')dq''}$ we may replace approximately by $c\delta(q_- - q')$ for $q' < q_+$. The similar arguments hold for the p-space. With these approximations equations (22) reduce to Eq. (21). The normalization constant $N'$ is

$$N' = \left(\int_{q_-}^{\infty} dq\, e^{-2\int_0^q W(q')dq'} \int_{p_-}^{\infty} dp\, e^{-2\int_0^{p'} V(p')dp'}\right)^{3/2}. \tag{23}$$


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The energy expectation value $\varepsilon = (\chi_1, H_{ext}\,\chi_1)$ gives the same result as that obtained for odd potentials by means of the $\sqrt{2\varepsilon_q}$, $\sqrt{2\varepsilon_p}$, and Eq. (22). Assuming the exponentials $e^{-2\int_{q_-}^{0} W(q)dq}$ and $e^{-2\int_{p_-}^{0} V(p)dp}$ to be small, which is correct to the same approximations underlying the $\varepsilon$, the difference is negligible and the integrals in both cases may be replaced by gaussians around $q_+$ and $p_+$, respectively. Hence, it is straightforward to obtain

$$\varepsilon = \frac{\hbar W_q'(q_+)}{2\pi}\, e^{-2\Delta W/\hbar} - \frac{\hbar V_p'(p_+)}{2\pi}\, e^{-2\Delta V/\hbar}, \tag{24}$$

which gives direct evidence for the SUSY breaking in the EPS QM system. Here we have reinstated $\hbar$ to show the order of adopted approximation and its non-perturbative nature, and denoted $\Delta W = \int_{q+}^{q-} W(q)\,dq$, $\quad \Delta V = \int_{p+}^{p-} V(p)\,dp$. However, a more practical measure for the SUSY breaking, in particular, in field theories is the expectation value of an auxiliary field, which can be replaced by its equation of motion right from the start: $< F >= (\chi_\uparrow,\, i\{Q_+,\, \sigma_-\}\chi_\uparrow)$ . Taking into account the relation $Q_+\chi_\uparrow = 0$, with $Q_+$ commuting with $H_{ext}$, which means that the intermediate state must have the same energy as $\chi$, the expectation value $< F >$ can be written in terms of a complete set of states as $< F >= i\,(\chi_\uparrow,\, Q_+\,\chi_\downarrow)\,(\chi_\downarrow,\, \sigma_-,\, \chi_\uparrow)$ . Note that

$$\begin{aligned}
&\varepsilon =< H_{ext} >= (\chi_\uparrow,\, H_{ext}\,\chi_\uparrow) = \frac{1}{2}\,(\chi_\uparrow,\, Q_+\,\chi_\downarrow)\,(\chi_\downarrow,\, Q_-\,\chi_\uparrow) = \\
&\varepsilon_q - \varepsilon_p =< H_q > - < H_p >= \quad (\psi_\uparrow,\, H_q\,\psi_\uparrow) - (\phi_\uparrow,\, H_p\,\phi_\uparrow) = \\
&\frac{1}{2}\,(\psi_\uparrow,\, Q_{q+}\,\psi_\downarrow)\,(\psi_\downarrow,\, Q_{q-}\,\psi_\uparrow) - \frac{1}{2}\,(\phi_\uparrow,\, Q_{p+}\,\phi_\downarrow)\,(\phi_\downarrow,\, Q_{p-}\,\phi_\uparrow)\,,
\end{aligned} \tag{25}$$

where $\chi_{\uparrow\downarrow} = \psi_{\uparrow\downarrow}\phi_{\uparrow\downarrow}$, and $\psi_\uparrow = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}$, $\quad \psi_\downarrow = \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}$, $\quad \phi_\uparrow = \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}$, $\quad \phi_\downarrow = \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix}$. By virtue of SUSY algebra, we have $\frac{1}{\sqrt{2\varepsilon}}\,Q_-\,\chi_\uparrow = \chi_\downarrow = \hat{\psi}_-\,\chi_\uparrow$, $\quad \frac{1}{\sqrt{2\varepsilon_q}}\,Q_{q-}\,\psi_\uparrow = \psi_\downarrow = \hat{\psi}_-\,\psi_\uparrow$, $\quad \frac{1}{\sqrt{2\varepsilon_p}}\,Q_{p-}\,\phi_\uparrow = \phi_\downarrow = \hat{\psi}_-\,\phi_\uparrow$, and the Eq. (25) reads

$$\begin{aligned}
&\sqrt{\varepsilon}\;(\chi_\uparrow,\, Q_+\,\chi_\downarrow)\left(\chi_\downarrow,\, \hat{\psi}_-\,\chi_\uparrow\right) = \sqrt{\varepsilon_q}\;(\psi_\uparrow,\, Q_{q+}\,\psi_\downarrow)\left(\psi_\downarrow,\, \hat{\psi}_-\,\psi_\uparrow\right) - \\
&\sqrt{\varepsilon_p}\;(\phi_\uparrow,\, Q_{p+}\,\phi_\downarrow)\left(\phi_\downarrow,\, \hat{\psi}_-\,\phi_\uparrow\right),
\end{aligned} \tag{26}$$

Using the matrix representations of $Q_{q+}, Q_{p+}$ and $\hat{\psi}_-$ and the wave functions Eq. (22), we obtain

$$(\chi_\uparrow,\, Q_+\,\chi_\downarrow)\left(\chi_\downarrow,\, \hat{\psi}_-\,\chi_\uparrow\right) = i\sqrt{\frac{\varepsilon_q}{\varepsilon}}\,\frac{W_q'(q)}{\pi}\,e^{-2\Delta W}\,\Delta q - i\sqrt{\frac{\varepsilon_p}{\varepsilon}}\,\frac{V_p'(p)}{\pi}\,e^{-2\Delta V}\,\Delta p, \tag{27}$$

where $\Delta q = q_+ - q_-$ and $\Delta p = p_+ - p_-$. Hence

$$< F >= -\sqrt{\frac{\varepsilon_q}{\varepsilon}}\,\frac{W_q'(q)}{\pi}\,e^{-2\Delta W}\,\Delta q + \sqrt{\frac{\varepsilon_p}{\varepsilon}}\,\frac{V_p'(p)}{\pi}\,e^{-2\Delta V}\,\Delta p. \tag{28}$$

It is remarkable that the expressions Eq. (27) and Eq. (28) can be obtained in the path integral formulation of the theory by calculating the matrix elements of tunneling between two classical vacua in an one-instanton background Polyakov (1977). That is, the matrix elements of $\hat{\psi}_\pm$, $Q_{q\pm}$ and $Q_{p\pm}$ can be calculated in the background of the classical solutions $\dot{q}_c = -W_c$ and $\dot{p}_c = -V_c$. In doing this we re-write the matrix element Eq. (26) in terms of eigenstates of the conjugate operator $\hat{\psi}_\pm$:

$$\begin{aligned}
&\sqrt{\varepsilon}\;< +0q_+p_+|Q_+|q_-p_-0- >< -0q_-p_-|\hat{\psi}_-|q_+p_+0+ >= \\
&\sqrt{\varepsilon_q}\;< +0q_+|Q_{q+}|q_-0- >< -0q_-|\hat{\psi}_-|q_+0+ > - \\
&\sqrt{\varepsilon_p}\;< +0p_+|Q_{p+}|p_-0- >< -0p_-|\hat{\psi}_-|p_+0+ > .
\end{aligned} \tag{29}$$

This, in turn, can be presented by path integrals defined in terms of anticommuting c-number operators $\zeta$ and $\eta$ where $\psi = \sqrt{\frac{1}{2}}\begin{pmatrix} \eta + \zeta \\ i(\eta - \zeta) \end{pmatrix}$, with Euclidean actions of the instantons $S_X^E$ in $q-$ and $p-$ spaces, respectively. Given $S_X^E$ one may calculate the matrix elements of $\hat{\psi}_\pm$ and $Q_{X\pm}$. The corresponding functional integrals include an integration over *instanton time* $\tau_0$ which is due to the problem of zero modes of the bilinear terms in Euclidean actions Salomonson & Van Holten (1982). The zero mode problem is solved by introducing a collective coordinate $\tau_0$ replacing the bosonic zero mode Balantekin (1998). In the case when


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the SUSY potentials in $q-$ and $p-$ spaces have more than two extrema $q_\nu$ and $p_\mu$, $\nu, \mu = 1, 2, \ldots, N$, one can put asymptotic conditions on the SUSY potentials Sobouti & Nasiri (1993) as

$$\begin{aligned} &\int_0^\infty W(q')dq' \to \infty \quad \text{at} \quad q \to \pm\infty \quad \text{for} \quad \psi_0^+, \\ &\int_0^\infty W(q')dq' \to -\infty \quad \text{at} \quad q \to \pm\infty \quad \text{for} \quad \psi_0^-, \end{aligned} \tag{30}$$

and similar ones for $V(p)$, that the extrema are well separated: $\int_{q_\nu}^{q_\nu+1} W(q')dq' \gg 1$, and $\int_{p_\mu}^{q_\mu+1} V(p')dp' \gg 1$. Around each of the classical minima $q_\nu$ and $p_\mu$ of the potentials $W^2(q)$ and $V^2(p)$, respectively, one can approximate the theory by a suppersymmetric harmonic oscillator. Then there are $N$ ground states which have zero energy. These states are described by upper or lower component of the wave function, depending on whether $\nu$ and $\mu$ are odd or even. With this provision the functional integrals are calculated in Salomonson & Van Holten (1982), hence

$$\begin{aligned} &< +0q_+|Q_{q+}|q_-0- >= i\sqrt{\frac{W_c'(q_+)}{\pi}}\, e^{-\Delta W_c}, \\ &< -0q_-|\hat{\psi}_-|q_+0+ >= \sqrt{\frac{W_c'(q_+)}{\pi}}\, e^{-\Delta W_c}\Delta q_c, \end{aligned} \tag{31}$$

etc. Inserting this in Eq. (29), we arrive at the Eq. (27)

$$\begin{aligned} &< +0q_+p_+|Q_+|q_-p_-0- >< -0q_-p_-|\hat{\psi}_-|q_+p_+0+ >= \\ &i\sqrt{\frac{\varepsilon_q}{\varepsilon}}\,\frac{W_c'(q)}{\pi}\, e^{-2\Delta W_c}\Delta q_c - i\sqrt{\frac{\varepsilon_p}{\varepsilon}}\,\frac{V_c'(p)}{\pi}\, e^{-2\Delta V_c}\Delta p_c, \end{aligned} \tag{32}$$

and, thus, of Eq. (28). This proves that the extended SUSY breaking has resulted from tunneling between the classical vacua of the theory. The corrections to this picture are due to higher order terms and quantum tunneling effects.

## 4. Spectrum generating algebra

An extended Hamiltonian $H_{ext}$ can be treated as a set of two ordinary two-dimensional partner Hamiltonians Sobouti & Nasiri (1993)

$$H_\pm = \frac{1}{2}\left[\pi_q^2 - \pi_p^2 + U_\pm(q,p)\right], \tag{33}$$

provided by partner potentials

$$\begin{aligned} &U_\pm(q,p) = U_{q\pm}(q) - U_{p\pm}(p), \quad U_{q\pm}(q) = W^2(q,\, a_q) \mp W_q'(q,\, a_q), \\ &U_{p\pm}(p) = V^2(p,\, a_p) \mp V_p'(p,\, a_p). \end{aligned} \tag{34}$$

A subset of the SUSY potentials for which the Schrödinger-like equations are exactly solvable share an integrability conditions of *shape-invariance* Balantekin (1998), Gangopadhyaya (1998), Gendenshtein (1983):

$$U_+(a_0,q,p) = U_-(a_1,q,p) + R(a_0), \quad a_1 = f(a_0), \tag{35}$$

where $a_0$ and $a_1$ are a set of parameters that specify phase-space-independent properties of the potentials, and the reminder $R(a_0)$ is independent of $(q,p)$.

Using the standard technique, we may construct a series of Hamiltonians $H_N, \quad N = 0, 1, 2, \ldots,$

$$H_N = \frac{1}{2}\left[\pi_q^2 - \pi_p^2 + U_-(a_N,q,p) + \sum_{k=1}^{N} R(a_k)\right], \tag{36}$$

where $a_N = f^{(N)}(a)$ ($N$ is the number of iterations). From Eqs. (34) and (35) we obtain $N = n + m$ coupled nonlinear differential equations which are the two recurrence relations of Riccati-type differential equations:

$$\begin{aligned} &W_{n+1}^2(q) + W_{q(n+1)}'(q) = W_n^2(q) - W_{qn}'(q) - \mu_n, \\ &V_{m+1}^2(p) + V_{p(m+1)}'(p) = V_m^2(p) - V_{pm}'(p) - \nu_m, \end{aligned} \tag{37}$$


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where we denote $W_n(q) \equiv W(q,\, a_{qn}),\quad V_m(p) \equiv V(p,\, a_{pm}),\ \mu_n \equiv R_q(a_{qn})$ and $\nu_m \equiv R_p(a_{pm})$. Here we admit that for unbroken SUSY, the eigenstates of the potentials $U_{q,p}$, respectively, are

$$E^{(-)}_{q0} = 0, \quad E^{(-)}_{qn} = \sum_{i=0}^{n-1} \mu_i, \quad E^{(-)}_{p0} = 0, \quad E^{(-)}_{pm} = \sum_{j=0}^{m-1} \nu_j, \tag{38}$$

that is, the ground states are at zero energies, characteristic of unbroken supersymmetry. The differential equations (37) can be investigated to find exactly solvable potentials.

The shape invariance condition of Eq. (35) can be expressed in terms of bosonic operators as

$$\begin{array}{l} B_+(x,\, a_0)B_-(x,\, a_0) - B_-(x,\, a_1)B_+(x,\, a_1) = R(a_0) = \mu_0 - \nu_0, \\ B_{X+}(X,\, a_{X0})B_{X-}(X,\, a_{X0}) - B_{X-}(X,\, a_{X1})B_{X+}(X,\, a_{X1}) = (\mu_0 \,\text{or}\, \nu_0), \end{array} \tag{39}$$

where $x(q,p)$-is the coordinate in $(q,p)-$ space. To classify algebras associated with the shape invariance, following Gangopadhyaya (1998) we introduce an auxiliary variables $\phi(\phi_q,\, \phi_p)$ and define the following creation and annihilation operators:

$$J_+ = e^{ik\phi} B_+(x,\, \chi(i\partial_\phi)), \quad J_- = B_-(x,\, \chi(i\partial_\phi))e^{-ik\phi}, \tag{40}$$

where $k(k_q,\, k_p)$ are an arbitrary real constants and $\chi(\chi_q,\, \chi_p)$ are an arbitrary real functions. Consequently, the creation and annihilation operators in $(q,p)-$spaces can be written as

$$J_{X+} = e^{ik_X\phi_X} B_{X+}(X,\, \chi_X(i\partial_{\phi_X})), \quad J_{X-} = B_{X-}(X,\, \chi_X(i\partial_{\phi_X}))e^{-ik_X\phi_X}. \tag{41}$$

The operators $B_\pm(x,\, \chi(i\partial_\phi))$ and $B_{X\pm}(X,\, \chi_X(i\partial_{\phi_X}))$ are the generalization of Eqs. (39), where $a_0 \to \chi(i\partial_\phi))$ and $a_{X0} \to \chi_X(i\partial_{\phi_X}))$. If we choose the function $\chi(i\partial_\phi)$ such that $\chi(i\partial_\phi + k) = f[\chi(i\partial_\phi)]$, then we have identified $a_0 \to \chi(i\partial_\phi))$, and $a_1 = f(a_0) \to f[\chi(i\partial_\phi))] = \chi(i\partial_\phi + k)$. From Eq. (39) we obtain

$$\begin{array}{l} B_+(x,\, \chi(i\partial_\phi))B_-(x,\, \chi(i\partial_\phi)) - B_-(x,\, \chi(i\partial_\phi + k))B_+(x,\, \chi(i\partial_\phi + k)) = R[\chi(i\partial_\phi))], \\ B_{X+}(X,\, \chi_X(i\partial_{\phi_X}))B_{X-}(X,\, \chi(i\partial_{\phi_X}))- \\ B_{X-}(X,\, \chi_X(i\partial_{\phi_X} + k_X))B_{X+}(X,\, \chi_X(i\partial_{\phi_X} + k_X)) = R_X[\chi_X(i\partial_{\phi_X}))]. \end{array} \tag{42}$$

Introducing the operators $J_3 = -\dfrac{i}{k}\partial_\phi$ and $J_{X3} = -\dfrac{i}{k_X}\partial_{\phi_X}$, and combining Eqs. (41) and (42), we arrive at a deformed Lie algebras as follows:

$$\begin{array}{l} [J_+,\, J_-] = [J_{q+},\, J_{q-}] - [J_{p+},\, J_{p-}] = \xi(J_3) = \xi_q(J_{q3}) - \xi_p(J_{p3}), \\ [J_3,\, J_\pm] = \pm J_\pm, \quad [J_{X+},\, J_{X-}] = \xi_X(J_{X3}), \quad [J_{X3},\, J_{X\pm}] = \pm J_{X\pm}, \end{array} \tag{43}$$

where $\xi(J_3) \equiv R[\chi(i\partial_\phi))]$ and $\xi_X(J_{X3}) \equiv R_X[\chi_X(i\partial_{\phi_X}))]$ define the deformations. Different $\chi$ functions in Eq. (42) define different reparametrizations corresponding to several models. For example:
1. The translational models $(a_1 = a_0 + k)$ correspond to $\chi(z) = z$. If $R$ is a linear function of $J_3$ the algebra becomes SO(2.1) or SO(3). Similar in many respects prediction is made in somewhat different method by Balantekin Balantekin (1998).
2. The scaling models $(a_1 = e^k a_0)$ correspond to $\chi(z) = e^z$, etc.

To find the energy spectrum of the partner SUSY Hamiltonians

$$\begin{array}{l} 2H_-(x,\, \chi(i\partial_\phi)) = B_-(x,\, \chi(i\partial_\phi + k))B_+(x,\, \chi(i\partial_\phi + k)), \\ 2H_{X-}(X,\, \chi_X(i\partial_{\phi_X})) = B_{X-}(X,\, \chi_X(i\partial_{\phi_X} + k_X))B_{X+}(X,\, \chi_X(i\partial_{\phi_X} + k_X)), \end{array} \tag{44}$$

one must construct the unitary representations of the deformed Lie algebra defined by Eq. (43) Gangopadhyaya (1998), E. (rf:). Using the standard technique, one defines up to additive constants the functions $g(J_3)$ and $g_X(J_{X3})$:

$$\xi(J_3) = g(J_3) - g(J_3 - 1), \quad \xi_X(J_{X3}) = g_X(J_{X3}) - g_X(J_{X3} - 1). \tag{45}$$

The Casimirs of this algebra can be written as $C_2 = J_-J_+ + g(J_3)$ and accordingly $C_{X2} = J_{X-}J_{X+} + g_X(J_{X3})$. In a basis in which $J_3$ and $C_2$ are diagonal, $J_-$ and $J_+$ are lowering and raising operators (the same holds for $X$-representations). Operating on an arbitrary state $|h>$ they yield

$$J_3|h> = h|h>, \quad J_-|h> = a(h)|h-1>, \quad J_+|h> = a^*(h+1)|h+1>, \tag{46}$$


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where

$$|a(h)|^2 - |a(h+1)|^2 = g(h) - g(h-1). \tag{47}$$

Similar arguments can be used for the operators $J_{(p,q)3}$, $C_{X2}$ and $J_{X\pm}$, which yield similar relations for the states $h_X$. The profile of $g(h)$ (and, thus, of $g_X(h_X)$) determines the dimension of the unitary representation. Having the representation of the algebra associated with a characteristic model, consequently we obtain the complete spectrum of the system. Let us obtain, for example, analytic expressions for the entire energy spectrum of extended Hamiltonian with *Self-Similar* potential without ever referring to underlying differential equation. A scaling change of parameters is given as $a_1 = Qa_0, \quad a_{X1} = Q_X a_{X0}$, at the simple choice $R(a_0) = -r_1 a_0$, where $r_1$ is a constant. That is,

$$\xi(J_3) \equiv -r_1 \exp(-kJ_3) = \xi_q(J_{q3}) - \xi_p(J_{p3}) = -r_{q1} \exp(-k_q J_{q3}) + r_{p1} \exp(-k_p J_{p3}). \tag{48}$$

This yields

$$[J_+,\, J_-] = \xi(J_3) = -r_1 \exp(-kJ_3), \quad [J_3,\, J_\pm] = \pm J_\pm, \tag{49}$$

which is a deformation of the standard $SO(2.1)$ Lie algebra. Therefore, one gets

$$\begin{aligned} g(h) &= \frac{r_1}{e^k - 1} e^{-kh} = -\frac{r_1}{1-Q} Q^{-h}, \quad Q = e^k, \\ g_X(h_X) &= \frac{r_{X1}}{e^{k_X} - 1} e^{-k_X h_X} = -\frac{r_{X1}}{1-Q_X} Q_X^{-h_X}, \quad Q_X = e^{k_X}. \end{aligned} \tag{50}$$

For scaling problems Gangopadhyaya (1998) one has $0 < q < 1$, which leads to $k < 0$. The unitary representation of this algebra for monotonically decreasing profile of the function $g(h)$, are infinite dimensional. Let the lowest weight state of the $J_3$ be $h_{min}$, then $a(h_{min}) = 0$. One can choose the coefficients $a(h)$ to be real. From Eq. (47), for an arbitrary $h = h_{min} + n, n = 0, 1, 2, \ldots$, we obtain

$$a(h)^2 = g(h-n-1) - g(h-1) = r_1 \frac{Q^n - 1}{Q-1} Q^{1-h}. \tag{51}$$

The spectrum of the extended Hamiltonian $H_-(x, a_1)$ reads

$$H_-|h> = a(h)^2|h> = r_1 \frac{Q^n - 1}{Q-1} Q^{1-h}|h>, \tag{52}$$

with the eigenenergies

$$E_n(h) = r_1 \alpha(h) \frac{Q^n - 1}{Q-1}, \quad \alpha(h) \equiv Q^{1-h}. \tag{53}$$

Similar expressions can be obtained for the $H_{X-}$ and eigenenergies $E_{qn_1}$ and $E_{pn_2}$ $(n \equiv (n_1, n_2), \quad n_{1,2} = 0, 1, 2, \ldots)$, as

$$\begin{aligned} H_{q-}|h_q> &= a_q(h_q)^2|h_q> = r_{q1} \frac{Q_q^{n_1} - 1}{Q_q - 1} Q_q^{1-h_q}|h_q>, \\ H_{p-}|h_p> &= a_p(h_p)^2|h_p> = r_{p1} \frac{Q_p^{n_2} - 1}{Q_p - 1} Q_p^{1-h_p}|h_p>, \end{aligned} \tag{54}$$

and

$$\begin{aligned} E_{qn_1}(h_q) &= r_{q1} \alpha_q(h_q) \frac{Q_q^{n_1} - 1}{Q_q - 1}, \quad \alpha_q(h_q) \equiv Q_q^{1-h_q}, \\ E_{pn_2}(h_p) &= r_{p1} \alpha_p(h_p) \frac{Q_p^{n_2} - 1}{Q_p - 1}, \quad \alpha_p(h_p) \equiv Q_p^{1-h_p}. \end{aligned} \tag{55}$$

Hence

$$E_n(h) = E_{qn_1}(h_q) - E_{pn_2}(h_p). \tag{56}$$

## 5. Concluding remarks

A considered classical system is characterized by the presence of two additional local in time odd integrals of motion being (N=2)-supersymmetry generators. We calculate the parameters which measure the breaking of EPS SUSY and analyze non-perturbative mechanism for EPS SUSY breaking in the instanton picture. We show that this has resulted from tunneling between the classical vacua of the theory. Using the factorization procedure we explore the algebraic property of shape invariance and SGA.


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