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On Conditionally Universal Functions with respect to the Walsh system

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Abstract

This paper proves theorems on the existence of conditionally **universal functions** and **universal triads** with respect to the Walsh system defined by the author. The proof method of these theorems provides a new approach to constructing universal series in the Walsh system: by varying its values on a certain set of arbitrarily small measure, any measurable almost everywhere finite function can be turned into a function such that, after choosing the corresponding signs for the terms of the Fourier-Walsh series of the changed function, we can achieve the fact that the obtained series is universal in the class of all measurable functions.

Keywords: Universal function, Fourier series, Walsh system
MSC (2010): 42B05, 42C10, 43A15

1. Introduction

This article continues the author's studies on establishing the existence and describing the structure of functions (**universal functions)**, which are universal with respect to a given classical system for various function classes (see [1-15]).

We also consider the problem of the existence of **universal pairs** and **universal triads** with respect to the Walsh system for the class of all Lebesgue measurable functions on $[0,1]$.

We need some standard notation.

Let $|E|$ be the Lebesgue measure of a measurable set $E \subseteq [a,b]$ ($[a,b]=[-\pi,\pi]$ or $[0,1]$).

Let $L^p(E)$, $p>0$ be the class of all measurable functions f on E with finite integral $\int_E |f(x)|^p\,dx$ and $L^0(E)$ be the class of all almost everywhere finite, Lebesgue measurable functions on E.

We denote by $M(E)$ the class of all Lebesgue measurable functions on E.

The sequence of functions $\{f_k(x)\}_{k=1}^{\infty} \subset L^0(E)$ is said to converge to f in $L^0(E)$(in $M(E)$), if $\{f_k(x)\}_{k=1}^{\infty}$ converges to $f(x)$ almost everywhere on E.

The sequence of functions $\{f_k\}_{k=1}^{\infty} \subset L^p(E)$ is said to converge to f in $L^p(E)$, if it converges to f in the $L^p(E)$ metric, that is

$$\lim_{k\to\infty} \int_E \left|f_k(x) - f(x)\right|^p dx = 0.$$

Let

$$a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx, \quad b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx, \quad k \geq 0$$

be the Fourier coefficients of function $f \in L^1[-\pi,\pi]$ with respect to the trigonometric system and let

$$S_N(x,f) = \frac{a_0(f)}{2} + \sum_{k=0}^{N-1} a_k(f)\cos kx + b_k(f)\sin kx$$

be the N-th partial sum of the function f of the Fourier series.

Let $\Phi := \{\varphi_k(x)\}$ be an orthonormal system on $[a,b]$ and let $f \in L^1[a,b]$.

We denote by $c_k(f)$ the Fourier coefficients of function $f \in L^1[a,b]$ with respect to the system Φ, that is

$$c_k(f) = \int_a^b f(x)\varphi_k(x)dx, \quad S_n(x,f) = \sum_{k=1}^{n-1} c_k(f)\varphi_k(x).$$

Below S will denote one of the spaces $M(E)$ or $L^p(E)$, $p \geq 0$.

Definition 1^0. A series $\sum_{k=0}^{\infty} f_k(x)$, ($f_k \subset S$, $k = 0,1,2,\ldots$), is called universal in S (universal in the usual sense), if for each function $f \in S$ there exists a growing subsequence of natural numbers $\{N_m\}_{m=1}^{\infty} \nearrow$, so that the subsequence $\{\sum_{k=0}^{N_m} f_k(x)\}_{m=1}^{\infty}$ of partial sums of that series converges to f in S.

The existence of functions and series which are universal (in one sense or another) in various classes of functions has been studied by many mathematicians working in the theory of functions of real or complex variables (see [16-29]).

The first example is due to Birkhoff [16] in 1929, who proved the existence of an entire function $f(z)$ with the property that for an arbitrary entire function $g(z)$ and every $r > 0$ there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of the natural numbers, such that $\{f(z+n_k)\}_{k=1}^{\infty}$ converges to $g(z)$, uniformly on the disc $\{z \in C : |z| \leq r\}$.

In 1935 Marcinkiewicz [17] proved that for any (converging to zero) sequence $h_n \to 0$ there exists a continuous function $F \in C[0,1]$ $F:$ $[0,1] \to \mathcal{R}$ having the property: for any measurable function $f(x)$: $[0,1]$, there is a subsequence $n_k \nearrow^{\infty}$ such that

$$\frac{F\left(x+h_{n_k}\right) - F(x)}{h_{n_k}} \to g(x)$$

as $k \to \infty$ almost everywhere on $[0,1]$ (see also [18]).

This continuous function F is called a universal primitive function with respect to the given sequence $\{h_n\}_{n=1}^{\infty}$.

In 1952 MacLane [19] proved the existence of a universal entire function $g(z)$ with respect to derivatives. Namely, for every entire function $f(z)$ and every $r>0$ one can choose an increasing sequence $\{n_k\}_{k=1}^{\infty}$ in such a way that the sequence of derivatives $\{g^{(n_k)}(z)\}_{k=1}^{\infty}$ converges to $f(z)$ uniformly on the disc $|z| \le r$.

In 1986 Luh [20] proved a theorem on the universality of power series $\sum_{k=0}^{\infty} c_k z^k$. Namely, if let $r \ge 0$, there exists a power series $\sum_{k=0}^{\infty} c_k z^k$ of radius of convergence r such that for every compact set K in $\{z \in C : |z| > r\}$ with connected complement and every function $h(z)$ that is continuous on K and holomorphic on the interior of K there exists an increasing sequence $\{N_m\}_{m=1}^{\infty}$ such that

$$\sum_{k=0}^{N_m} c_k z^k \to h(z)$$

as $m \to \infty$ uniformly on K.

We note that the first universal real power series was constructed as early as 1914 by Fekete [21], who in particular proved the existence of a real power series $\sum\limits_{k=0}^{\infty} a_k x^k$ with the following property. For every continuous function $g(x)$ on $[-1,1]$ with $g(0)=0$ there is an increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that $\sum_{j=0}^{n_k} a_j x^j$ converges to $g(x)$ uniformly on $[-1,1]$.

In 1987 Grosse-Erdman [22] proved the existence of an infinitely differentiable function $h(x)$ with universal Taylor expansion. Namely, there exists an infinitely differentiable function h on $(-\infty,\infty)$ with $h(0)=0$ function $h(x) \in C^{\infty}(-\infty,\infty)$ with $h(0)=0$ such that the Taylor series at $x_0 = 0$ is locally uniformly universal in $C(R)$; that is for each function $f(x) \in C(R) = C(-\infty,\infty)$ with $f(0)=0$ and any number $r>0$ there exists a subsequence

$$S_{n_k}(h,0) = \sum_{m=1}^{n_k} \frac{h^{(m)}(0)}{m!} x^m$$

of partial sums of the Taylor series $\sum_{m=1}^{\infty} \frac{h^{(m)}(0)}{m!} x^m$ of $h(x)$ which converges to $f(x)$ uniformly on the $[-r,r]$.

It is not difficult to see that from this result it follows that for every measurable function $f(x)$ on $[-\pi,\pi]$ there is an increasing sequence $\{m_q\} \nearrow \infty$ of positive integers such that the subsequence of partial sums

$$\sum_{m=1}^{m_q} \frac{h^{(m)}(0)}{m!} x^m$$

converges to $f(x)$ as $q \to \infty$ almost everywhere on $[-\pi, \pi]$.

The notion of a universal series in $M[-\pi,\pi]$ in the trigonometric and by general orthonormal systems is due to Men'shov [23] in 1947 and Talalyan [24] in 1957 (see also [25]-[30]). In this direction, important results were obtained by them and their students.

The first construction of universal trigonometric series in the class of all measurable functions in the sense of convergence almost everywhere was given by Men'shov. He proved the following fundamental theorem.

Theorem (Men'shov). There is a trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx, \quad |a_k| + |b_k| \to 0 \text{ as } k \to \infty$$

with the following property. For every measurable function $f(x)$ on $[-\pi,\pi]$ there is an increasing sequence $\{m_q\}_{q=1}^{\infty}$ of positive integers such that the subsequence of partial sums

$$\frac{a_0}{2} + \sum_{k=1}^{m_q} a_k \cos kx + b_k \sin kx,$$

converges to $f(x)$ as $q \to \infty$ almost everywhere on $[-\pi,\pi]$(that is this series is universal in $M[-\pi,\pi]$.)

Theorem (Talalyan). Let $\Phi := \{\varphi_k(x)\}_{k=1}^{\infty}, x \in [0,1]$ be an orthonormal system. There is a series in the system $\{\varphi_k(x)\}_{k=1}^{\infty}$:

$$\sum_{k=1}^{\infty} d_k \varphi_k(x), \; d_k \to 0 \text{ as } k \to \infty,$$

which is universal in $M[0,1]$.

Remark 1. As noted above, there is an infinitely differentiable function with universal Taylor series, **but there is no function** $U \in L^1[-\pi,\pi]$, whose Fourier series in the trigonometric system is universal in $M[-\pi,\pi]$:

Otherwise, if there were a function $U \in L^1[-\pi,\pi]$ whose Fourier series in the trigonometric system is universal in $M[-\pi,\pi]$, then for the function $f(x) = 2U(x)$ one could find a growing subsequence of natural numbers $\{N_m\} \nearrow \infty$ such that

$$S_{N_m}(x,U) = \frac{a_0(U)}{2} + \sum_{k=1}^{N_m - 1} a_k(U) \cos kx + b_k(U) \sin kx$$

converges to $2U(x)$ as $q \to \infty$ almost everywhere on $[-\pi,\pi]$.

On the other hand, from a **well-known theorem of Kolmogorov [31] (the Fourier series in the trigonometric system of any integrable function converges to it in** $L^p[-\pi,\pi], p \in (0,1)$, in particular $\lim_{m\to\infty}\int_{-\pi}^{\pi}\left|S_{N_m}(x,U)-U(x)\right|^p dx=0$), it follows that for some subsequence $\{N_{m_q}\}_{q=1}^{\infty}$ of sequence $\{N_m\}_{m=1}^{\infty}$ the subsequence

$$S_{N_{m_q}}(x,U)=\frac{a_0(U)}{2}+\sum_{k=1}^{N_{m_q}}a_k(U)\cos x+b_k(U)\sin kx$$

converges to $U(x)$ as $q\to\infty$ almost everywhere on $[-\pi,\pi]$.

Hence, we see that $U(x)=2U(x)$ almost everywhere on $[-\pi,\pi]$.

This contradiction shows that there does not exist a function $U\in L^1[-\pi,\pi]$ which is universal for the class $M[-\pi,\pi]$ with respect to the trigonometric system.

Remark 2. A similar analysis shows that there does not exist an integrable function U, whose Fourier series in the other classical systems (Walsh system, Haar system, Franklin system, Vilenkin system) is universal in $M[0,1]$.

The above considerations suggest the following question, the answer to which is unknown:

Question 1. Do there exist an orthonormal system of bounded functions and an integrable function U(x) that is universal for the space $M[0,1]$ with respect to the system $\{\varphi_k(x)\}_{k=1}^{\infty}$?

Despite the fact that, as Men'shov proved, there is a universal trigonometric series in the class $M[-\pi,\pi]$ and (as we indicated above) there is no function $U\in L^1[-\pi,\pi]$ whose Fourier series in the trigonometric system is universal in the class $M[-\pi,\pi]$.

Nevertheless, we managed to construct an integrable function U and prove that after a suitable choice of signs $\{\delta_k;\ \delta_k=\pm1\}_{k=0}^{\infty}$ for the Fourier coefficients of this function U, it is possible to achieve that the newly obtained series $\frac{a_0(U)}{2}+\sum_{k=0}^{\infty}\delta_k\left(a_k(U)\cos kx+b_k(U)\sin kx\right)$ would be universal in $M[-\pi,\pi]$.

Before formulating the main result of the paper, we give corresponding definitions.

Let $\Phi:=\{\varphi_k(x)\}$ be an orthonormal system on $[a,b]$.

Definition 1. We say that a function $U\in L^1[a,b]$ and a sequence $\delta=\{\delta_k=\pm1,k=0,1,2,\dots\}$ of signs form **universal pairs:** (U,δ) for space S with respect to this system $\{\varphi_k(x)\}_{k=1}^{\infty}$ in sense of universal series, if the series $\sum_{k=0}^{\infty}\delta_k c_k(f)\varphi_k(x)$ is universal in S.

Definition 2. We say that a function $U\in L^1[a,b]$ and a measurable set $E\subset[a,b]$ form **universal pairs:** (U,E) with respect to the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ in sense of modification, if for each function $f\in L^1[a,b]$ one can find such a function $g\in L^1[a,b]$ coinciding with f on E, and such that

$$|c_k(g)|=|c_k(U)|,\quad k=0,1,2,\dots$$

Definition 3. We say that a function $U\in L^1[a,b]$, a sequence $\delta=\{\delta_k=\pm1,k=0,1,2,\dots\}$ of signs and a measurable set $E\subset[a,b]$ form **universal triads:** (U,δ,E) for space S with respect to the system $\{\varphi_k(x)\}_{k=0}^{\infty}$, if partial sums of the series $\sum_{k=0}^{\infty}\delta_k c_k(f)\varphi_k(x)$ are universal in S and for each function $f\in L^1[a,b]$ one can find such a function $g\in L^1[a,b]$ coinciding with f on E, and such that

$$|c_k(g)|=|c_k(U)|,\quad k=0,1,2,...$$

Definition 4. We say that a function $U\in L^1[a,b]$ is

1) **universal for** a space S with respect to the system $\{\varphi_k(x)\}_{k=0}^{\infty}$, if the Fourier series of the function U with respect to this system is universal in S,

2) **conditionally universal** for a space S with respect to the system $\{\varphi_k(x)\}_{k=0}^{\infty}$, if there exists a sequence of signs $\delta_k=\pm1$ such that the series $\sum\limits_{k=0}^{\infty}\delta_k c_k(U)\varphi_k(x)$ is universal in S,

3) **quasiuniversal** for a space S with respect to the system $\{\varphi_k(x)\}_{k=0}^{\infty}$, if there exists a sequence of signs $\delta_k=\pm1$, with $d_\Lambda(\Omega^+)=1$, such that the series $\sum_{k=0}^{\infty}\delta_k c_k(U)\varphi_k(x)$ is universal in S, where

$$d_\Lambda(\Omega^+):=\lim_{n\to\infty}\sup\frac{\#(\Omega^+\cap[0,n))}{\#(\Lambda\cap[0,n))}$$

is the upper density of the subset $\Omega^+=\{k\in N\cup\{0\},\delta_k=1\}$ with respect to the set $\Lambda=spec(U)=\{k\in N\cup\{0\},\,c_k(U)\neq0\}$, and $\#(E)$ is the number of elements of a finite set E,

4) **almost universal** for a space S with respect to the system $\{\varphi_k(x)\}_{k=0}^{\infty}$, if there exists a sequence of signs $\delta_k=\pm1$ with $\rho_\Lambda(\Omega)=0$, such that the series $\sum_{k=0}^{\infty}\delta_k c_k(U)\varphi_k(x)$ is universal in S, where

$$\rho_\Lambda(\Omega):=\lim_{n\to\infty}\frac{\#(\Omega\cap[0,n))}{\#(\Lambda\cap[0,n))}$$

is the density of the subset $\Omega=\{k\in N\cup\{0\},\,\delta_k=-1\}$ with respect to the set $\Lambda=spec(U)=\{k\in N\cup\{0\},\,c_k(U)\neq0\}$,

5) **universal in sense of signs**, if its Fourier series $\sum_{k=0}^{\infty}c_k(U)\varphi_k(x)$ in the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ is universal in S **in sense of signs**: that is for each function $f\in S$ one can find a sequence of signs $\delta_k=\pm1$ such that the series $\sum_{k=0}^{\infty}\delta_k c_k(U)\varphi_k(x)$ converges to f in S,

6) **universal in sense of rearrangements**, if its Fourier series $\sum_{k=0}^{\infty}c_k(U)\varphi_k(x)$ in the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ is universal in S in sense of rearrangements: that is for each function $f\in S$ one can

find $\{\sigma(k)\}_{k=1}^{\infty}$ some permutation of the natural numbers, such that the series $\sum_{k=1}^{\infty} c_{\sigma(k)}(U)\varphi_{\sigma(k)}(x)$ converges to f in S.

In recent years, we have obtained several results related to the existence and description of the structure of functions (universal functions), whose Fourier series with respect to a given classical system are universal in a certain sense for different function classes.

In particular, in [2], [3], [11], and [15] the following theorems were proved:

Theorem 1. There exists an integrable function U with monotonically decreasing Fourier–Walsh coefficients, which is universal in the sense of signs with respect to the Walsh system for the spaces $L^p[0,1]$ for all $p\in(0,1)$. In addition, the Fourier–Walsh series of the function U converges to it in $L^1[0,1]$.

Theorem 2. There exist an integrable function $U\in L^1[-\pi,\pi]$ and a sequence $\delta=\{\delta_k=\pm1, k=0,1,2,\ldots\}$ of signs, which are form **universal pairs:** (U,δ) for space $L^p[-\pi,\pi]$, $p\in(0,1)$ with respect to the trigonometric system.

Theorem 3. For any $p\in(0,1)$ there exists an integrable function U that is both conditional universal and universal in the sense of signs with respect to the trigonometric system for the class $L^p[-\pi,\pi]$.

Theorem 4. There exists an integrable function U, with monotonically decreasing Fourier–Walsh coefficients, which is universal in the sense of signs with respect to the Walsh system for the space $L^0[0,1]$. In addition, the Fourier–Walsh series of the function U converges to it in $L^1[0,1]$.

Remark 3. It should be noted (this also follows from the above) that the existence of universal functions and universal triads depends on the type (sense) of universality, on the system, and on the space S. Therefore, questions in this direction are very comprehensive.

In this article we will prove

Theorem 5. There exist an integrable function $U\in L^1[0,1]$ and a sequence $\delta=\{\delta_k=\pm1, k=0,1,2,\ldots\}$ of signs and a measurable set $E\subset[a,b]$ form **universal triads:** (U,δ,E) for space $L^0[0,1]$ with respect to the Walsh system.

Moreover, the following statement holds

Theorem 6. For any $\varepsilon\in(0,1)$ there exists a function $U\in L^1[0,1]$, with $\sup(U)\subset[0,\varepsilon]$, which has the following properties:

a) the Fourier coefficients of the function U in the Walsh system are positive and monotonically decreasing,

b) the Fourier–Walsh series of the function U converges to it in $L^1[0,1]$,

c) the function U is conditional universal for the class $M[0,1]$ with respect to the Walsh system,

d) for any $\delta\in(0,1)$ exists a measurable set $E\subset[0,1]$ with $|E|>1-\delta$, so that for each function $g\in L^1[0,1]$ one can find such a function $f\in L^1[0,1]$ coinciding with $g(x)$ on E, and such that

$$|c_k(f)|=c_k(U),\quad k=0,1,2,\ldots,$$

e) the Fourier–Walsh series of the corrected function f converges to it in $L^1[0,1]$.

Remark 4. The proof method of Theorem 5 allows obtaining a new approach to constructing universal series in the Walsh system: by varying its values on a certain set of arbitrarily small measure, any measurable almost everywhere finite function can be turned into a function such that, after choosing the corresponding signs for the terms of the Fourier–Walsh series of the changed function, we can achieve the fact that the obtained series is universal in $M[-\pi,\pi]$.

The following questions arise, the answer to which is still unknown:

Question 2. Do there exist an orthonormal system $\{\varphi_k(x)\}_{k=1}^{\infty}$ of bounded functions and an integrable function $U(x)$ that is universal for the space $L^p[0,1]$ for some $p\in[0,1)$ (or at least for the space $M[0,1]$) with respect to the system $\{\varphi_k(x)\}_{k=1}^{\infty}$?

Question 3. Are the Theorems 4 and 6 true for the trigonometric system?

Question 4. Are the theorems 1-7 true for the Vilenkin system?

Question 5. Is it possible to construct a function that is universal for the space $L^0[0,1]$ with respect to the Walsh system in sense of rearrangements?

Question 6. Does a function $U\in L^1[-\pi,\pi]$ exist that, for the class $L^p[-\pi,\pi]$, with respect to the trigonometric system, is universal in the sense of permutations?

Question 7. Are the theorems 1-7 true for the Franklin system (for the Haar system)?

Question 8. Are the theorems 1-5 true for spherical harmonics?

In connection with Question 3, note that we can prove the following statement:

Theorem 7. There exist an integrable function $U\in L^1[0,1]$ and a sequence $\delta=\{\delta_k=\pm1, k=0,1,2,\dots\}$ of signs and a measurable set $E\subset[a,b]$ form **universal triads:** (U,δ,E) for space $L^0[0,1]$ with respect to the trigonometric system.

The proof of this theorem will be given in another paper by the author. In this paper, we will prove Theorem 6.

2. Auxiliary facts

The Walsh system, an extension of the Rademacher system, may be obtained in the following manner. Let r be the periodic function of least period 1 defined on $[0,1)$ by

$$r(x)=\chi_{[0,1/2)}(x)-\chi_{[1/2,1)}(x).$$

The Rademacher system $\{r_n(x)\}_{n=0}^{\infty}$ is defined by the conditions:

$$r_n(x)=r\left(2^n x\right),\ \ \forall x\in R,\ \ n=0,1,...$$

and, in the ordering employed by Paley (see [32], [33]), the n-th element of the Walsh system $\{\varphi_n(x)\}_{n=0}^{\infty}$ is given by

$$\varphi_n(x)=\prod_{k=0}^{\infty}\left(r_k(x)\right)^{\theta_k(n)}, \tag{2.1}$$

where $\sum_{k=0}^{\infty}\theta_k(n)2^k$ is the unique binary expansion of n, with each $\theta_k(n)$ either 0 or 1.

Note that (see [34]) the Walsh system is a basis for all $L^p[0,1]$ $p\in(1,\infty)$, i.e., every function $f(x)$ is $L^p[0,1]$ $p\in(1,\infty)$ uniquely representable by the series $\sum_{k=0}^{\infty}c_k\varphi_k(x)$ in the Walsh system by norm $L^p[0,1]$ convergent to $f(x)$.

Let $f\in L^1[0,1]$ and let

$$S_m(x,f)=\sum_{k=0}^{m-1}\quad c_k(f)\varphi_k(x), \tag{2.2}$$

where

$$c_k(f)=\int_0^1 f(x)\varphi_k(x)dx. \tag{2.3}$$

In this article, we use the following lemma, which was proved in [4].

Lemma 1. Let numbers $n_0\in N, \left(N_0=2^{n_0}\right), 0<p_1<p_2<1, \varepsilon\in(0,1)$ and polynomial $W(x)$ in the Walsh system $\{\varphi_k(x)\}_{k=0}^{\infty}$ are given. Then there exist polynomials $U(x)$ and $B(x)$ in the Walsh system of the following form

$$U(x)=\sum_{k=2^{n_0}}^{2^n-1}b_k\varphi_k(x)\,,\ B(x)=\sum_{k=N_0}^{N-1}\varepsilon_k b_k\varphi_k(x),\quad N_0=2^{n_0},\quad N=2^n,$$

which satisfy the following conditions:

$$0<b_{k+1}<b_k<\varepsilon,\quad \varepsilon_k=\pm1,\ \ \forall k\in\left[2^{n_0},2^n\right)=\left[N_0,N\right), \tag{1}$$

$$B(x)=W(x)\ \forall x\in G,\ \ |G|\ge 1-\varepsilon-2^{-n_0}\,, \tag{2}$$

$$U(x)\chi_{[2^{-n_0},1)}(x)=0, \tag{3}$$

$$\max_{m\in[N_0,N)}\int_0^1\left|\sum_{k=N_0}^{m}\varepsilon_k b_k\varphi_k(x)\right|^p dx<4\int_0^1|W(x)|^p\,dx,\ \forall p\in(p_1,p_2), \tag{4}$$

$$\max_{m\in[N_0,N)}\int_0^1\left|\sum_{k=N_0}^{m}\varepsilon_k b_k\varphi_k(x)\right|dx<5\int_0^1|W(x)|dx, \tag{5}$$

$$\int_0^1|W(x)-B(x)|^p\,dx<\varepsilon\quad \forall p\in(p_1,p_2), \tag{6}$$

$$\max_{m\in[N_0,N)}\int_0^1\left|\sum_{k=N_0}^{m}a_k\varphi_k(x)\right|dx<\varepsilon. \tag{7}$$

We also use the following elementary result:

Lemma 2. For each function $g(x)\in M[0,1)$ one can find a sequence of polynomials $\{w_k(x)\}_{k=1}^{\infty}$ in the Walsh system with rational coefficients, which converges to $g(x)$ almost everywhere on $[0,1)$.

Proof of Lemma 2. We set

$$G_n=\left\{x\in[0,1);|g(x)|\le n\right\},\ G=\left\{x\in[0,1);\ |g(x)|<\infty\right\}, \tag{2.6}$$

$$G^{\pm\infty} = \left\{x \in [0,1);\ g(x) = \pm\infty\right\}, \tag{2.7}$$

$$g_n(x) = \begin{cases} g(x), & x \in G_n \\ sign\left\{g(x)\right\} n, & x \in G \setminus G_n \\ \pm n, & x \in G^{\pm\infty} \end{cases} \tag{2.8}$$

By (2.6)-(2.8), we have

$$G = \bigcup_{n=1}^{\infty} G_n,\ [0,1) = G \bigcup G^{+\infty} \bigcup G^{-\infty} \tag{2.9}$$

and

$$\lim_{n\to\infty} g_n(x) = g(x),\ \ x \in G,\ \ \lim_{n\to\infty} g_n(x) = \pm\infty,\ \ x \in G^{\pm\infty}. \tag{2.10}$$

For every natural number n one can find a polynomial $w_n(x)$ in the Walsh system with rational coefficients, such that

$$\left|\left\{x \in [0,1);\ \left|g_n(x) - w_n(x)\right| \le \frac{1}{n}\right\}\right| \ge 1 - 2^{-n}.$$

We set

$$E = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \left\{x \in [0,1);\ \ \left|g_n(x) - w_n(x)\right| \le \frac{1}{n}\right\}.$$

Taking into account (2.9) and (2.10), we get that $|E| = 1$ and on E

$$\lim_{n\to\infty} \left|g_n(x) - w_n(x)\right| = 0.$$

From this and from (2.8) it follows that the sequence of polynomials $\{w_k(x)\}_{k=1}^{\infty}$ in the Walsh system with rational coefficients converges to $g(x)$ almost everywhere on $[0,1]$.

3. Proof of Theorem 6

Let

$$\{W_n(x)\}_{n=1}^{\infty} \tag{3.1}$$

be the sequence of all polynomials in the Walsh system $\{\varphi_k(x)\}_{k=0}^{\infty}$ with rational coefficients. It is easy to see the sequence $\{W_n(x)\}_{k=1}^{\infty}$ is dense in $L^p[0,1]$, $p \in (0,\infty)$. Let $\theta, \delta \in (0,1)$ and let

$$m_1 = [-\log_2 \theta] + 2\ . \tag{3.2}$$

We use Lemma 1, with

$$n_0 = m_1\left(N_0 = M_1 = 2^{m_1}\right),\ \ \theta = 2^{-8(1+2)},\ \ p_1 = \frac{1}{4},\ \ p_2 = \frac{3}{4},\ \ W(x) = W_1(x)$$

in its formulation. Then there exist polynomials $U_1^{(1)}(x)$ and $B_1^{(1)}(x)$:

$$U_1^{(1)}(x) = \sum_{k=M_1}^{M_2-1} b_k^{(1)} \varphi_k(x)\ ,\ B_1^{(1)}(x) = \sum_{k=M_1}^{M_2-1} \varepsilon_k^{(1)} b_k^{(1)} \varphi_k(x),\ \ M_1 = 2^{m_1},\ \ M_2 = 2^{m_2}$$

satisfying the following conditions:

$$0<b_{k+1}^{(1)}<b_k^{(1)},\ \ k\in\left[M_1,M_2-1\right),\ \ \varepsilon_k^{(1)}=\pm 1, k\in\left[M_1,M_2\right),$$

$$B_1^{(1)}(x)=W_1(x)\,\forall x\in G_1,\ \ \left|G_1\right|\ge 1-\frac{\delta}{2^2}-2^{-n_1}\ge 1-\frac{\delta}{2^1},$$

$$U_1^{(1)}(x)\,\chi_{[\beta_1,1)}(x)=0,\ \ \beta_1=2^{-m_1}<\frac{\theta}{2},$$

$$\max_{m\in[M_1,M_2)}\int_0^1\left|\sum_{k=M_1}^{m}\varepsilon_k^{(1)}b_k^{(1)}\varphi_k(x)\right|dx<5\int_0^1\left|W_1(x)\right|dx,$$

$$\int_0^1\left|U_1^{(1)}(x)\right|dx<\max_{m\in[M_1,M_2)}\int_0^1\left|\sum_{k=M_1}^{m}b_k^{(1)}\varphi_k(x)\right|dx<4^{-2-1}.$$

Again, we use Lemma 1 with

$$n_0=\log_2 M_2,\ \ \theta=\min\left\{b_{M_2-1}^{(1)},2^{-8(1+2)}\right\},\ \ p_1=\frac{1}{4},\ \ p_2=\frac{3}{4},\ \ W(x)=\left\{W_1(x)-U_1^{(1)}(x)\right\}$$

in its formulation. Then, we determine polynomials $U_1^{(2)}(x)$ and $B_1^{(2)}(x)$ of the form

$$U_1^{(2)}(x)=\sum_{k=M_2}^{M_3-1}b_k^{(1)}\varphi_k(x)\,,\ \ B_1^{(2)}(x)=\sum_{k=M_2}^{M_3-1}\varepsilon_k^{(1)}b_k^{(1)}\varphi_k(x)$$

satisfying the following conditions:

$$0<b_{k+1}^{(1)}<b_k^{(1)}<b_{M_{2,}-1}^{(1)},\ \ k\in\left[M_2,M_3-1\right),\ \ \varepsilon_k^{(1)}=\pm 1,\ \ k\in\left[M_2,M_3\right),$$

$$U_1^{(2)}(x)\,\chi_{[\beta_2,1)}(x)=0,\ \ \beta_2=\frac{1}{M_2},$$

$$\int_0^1\left|\left\{W_1(x)-U_1^{(1)}(x)\right\}-B_1^{(2)}(x)\right|^p dx<4^{-8(1+2)},\ \ p\in\left(\frac{1}{4},\frac{3}{4}\right),$$

$$\int_0^1\left|U_1^{(2)}\right|dx<\max_{m\in[M_2,M_3)}\int_0^1\left|\sum_{k=M_2}^{m}b_k^{(1)}\varphi_k(x)\right|dx<4^{-2-1},$$

It is clear that by using an induction, one can determine a sequence of polynomials $\{U_n^{(1)}(x)\}_{n=1}^{\infty}$, $\{U_n^{(2)}(x)\}_{n=1}^{\infty}$ $\{Q_n^{(1)}(x)\}_{n=1}^{\infty}$, $\{Q_n^{(2)}(x)\}_{n=1}^{\infty}$ of the form

$$U_n^{(1)}(x)=\sum_{k=M_{2n-1}}^{M_{2n}-1}b_k^{(n)}\varphi_k(x)\,,\ \ U_n^{(2)}(x)=\sum_{k=M_{2n}}^{M_{2n+1}-1}b_k^{(n)}\varphi_k(x), \tag{3.3}$$

$$B_n^{(1)}(x)=\sum_{k=M_{2n-1}}^{M_{2n}-1}\varepsilon_k^{(n)}b_k^{(n)}\varphi_k(x)\,,\ \ B_n^{(2)}(x)=\sum_{k=M_{2n}}^{M_{2n+1}-1}\varepsilon_k^{(n)}b_k^{(n)}\varphi_k(x), \tag{3.4}$$

which, for all $n=1,2,..$, satisfy the conditions:

$$M_n=2^{m_n},\ \ \{m_n\}_{n=1}^{\infty}\nearrow\infty,\ \ \varepsilon_k^{(n)}=\pm 1,\ \ k\in\left[M_{2n-1},M_{2n+1}\right), \tag{3.5}$$

$$0<b_{M_{2n+1}}^{(n)}<\ldots b_{k+1}^{(n)}\le b_k^{(n)}<\ldots\le b_{M_{2n-1}}^{(n)},\ \ k\in\left(M_{2n-1},M_{2n+1}-1\right), \tag{3.6}$$

$$B_n^{(1)}(x)=W_n(x),\quad \forall x\in G_n,\ \left|G_n\right|\ge 1-\frac{\delta}{2^n}, \tag{3.7}$$

$$U_n^{(1)}(x)\chi_{[\beta_{2n-1},1)}(x)=U_n^{(2)}(x)\chi_{[\beta_{2n},1)}(x)=0,\ \ \beta_j=\frac{1}{M_j}\le\theta, \tag{3.8}$$

$$\max_{m\in[M_{2n-1},M_{2n})}\int_0^1\left|\sum_{k=M_{2n-1}}^{m} b_k^{(n)}\varphi_k(x)\right|dx+\max_{m\in[M_{2n},M_{2n+1})}\int_0^1\left|\sum_{k=M_{2n},}^{m} b_k^{(n)}\varphi_k(x)\right|dx<4^{-n}, \tag{3.9}$$

$$\max_{m\in[M_{2n-1},M_{2n})}\int_0^1\left|\sum_{k=M_{2n-1}}^{m}\varepsilon_k^{(n)}b_k^{(n)}\varphi_k(x)\right|dx\le 5\int_0^1\left|W_n(x)\right|dx, \tag{3.10}$$

$$\int_0^1\left|W_n(x)-\sum_{j=1}^{n}\left(U_j^{(1)}(x)+B_j^{(2)}(x)\right)\right|^p dx<2^{-8n},\quad p\in\left(\frac{1}{4^n},1-\frac{1}{4^n}\right). \tag{3.11}$$

It is clear that (see (3.3), (3.9))

$$\sum_{n=1}^{\infty}\left(\int_0^1\left|U_n^{(1)}(x)\right|dx+\int_0^1\left|U_n^{(2)}(x)\right|dx\right)<\sum_{n=1}^{\infty}2^{-n}. \tag{3.12}$$

We define the function $U(x)$ and the sequence of numbers $\{b_k\}_{k=0}^{\infty}$ in the following way:

$$U(x)=U_0(x)+\sum_{n=1}^{\infty}\left(U_n^{(1)}(x)+U_n^{(2)}(x)\right)=\sum_{n=0}^{\infty}b_k\varphi_k(x), \tag{3.13}$$

where

$$U_0(x)=\sum_{k=0}^{M_{1,}-1}\varphi_k(x), \tag{3.14}$$

$$b_k=1,\ \ \forall k\in[0,M_1),\ \ b_k=b_k^{(n)},\ \ k\in\left[M_{2n-1,}M_{2n+1}\right),\ \ n=1,2,... \tag{3.15}$$

Using (3.3), (3.6) (3.8) and (3.12) – (3.15), we get that

$U(x)\in L^1[0,1]$, $U(x)=0,\ x\in[\theta,1)$, $b_k>0,\ \forall k\in\mathbf{N}\cup\{0\}$ and $\{b_k\}_{k=0}^{\infty}\searrow$.

From the conditions (3.3), (3.9) and (3.13) – (3.15) for all $m\in[M_{2n-1},M_{2n+1})$, $n=1,2,...$ it follows

$$\int_0^1\left|\sum_{k=0}^{m}b_k\varphi_k(x)-U(x)\right|dx\le\sum_{j=n+1}^{\infty}\left(\int_0^1\left|U_j^{(1)}(x)\right|dx+\int_0^1\left|U_j^{(2)}(x)\right|dx\right)\le$$

$$\le\max_{m\in[M_{2n-1},M_{2n})}\int_0^1\left|\sum_{k=M_{2n-1}}^{m}b_k^{(n)}\varphi_k(x)\right|dx+\max_{m\in[M_{2n},M_{2n+1})}\int_0^1\left|\sum_{k=M_{2n},}^{m}b_k^{(n)}\varphi_k(x)\right|dx<2^{-n}.$$

From this it follows that the series $\sum_{k=0}^{\infty}b_k\varphi_k(x)$ converges to $U(x)$ in $L^1[0,1]$, and therefore

$$b_k=c_k(U),\ \ k=0,1,2,... \ . \tag{3.16}$$

Thus, the Fourier series of the function $U(x)$ in the Walsh system converges in $L^1[0,1]$, and $\{c_k(U)\}_{k=0}^{\infty} \searrow 0$.

We set

$$\varepsilon_k = \begin{cases} 1, \text{if } k \in [0, M_1) \cup \left(\bigcup\limits_{n=1}^{\infty} [M_{2n-1}, M_{2n}) \right), \\ \varepsilon_k^{(n)}, \text{ if } k \in [M_{2n}, M_{2n+1}), n \geq 1. \end{cases} \tag{3.17}$$

We will prove that the series (see (3.3), (3.4), (3.14) – (3.17))

$$\sum_{k=0}^{\infty} \varepsilon_k c_k(U) \varphi_k(x) = U_0(x) + \sum_{j=1}^{\infty} \left(U_j^{(1)}(x) + B_j^{(2)}(x) \right), \varepsilon_k = \pm 1 \tag{3.18}$$

is universal in $M[0,1)$ in the usual sense.

We set

$$E_n = \left\{ x \in [0,1]; \left| W_n(x) - \sum_{j=1}^{n} \left(U_j^{(1)}(x) + B_j^{(2)}(x) \right) \right| \leq 2^{-n} \right\}. \tag{3.19}$$

By (3.11) and (3.19) it follows that

$$2^{-n} \left| [0,1] \setminus E_n \right| \leq \int_0^1 \left| W_n(x) - \sum_{j=1}^{n} \left(U_j^{(1)}(x) + B_j^{(2)}(x) \right) \right|^{\frac{1}{2}} dx \leq$$

$$\leq \int_0^1 \left| W_n(x) - \sum_{j=1}^{n} \left(U_j^{(1)}(x) + B_j^{(2)}(x) \right) \right|^{\frac{1}{2}} dx \leq 2^{-3n}.$$

From this we have

$$|E_n| \geq 1 - 2^{-2n+1}. \tag{3.20}$$

We put

$$E = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n. \tag{3.21}$$

It is clear that (see (3.20) and (3.21)) $|E| = 1$.

It is not hard to see that for all $x \in E$

$$\lim_{n \to \infty} \left(W_n(x) - \sum_{j=1}^{n} \left(U_j^{(1)}(x) + B_j^{(2)}(x) \right) \right) = 0\,. \tag{3.22}$$

Let $f(x) \in M[0,1)$.

Applying Lemma 2, one can find a subsequence $\{W_{n_q}(x)\}_{q=1}^{\infty}$ from the sequence (3.1), such that

$$\lim_{q \to \infty} W_{n_q}(x) = f(x) - U_0(x) \tag{3.23}$$

almost everywhere on $[0,1]$.

By (3.22) we have

$$\lim_{q\to\infty}\left(W_{n_q}(x)-\sum_{j=1}^{n_q}\left(U_j^{(1)}(x)+B_j^{(2)}(x)\right)\right)=0. \qquad (3.24)$$

almost everywhere on $[0,1]$.

We put $N_q = M_{n_q} - 1$. Taking into account (3.3), (3.4), (3.17), (3.23) and (3.24), it follows that

$$\sum_{k=0}^{N_q}\varepsilon_k c_k(U)\varphi_k(x)=U_0(x)+\sum_{j=1}^{n_q}\left(U_j^{(1)}(x)+B_j^{(2)}(x)\right)$$

converges to $f(x)$ almost everywhere on $[0,1]$. Thus, the series (3.18) is universal in $M[0,1)$, that is the function U is conditional universal for the class $M[0,1]$ with respect to the Walsh system.

Now we will prove assertions $c)$ and $d)$ of Theorem 6. We put

$$G=\bigcap_{n=1}^{\infty}G_n, \qquad (3.25)$$

Obviously (see (3.7), (3.25))

$$|G|\ge 1-\delta. \qquad (3.26)$$

Let $g(x)\in L^1[0,1)$. In the sequence (3.1) we select a subsequence $\{W_{k_q}(x)\}_{q=1}^{\infty}$ such that the following conditions

$$\lim_{N\to\infty}\int_0^1\left|\sum_{q=1}^{N}W_{k_q}(x)-\left(g(x)-U_0(x)\right)\right|dx=0, \qquad (2.27)$$

$$\int_0^1\left|W_{k_q}(x)\right|dx<2^{-8q}, \quad \forall q\ge 2 \qquad (3.28)$$

are satisfied.

Assume that the numbers $\nu_1 = k_1 < \ldots < \nu_{q-1}$, the functions $W_{\nu_1}(x),\ldots,W_{\nu_{q-1}}(x)$, $f_1(x),\ldots,f_{q-1}(x)$, and the polynomials $B_{\nu_1}^{(1)}(x), U_{\nu_1}^{(2)}(x)\ldots B_{\nu_{q-1}}^{(1)}(x), U_{\nu_{q-1}}^{(2)}(x)$ satisfying the following conditions:

$$f_n(x)=W_{k_n}(x),\ x\in G, 1\le n\le q-1,$$

$$\int_a^b\left|f_n(x)\right|dx<2^{-(n+1)},\ 1<n\le q-1,$$

$$\int_0^1\left|\sum_{j=1}^{l}\left[f_j(x)-\left(\sum_{n=\nu_{j-1}+1}^{\nu_j-1}U_n(x)+B_{\nu_j}^{(1)}(x)+U_{\nu_j}^{(2)}(x)\right)\right]\right|dx< \qquad (3.29)$$

$$<2^{-2(l+1)},\ 1<l\le q-1,\ \left(\nu_0=0\right).$$

are already determined, where

$$U_n(x)=U_n^{(1)}(x)+U_n^{(2)}(x). \qquad (3.30)$$

We choose a function $W_{\nu_q}(x),\ \nu_q>\nu_{q-1}$ from the sequence (3.1) so that

$$\int_0^1 \left| \left\{ W_{k_q}(x) - \sum_{j=1}^{q-1} \left[f_j(x) - \left(\sum_{n=\nu_{j-1}+1}^{\nu_j-1} U_n(x) + B_{\nu_j}^{(1)}(x) + U_{\nu_j}^{(2)}(x) \right) \right] \right\} - W_{\nu_q}(x) \right| dx \le \tag{3.31}$$
$$\le 2^{-3(q+3)}.$$

By (3.28), (3.29) and (3.31), we obtain

$$\int_a^b \left| W_{\nu_q}(x) \right| dx < 2^{-(q+1)}. \tag{3.32}$$

From this and (3.10) it follows that

$$\int_0^1 \left| B_{\nu_q}^{(1)}(x) \right| dx \le \max_{m \in \left[M_{2\nu_q-1}, M_{2\nu_q} \right)} \int_0^1 \left| \sum_{k=M_{2\nu_q-1}}^{m} \varepsilon_k^{(\nu_q)} b_k^{(\nu_q)} \varphi_k(x) \right| dx \le 5 \int_0^1 \left| W_{\nu_q}(x) \right| dx < \tag{3.33}$$
$$< 2^{-(q+1)}.$$

We put

$$f_q(x) = W_{k_q}(x) + \left[B_{\nu_q}^{(1)}(x) - W_{\nu_q}(x) \right], \quad q \ge 1. \tag{3.34}$$

From (3.7) and (3.34) we have

$$f_q(x) = W_{k_q}(x), \quad x \in G, \quad q \ge 1. \tag{3.35}$$

Next, taking (3.3), (3.9), (3.29) – (3.31) and (3.34) into account for all $q \ge 1$ we find that

$$\left(\int_0^1 \left| \sum_{j=1}^{q} \left[f_j(x) - \left(\sum_{n=\nu_{j-1}+1}^{\nu_j-1} U_n(x) + B_{\nu_j}^{(1)}(x) + U_{\nu_j}^{(2)}(x) \right) \right] \right| dx \right) \le$$
$$\le \int_0^1 \left| \left\{ W_{k_q}(x) - \sum_{j=1}^{q-1} \left[f_j(x) - \left(\sum_{n=\nu_{j-1}+1}^{\nu_j-1} U_n(x) + B_{\nu_j}^{(1)}(x) + U_{\nu_j}^{(2)}(x) \right) \right] \right\} - W_{\nu_q}(x) \right| dx +$$
$$+ \left(\int_0^1 \left| \sum_{j=1}^{q-1} \left[f_j(x) - \left(\sum_{n=\nu_{j-1}+1}^{\nu_j-1} U_n(x) + B_{\nu_j}^{(1)}(x) + U_{\nu_j}^{(2)}(x) \right) \right] \right| dx \right) +$$
$$+ \sum_{k=\nu_{q-1}+1}^{\nu_q-1} \left(\int_0^1 \left| U_k^{(1)}(x) \right| dx + \int_0^1 \left| U_k^{(2)}(x) \right| dx \right) + \int_0^1 \left| U_{\nu_q}^{(2)}(x) \right| dx < 2^{-2q-2}. \tag{3.36}$$

Finally, it follows from (3.28), (3.32) – (3.34) for all $q \ge 1$ that

$$\int_a^b \left| f_q(x) \right| dx \le \int_a^b \left| W_{k_q}(x) \right| dx + \int_a^b \left| B_{\nu_q}^{(1)}(x) \right| dx + \int_a^b \left| W_{\nu_q}(x) \right| dx <$$
$$< 2^{-8q} + 6 \int_a^b \left| W_{\nu_q}(x) \right| dx < 2^{-q-1}. \tag{3.37}$$

Clearly, using induction we can define increasing natural numbers $\nu_1 < \ldots < \nu_q < \ldots$, functions $W_{\nu_1}(x), \ldots, W_{\nu_q}(x) \ldots$, $g_1(x), \ldots, g_q(x) \ldots$, and polynomials $U_{\nu_1}^{(1)}(x), \ldots, U_{\nu_q}^{(1)}(x), \ldots$ to satisfy (3.31) – (3.37) for any $q \ge 1$.

It is clear that (see (3.37))

$$\sum_{q=1}^{\infty}\int_0^1\left|f_q(x)\right|dx<\infty. \tag{3.38}$$

We now determine the required function $f(x)$ and the sequence of numbers $\{\eta_k\}_{k=1}^{\infty}$ as follows:

$$f(x)=U_0(x)+\sum_{q=1}^{\infty}f_q(x), \tag{3.39}$$

$$\eta_k=\begin{cases}1, \ k\notin\bigcup\limits_{q=1}^{\infty}\left[M_{2\nu_q-1},M_{2\nu_q}\right)\\ \varepsilon_k^{(\nu_q)}, \ k\in\left[M_{2\nu_q-1},M_{2\nu_q}\right), \quad q=1,2,...\end{cases} \tag{3.40}$$

Using (3.27), (3.35), (3.38), and (3.39) we conclude that

$$f\in L^1[0,1]; \ \ f(x)=g(x), \quad x\in G.$$

Taking (3.3), (3.4), (3.13) – (3.16), (3.30), (3.36), (3.37), (3.39), and (3.40) into account, we find that

$$\left(\int_0^1\left|\sum_{k=0}^{M_{2\nu_q+1}-1}\eta_k c_k(U)\varphi_k(x)-f(x)\right|dx\right)\le$$

$$\le\left(\int_0^1\left|\sum_{j=1}^{q}\left[f_j(x)-\left(\sum_{n=\nu_{j-1}+1}^{\nu_j-1}U_n(x)+B_{\nu_j}^{(1)}(x)+U_{\nu_j}^{(2)}(x)\right)\right]\right|dx\right)+ \tag{3.41}$$

$$+\sum_{j=q+1}^{\infty}\left(\int_0^1\left|f_j(x)\right|dx\right)<2^{-q},$$

from this and (3.5) and (3.40) we have

$$\eta_k c_k(U)=c_k(f), \quad \eta_k=\pm1, \quad \left(\left|c_k(f)\right|=c_k(U)\right), \quad k=0,1,2,... \ . \tag{3.42}$$

Let $m\ge1$. Then for some $q\ge1$, $m\in\left[M_{2\nu_q-1},M_{2\nu_q+1}\right)$, and therefore

$$\sum_{k=0}^{m}c_k(f)\varphi_k(x)=\sum_{k=0}^{M_{2\nu_q+1}}c_k(f)\varphi_k(x)+\sum_{k=M_{2\nu_q-1}}^{m}c_k(f)\varphi_k(x). \tag{3.43}$$

Taking (3.3), (3.4), (3.29), (3.30), (3.34), and (3.40) – (3.43) into account, we get

$$\left(\int_0^1\left|\sum_{k=0}^{m}c_k(f)\varphi_k(x)-f(x)\right|dx\right)\le\left(\int_0^1\left|\sum_{k=0}^{M_{2\nu_q+1}}\eta_k c_k(U)\varphi_k(x)-f(x)\right|dx\right)+$$

$$+\left(\int_0^1\left|\sum_{k=M_{2\nu_q-1}}^{m}\eta_k c_k(U)\varphi_k(x)\right|dx\right)\le2^{-q}+\left(\int_0^1\left|\sum_{k=M_{2\nu_q-1}}^{m}\varepsilon_k^{(\nu_q)}b_k^{(\nu_q)}\varphi_k(x)\right|dx\right)+$$

$$\sum_{n=\nu_{q-1}+1}^{\nu_q-1}\left(\max_{m\in[M_{2n-1},M_{2n})}\int_0^1\left|\sum_{k=M_{2n-1}}^{m} b_k^{(n)}\varphi_k(x)\right|dx+\max_{m\in[M_{2n},M_{2n}+1)}\int_0^1\left|\sum_{k=M_{2n}}^{m} b_k^{(n)}\varphi_k(x)\right|dx\right)\le 2^{-q-1}$$

From this (since if $m\to\infty$ then as $q\to\infty$) it follows that the Fourier–Walsh series of the corrected function f converges to it in $L^1[0,1]$.

Theorem 6 is proved.

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ԱՄՓՈՓԱԳԻՐ

Ուոլշի համակարգի նկատմամբ պայմանական ունիվերսալ ֆունկցիաների մասին

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Հոդվածում ապացուցվում է հեղինակի կողմից սահմանված պայմանական ունիվերսալ ֆունկցիաների և ունիվերսալ եռյակների գոյության վերաբերյալ թեորեմներ, որոնց ապացուցման համար կիրառված մեթոդը հանդիսանում է նոր մոտեցում չափելի ֆունկցիաների դասում Ուոլշի համակարգով ունիվերսալ շարքեր կառուցելու համար: Փոքր չափի բազմության վրա փոփոխելով գրեթե ամենուրեք վերջավոր, չափելի ցանկացած ֆունկցիայի արժեքները՝ ստանալ նոր ֆունկցիա, և այդ ուղղված ֆունկցիայի Ուոլշի համակարգով Ֆուրիեի շարքի անդամների համար համապատասխան նշաններ ընտրելով՝ այն դարձնել ունիվերսալ շարք չափելի ֆունկցիաների դասում:

Բանալի բառեր՝ Ունիվերսալ ֆունկցիա, Ֆուրիեի շարք, Ուոլշի համակարգ