ON COMPUTATIONS OF RELIABILITIES INTERDEPENDENCIES FOR OPTIMAL HYPOTHESIS TESTING OF DISTRIBUTIONS FOR STOCHASTICALLY DEPENDENT OBJECTS

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It is considered problem of hypotheses optiaml testing for a model consisting of two stochastically dependent objects. It is supposed that L_1 probability distributions are known for the first object and the second object dependent on the first can be distributed according to one of $L_1 \times L_2$ given conditional distributions. In particular case optimal interdependencies (reliability-reliability functions) of pairs of the error probability exponents (reliabilities) are calculated and graphically presented.

Keywords: Hypothesis testing; Reliabilities.

1. Problem Statement and Preliminary Results.

Let X_1 and X_2 be random variables taking values in the finite set X. Let $P(X)$ be the space of all possible probability distributions (PD) on X. The object characterized by X_1 can have one of given L_1 PDs and X_2 can have one of $L_1 \times L_2$ conditional PDs $G_{l_2/l_1} = \{G_{l_2/l_1}(x^2/x^1), x^1, x^2 \in X\}$ $l_1 = \overline{1, L_1}, l_2 = \overline{1, L_2}$.

Let $(x_1, x_2) = ((x_1^1, x_1^2), (x_2^1, x_2^2), \ldots (x_N^1, x_N^2))$ be a sequence of results of N independent observations of the vector (X_1, X_2). The test, which we denote by Φ^N, is a procedure of making decision on the base of these N observations of both objects. For this model the vector (X_1, X_2) can have one of $L_1 \times L_2$ probability distributions $G_{l_1,l_2}=\{G_{l_1,l_2}(x^1,x^2),x^1,x^2\in X\}$, $l_1=\overline{1,L_1},l_2=\overline{1,L_2}$, where $G_{l_1,l_2}(x^1,x^2)=G_{l_1}(x^1)G_{l_2/l_1}(x^2/x^1)$.

The notations and results for the first and the second objects can be found in the corresponding articles [3]-[7]. Here we use only the notations and formulation of theorem for two stochastically dependent objects [6], which will be useful for interpretation of graphical presentations. We study the probability $\alpha_{l_1,l_2|m_1,m_2}(\Phi^N)$ of the erroneous acceptance by the sequence of tests Φ of the pair of PDs $(G_{l_1},G_{l_2/l_1})$ provided that the pair $(G_{m_1},G_{m_2/m_1})$ is true, where $(m_1,m_2)\neq(l_1,l_2)$, $m_1, l_1=\overline{1,L_1}$, $m_2, l_2=\overline{1,L_2}$. The probability to reject a true pair of PDs $(G_{m_1},G_{m_2/m_1})$, is defined as follows

$$\alpha^N_{m_1,m_2|m_1,m_2}(\Phi^N)=\sum_{(l_1,l_2)\neq(m_1,m_2)}\alpha^N_{l_1,l_2|m_1,m_2}(\Phi^N),\ m_i,l_i=1,2,\ i=1,2.$$

The reliabilities of the sequence of tests Φ are the following

$$E_{l_1,l_2|m_1,m_2}(\Phi)\overset{\Delta}{=}\overline{\lim_{N\to\infty}}-\frac{1}{N}\log\alpha^N_{l_1,l_2|m_1,m_2}(\Phi^N),\ m_i,l_i=1,2,\ i=1,2.$$

From last definitions it follows that

$$E_{m_1,m_2|m_1,m_2}(\Phi)=\min_{(l_1,l_2)\neq(m_1,m_2)}E_{l_1,l_2|m_1,m_2}(\Phi),m_i,l_i=1,2,\ i=1,2.$$

The matrix $E(\Phi)=\{E_{l_1,l_2|m_1,m_2}(\Phi),m_i,l_i=1,2,\ i=1,2\}$ is called the reliability matrix of the sequence of tests Φ. $F(\Phi)$ is corresponding lower estimate matrix of $E(\Phi)$.

In general case let us define the following subsets of $P(X)$ for given strictly positive elements $E_{L_1,l_2|l_1,l_2}$, $F_{l_1,L_2|l_1,l_2}$, $l_1=\overline{1,L_1-1},\ l_2=\overline{1,L_2-1}$:

$$R_{l_1} \overset{\Delta}{=} \{Q : D(Q \| G_{l_1}) \leq E_{L_1,l_2|l_1,l_2}\},\ l_1 = \overline{1, L_1 - 1},\ l_2 = \overline{1, L_2 - 1},$$

$$R_{l_2/l_1}(Q) \overset{\Delta}{=} \{V : D(V \| G_{l_2/l_1} | Q) \leq F_{l_1,L_2|l_1,l_2}\},\ l_1 = \overline{1, L_1 - 1},$$

$l_2 = \overline{1, L_2 - 1}$,

$$R_{L_1} \overset{\Delta}{=} \{Q : D(Q \| G_{l_1}) > E_{L_1,l_2|l_1,l_2},\ l_1 = \overline{1, L_1 - 1},\ l_2 = \overline{1, L_2 - 1}\},$$

$$R_{L_2/l_1}(Q) \overset{\Delta}{=} \{V : D(V \| G_{l_2/l_1} | Q) > F_{l_1,L_2|l_1,l_2},\ l_1 = \overline{1, L_1 - 1},$$

$l_2 = \overline{1, L_2 - 1}\}$.

Assume also

$$F^*_{l_1,L_2|l_1,l_2} \overset{\Delta}{=} F_{l_1,L_2|l_1,l_2},\ E^*_{L_1,l_2|l_1,l_2} \overset{\Delta}{=} E_{L_1,l_2|l_1,l_2},\ l_1 = \overline{1, L_1 - 1},$$

$l_2 = \overline{1, L_2 - 1}$, (1.$a$)

$$E^*_{l_1,l_2|m_1,l_2} \overset{\Delta}{=} \inf_{Q:Q \in R_{l_1}} D(Q \| G_{m_1}),\ m_1 \neq l_1 \quad (1.b)$$

$$F^*_{l_1,l_2|l_1,m_2} \overset{\Delta}{=} \inf_{Q \in R_{l_1}} \inf_{V:V \in R_{l_2/l_1}(Q)} D(V \| G_{m_2/m_1} | Q),\ m_2 \neq l_2 \quad (1.c)$$

$$F^*_{l_1,l_2|m_1,m_2} \overset{\Delta}{=} F^*_{m_1,l_2|m_1,m_2} + E^*_{l_1,m_2|m_1,m_2},\ m_i \neq l_i,\ i = 1,2, \quad (1.d)$$

$$F^*_{m_1,m_2|m_1,m_2} \overset{\Delta}{=} \min_{(l_1,l_2) \neq (m_1,m_2)} F^*_{l_1,l_2|m_1,m_2}. \qquad (1.e)$$

Theorem [8]: *If all distributions* G_{m_1}, $m_1 = \overline{1, L_1}$, *are different, that is* $D(G_{l_1} \| G_{m_1}) > 0$, $l_1 \neq m_1$, $l_1, m_1 = \overline{1, L_1}$, *and all conditional distributions* G_{l_2/l_1}, $l_2 = \overline{1, L_2}$, *are also different for all* $l_1 = \overline{1, L_1}$, *in the sense that* $D(G_{l_2/l_1} \| G_{m_2/l_1} | Q) > 0$, $l_2 \neq m_2$, *then the following statements are valid.*

When given elements $E_{L_1,l_2|l_1,l_2}$ and $F_{l_1,L_2|l_1,l_2}$, $l_1 = \overline{1, L_1 - 1}$,

$l_2 = \overline{1, L_2 - 1}$, meet the following conditions

$$0 < E_{L_1, l_2 | 1, l_2} < \min_{l_1 = \overline{2, L_1}} D(G_{l_1} \| G_1), \qquad (2.a)$$

$$0 < F_{l_1, L_2 | l_1, 1} < \min_{l_2 = \overline{2, L_2}} \inf_{Q \in R_{l_1}} D(G_{l_2/l_1} \| G_{1/m_1} | Q), \qquad (2.b)$$

$$0 < E_{L_1, l_2 | l_1, l_2} < \min[\min_{l_1 = \overline{1, m_1 - 1}} E^*_{l_1, l_2 | m_1, l_2}, \min_{l_1 = \overline{m_1 + 1, L_1}} D(G_{l_1} \| G_{m_1})],$$

$l_1 = \overline{2, L_1 - 1}$, $\qquad (2.c)$

$$0 < F_{l_1, L_2 | l_1, l_2} < \min[\min_{l_2 = \overline{1, m_2 - 1}} F^*_{l_1, l_2 | l_1, m_2}, \min_{l_2 = \overline{m_2 + 1, L_2}} \inf_{Q \in R_{l_1}} D(G_{l_2/l_1} \| G_{m_2/m_1} | Q)],$$

$l_2 = \overline{2, L_2 - 1}$, $\qquad (2.d)$

then there exists a LAO test sequence Φ^*, the lower estimate matrix of which $F(\Phi^*) = \{F_{l_1, l_2 | m_1, m_2}(\Phi^*)\}$ is defined in (1) and all elements of it are positive.

When even one of the inequalities (2) is violated, then at least one element of the lower estimate matrix $F(\Phi^*)$ is equal to 0.

2. Example

In the case of $L_1 = 2, L_2 = 2$ let us consider the set of two elements $X = \{0,1\}$ and the following probability distributions given on X: $G_1 = \{0.835, 0.165\}$, $G_2 = \{0.24, 0.76\}$

$$G_{1/1} = \begin{pmatrix} 0.39 & 0.61 \\ 0.51 & 0.49 \end{pmatrix}, G_{1/2} = \begin{pmatrix} 0.27 & 0.73 \\ 0.45 & 0.55 \end{pmatrix},$$

$$G_{2/1} = \begin{pmatrix} 0.21 & 0.79 \\ 0.59 & 0.41 \end{pmatrix}, G_{2/2} = \begin{pmatrix} 0.32 & 0.68 \\ 0.39 & 0.61 \end{pmatrix}$$

In Fig.1 the results of calculations of functions $E_{1,1|2,1}(E_{2,1|1,1})$ are presented.

We have $D(G_2 \| G_1) \approx 1.24$. We see that when first inequality of theorem is violated then $E_{1,1|2,1} = 0$. All calculations are made using package Mathematica.

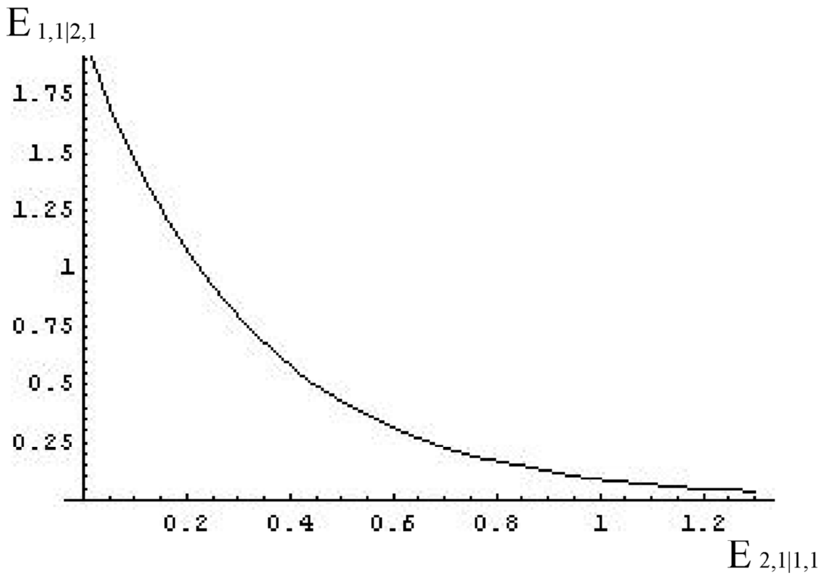

Fig.

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ՍՏՈԽԱՍՏԻԿՈՐԵՆ ԿԱԽՅԱԼ ՕԲՅԵԿՏՆԵՐԻ ՆԿԱՏՄԱՄԲ ՎԱՐԿԱԾՆԵՐԻ ՕՊՏԻՄԱԼ ԹԵՍՏԱՎՈՐՄԱՆ ԴԵՊՔՈՒՄ ՀՈՒՍԱԼԻՈՒԹՅՈՒՆՆԵՐԻ ՓՈԽԿԱԽՎԱԾՈՒԹՅՈՒՆՆԵՐԻ ՀԱՇՎԱՐԿՄԱՆ ՄԱՍԻՆ

Ամփոփում

Դիտարկվել է երկու ստոխաստիկորեն կախյալ օբյեկտներից բաղկացած մասնակի մոդելի դեպք: Ենթադրվում է, որ առաջին օբյեկտը կարող է բաշխված լինել տրված հավանականային բաշխումներից մեկով, իսկ երկրորդը առաջինից կախյալ, տրված պայմանական հավանականային բաշխում-

ների մեկով: Սխալի հավանականությունների ցուցիչների (հուսալիությունների) օպտիմալ փոխկախվածությունները հաշվարկվել և ներկայացվել են գրաֆիկորեն:

Առանցքային բառեր: Վարկածների ստուգում, հուսալիություն:

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О ВЫЧИСЛЕНИИ ЗАВИСИМОСТЕЙ МЕЖДУ НАДЕЖНОСТЯМИ ПРИ ОПТИМАЛЬНОМ ТЕСТИРОВАНИИ РАСПРЕДЕЛЕНИЙ ПАРЫ СТОХАСТИЧЕСКИ ЗАВИСИМЫХ ОБЪЕКТОВ

Резюме

Рассматривается пример задачи оптимального тестирования модели, состоящей из двух стохастически зависимых объектов. Предполагаетеся, что первый объект может быть распределен согласно одному из заданных распределений, а второй, зависящий от первого, согласно одному из заданных условных распределений. Оптимал взаимозависимости пар экспонент вероятностей ошибок (надежностей) вычеслены и представлены графиками.

Ключевые слова: Проверка гипотез, надежность.