

# Parametrized Version of the Generalized Aubry-André Model

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**Abstract.** A recently introduced recurrence-relation ansatz applied to the Bose-Hubbard model is here used in the generalized Aubry-André model. The resulting modified Aubry-André model allows for a simple parametrization of the solutions in terms of three parameters, viz., the system energy when the quasiperiodicity amplitude  $\Delta = 0$ , the site  $\mu$  where the particle is initially localized, and the tuning parameter  $\alpha \in (-1, 1)$  that determines the regions of localized or extended states. The standard Aubry-André form corresponds to  $\alpha = 0$ .

**Keywords:** Bose-Hubbard model, generalized Aubry-André model, recurrence-relation ansatz, quasiperiodic site energies, mobility edge, extended and localized states

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## 1. Introduction

The Aubry-André (AA) model [1] is a one-dimensional crystal with incommensurate on-site energies that exhibits a localized ( $\Delta/J > 2$ ) state to an extended ( $\Delta/J < 2$ ) state transition, where  $\Delta$  is the amplitude of the on-site energy and  $J$  is the hopping energy. The AA model may be used to understand both quasicrystals and the Anderson localization metal-insulator transition in disordered systems. However, the AA model does not give rise to a mobility edge (ME), viz., an energy-dependent localization transition that demarcates localized from extended states as a function of energy. In order to generate a model that exhibits a ME, the AA model quasiperiodic site energies are generalized by introducing a tuning parameter  $\alpha$ , with  $-1 < \alpha < 1$  that controls the shape of the potential and the distribution of site energies [2]. The generalized Aubry-André (GAA) model has been experimentally realized exhibiting an exact ME [3].

This paper is structured as follows. In Sec. 2, we present the GAA model on an infinite, one-dimensional lattice. In Sec. 3, we introduce a recurrence-relation ansatz for the term associated with the hopping term. In Sec. 4, we present our numerical calculations and compare them to experiments of nearest-neighbor, tight-binding models having a quasiperiodic site energy. Finally, Sec. 5 summarizes our results.

## 2. Generalized Aubry-André model

The GAA Hamiltonian describes a single particle moving in a disordered lattice in one dimension with hopping between nearest-neighbor sites and quasiperiodic site energies,

$$\hat{H}_{GAA} = -J \sum_n (\hat{c}_{n+1}^\dagger \hat{c}_n + \hat{c}_n^\dagger \hat{c}_{n+1}) + \sum_n \epsilon_n \hat{c}_n^\dagger \hat{c}_n, \quad (1)$$

where  $\hat{c}_n^\dagger$  and  $\hat{c}_n$  represent the creation and annihilation operators at site  $n$ , respectively, and  $J$  is the hopping amplitude. The quasiperiodic site energy is

$$\epsilon_n = \Delta \frac{\cos(2\pi nb + \phi)}{1 - \alpha \cos(2\pi nb + \phi)}. \quad (2)$$

The irrational number is chosen as  $b = (\sqrt{5} - 1)/2$  with amplitude  $\Delta$ , phase  $\phi$ , and tuning parameter  $\alpha \in (-1, 1)$ . One obtains the AA model for  $\alpha = 0$ .

### 3. Ansatz

In order to obtain a simple parametrization of the GAA model, consider the following recurrence-relation ansatz [4] for the term associated with the hopping term of the  $n$ -th lattice site in (1)

$$\hat{c}_{n+1} = C(\hat{c}_n - \hat{c}_{n-1}). \quad (3)$$

and so

$$\sum_n (\hat{c}_{n+1}^\dagger \hat{c}_n + \hat{c}_n^\dagger \hat{c}_{n+1}) = B \sum_n \hat{c}_n^\dagger \hat{c}_n, \quad (4)$$

where we have considered an infinite long lattice and  $B = 2C/(1 + C)$  with  $B \in (-\infty, \infty)$ .

The GAA model (1) is thus reduced to

$$\hat{H}_{GAA} = \sum_n (\epsilon_n - JB) \hat{c}_n^\dagger \hat{c}_n. \quad (5)$$

One can add the atomic interaction energy that corresponds to an onsite interaction, where the atoms only see each other whenever they are in the same lattice site and so

$$\hat{H}_{GAA} = \sum_n (\epsilon_n - JB) \hat{c}_n^\dagger \hat{c}_n + U \sum_n \hat{c}_n^\dagger \hat{c}_n (\hat{c}_n^\dagger \hat{c}_n - 1), \quad (6)$$

with

$$U = \frac{4\pi\hbar^2 a_s}{m} \int |\omega(x)|^4 dx, \quad (7)$$

where  $a_s$  is the  $s$ -wave scattering length,  $m$  is the mass of the particle and  $\omega(x)$  is a Wannier function.

In order to determine the localization properties, localization is quantified by the participation ratio

$$PR = \frac{1}{\sum_n P_n^2}, \quad (8)$$

where  $P_n$  is the normalized atom population at site  $n$  or the inverse participation ratio

$$IPR = \sum_n P_n^2, \quad (9)$$

with  $\sum_n P_n = 1$ .

We consider a simple Lorentzian distribution for the normalized atom population  $P_n$  at site  $\mu$ , viz.,

$$P_n = \frac{1}{\sum_k \frac{J/\Delta}{(k-\mu)^2 + (J/\Delta)^2}} \frac{J/\Delta}{(n-\mu)^2 + (J/\Delta)^2}, \quad (10)$$

with  $0 \leq \Delta/J < \infty$ , where small values of  $J/\Delta$  correspond to a localized state, whereas large values of  $J/\Delta$  correspond to an extended state. The atomic interaction in (6) is thus given by

$$-1 + \frac{1}{N} \leq \sum_{n=1}^N P_n(P_n - 1) \leq 0, \quad (11)$$

where the upper bound corresponds to a localized state, whereas the lower bound corresponds to an extended state and  $N$  is the number of sites. Accordingly, the atomic interaction gives rise to a positive (negative) contribution to the overall energy for a negative (positive) scattering length. The former corresponds to an attractive potential, whereas the latter corresponds to a repulsive potential.

The constant  $B$  in (5) determines the value of  $E/J$  at  $\Delta/J = 0$  and is assumed to be a function of  $\mu$ , which is determined by the fact that the curves  $E/J$  do not cross except possibly at  $\Delta/J = 0$ .

#### 4. Numerical calculations

Consider the following expression for the energy in units of  $J$ ,

$$E(\Delta/J, B, \alpha, \mu)/J = \sum_{k=1}^N \left( \frac{\Delta}{J} \frac{\cos\left(\frac{2\pi(\sqrt{5}-1)k}{2} + \phi\right)}{1 - \alpha \cos\left(\frac{2\pi(\sqrt{5}-1)k}{2} + \phi\right)} - B(\mu) \right) P_k(\mu), \quad (12)$$

where  $P_k(\mu)$  is given in (10) and we have set  $U = 0$ , viz., zero scattering length. Note that  $E/J \rightarrow -B(\mu)$  as  $\Delta/J \rightarrow 0$ . The GAA model exhibits an exact ME at energy  $E$  for  $\alpha \neq 0$  for both  $J > 0$  and  $\Delta > 0$  given by [2]

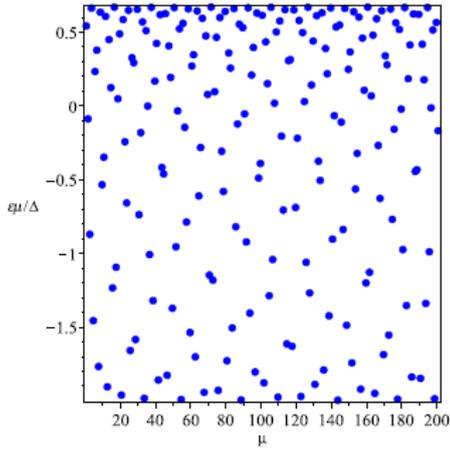
$$\alpha E = 2J - \Delta. \quad (13)$$

There ought to be no level crossing in (12) except possibly at  $\Delta/J = 0$ . The intersection of two energies would require

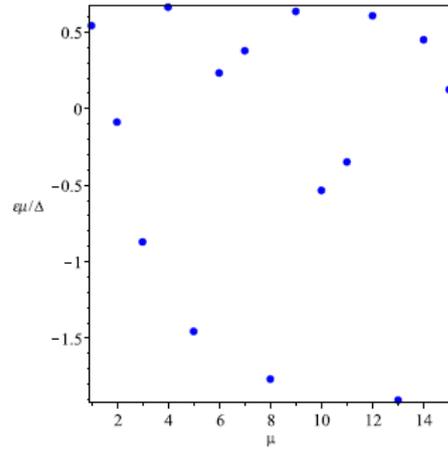
$$\sum_{k=1}^N \left( \frac{\Delta}{J} \frac{\cos\left(\frac{2\pi(\sqrt{5}-1)k}{2} + \phi\right)}{1 - \alpha \cos\left(\frac{2\pi(\sqrt{5}-1)k}{2} + \phi\right)} \right) (P_k(\mu) - P_k(\mu')) = B - B'. \quad (14)$$

The right-hand-side of (14) and  $P_k(\mu)$  are rational numbers, whereas the left-hand-side of (14) is an irrational number for  $\Delta/J$  and  $\alpha$  rational. Accordingly, our ansatz (3) preserves the non-crossing of energy eigenstates in (12) as is in (1) except possible at  $\Delta/J=0$ . Therefore,  $\epsilon_{\mu'} > \epsilon_{\mu}$  implies that  $B' > B$ .

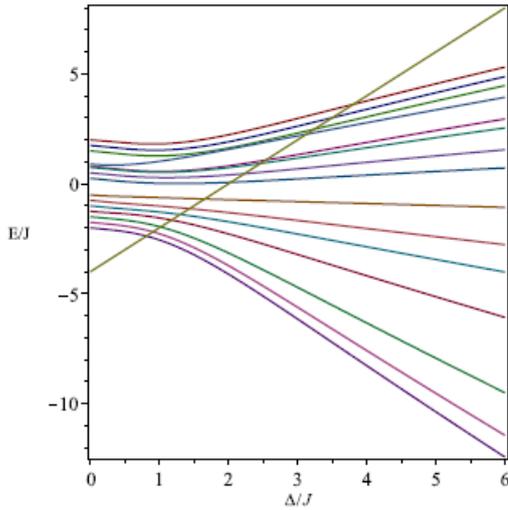
Consider atoms initially localized, viz.,  $\Delta/J \rightarrow \infty$ , in the GAA model for  $\alpha = -0.5$ ,  $\phi = \pi$  and  $N=201$ . Fig1(a) shows the pointplot of  $\epsilon_{\mu}/\Delta$  vs.  $\mu$  from (2) of initially localized states plotted up to  $\mu=201$ , whereas we have only plotted up to  $\mu = 15$  in Fig. 1(b). Note for  $\Delta/J \gg 1$ , the value of  $E/J$  is actually independent of  $B$ . Accordingly, we consider atoms initially localized for  $\Delta/J = \infty$  are slowly loaded into an eigenstate of the GAA model at a final quasiperiodicity-to-tunneling ratio  $\Delta/J$ . In Fig. 1(c), the plot shows  $E/J$  vs.  $\Delta/J$  for the range  $-2 \leq B \leq 2$  for the first fifteen values of  $\mu$  and choosing the value of  $B$  guided by the principle of no level crossing. Clearly, we could have included all  $N = 201$  initial states and thus compare our results to those of Ref. 3 more closely. However, the present results are quite consistent with those of Ref. 3. The region  $E/J < -4 + 2\Delta/J$  represents extended states; whereas the region  $E/J > -4 + 2\Delta/J$  represents localized states. Fig. 1(d) shows the participation ratio per unit site for  $\mu = 100$  and  $N = 201$ , which is actually independent of the value of the tuning parameter  $\alpha$ . The plot shows a maximum at  $\Delta/J = 0$  and is evaluated at the center of the crystal, viz.  $\mu = 100$ , but the result is not strongly dependent on the value of the site  $\mu$ . Our result shares the behavior shown in Fig. 2 (c) of Ref. 3.



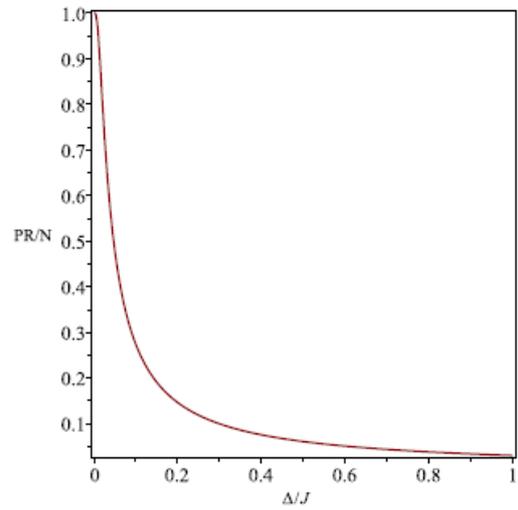
(a) Fig. 1. Pointplot of  $\epsilon_{\mu}/\Delta$  vs.  $\mu$  from (2) for  $\alpha = -0.5$ ,  $\phi = \pi$ , and  $N = 201$ .



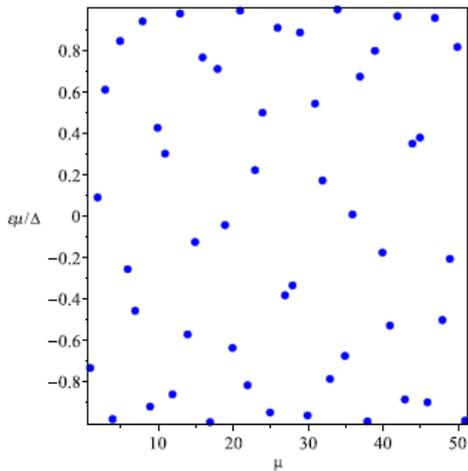
(b) Fig. 1. Same as Fig. 1(a) but including only the first fifteen values of  $\mu$ .



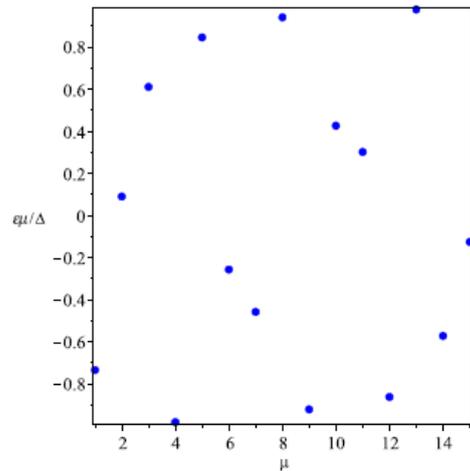
(c) Fig. 1. Eigenenergies  $E/J$  vs.  $\Delta/J$  of the GAA model for  $\alpha = -0.5$ ,  $\phi = \pi$  and  $N = 201$  for the first fifteen sites as shown in Fig. 1(b). The ME edge is given by  $E/J = -4 + 2\Delta/J$  (green).



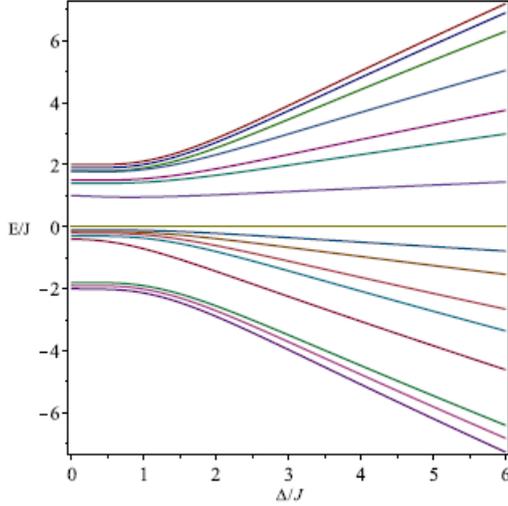
(d) Fig. 1. Participation ratio per unit site  $PR/N$  vs.  $\Delta/J$  for  $\mu = 100$  and  $N = 201$ .



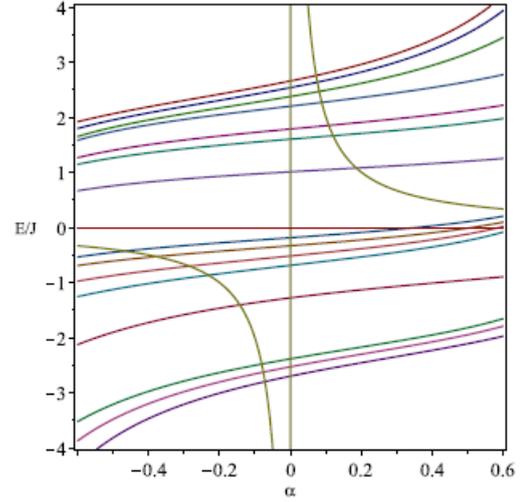
(a) Fig. 2. Pointplot of  $\epsilon_{\mu}/\Delta$  vs.  $\mu$  from (2) for  $\alpha = 0$ ,  $\phi = 0$ , and  $N = 51$ .



(b) Fig. 2. Same as Fig. 2(a) but including only the first fifteen values of  $\mu$ .



(c) Fig. 2. Eigenenergies  $E/J$  vs.  $\Delta/J$  of the GAA model for  $\alpha = 0$ ,  $\phi = 0$  and  $N = 51$  for the first fifteen sites as shown in Fig. 2(b).



(d) Fig. 2. Eigenenergies  $E/J$  (12) vs.  $\alpha$  of the GAA model for  $\Delta/J = 1.8$  and  $N = 51$  for the first fifteen sites as shown in Fig. 2(b). The values of  $\mu$  and  $B$  are those obtained from Fig. 2(c). The ME is given by  $E/J = .2/\alpha$  (green).

Results in Fig. 2(a)-2(c) are obtained similarly as those shown in Figs. 1(a)-1(c), albeit, for  $\alpha=0$ ,  $\phi = 0$ , and  $N = 51$ . The plot in Fig. 2(d) shows the eigenenergies just below the critical quasiperiodicity strength at  $\Delta/J = 1.8$ . The ME, viz.,  $E/J = .2/\alpha$ , demarcates the extended from the localized states. The localized states are in the region  $E(\alpha)/J > .2/\alpha$  for  $\alpha > 0$  and  $E(\alpha)/J < -.2/\alpha$  for  $\alpha < 0$ . The complementary region,  $-.2/\alpha < E/J < .2/\alpha$ , corresponds to extended states. The results qualitatively agree with those of Ref. 3.

## 5. Conclusions

We have presented a parametrized version of the generalized Aubry-André model and evaluated the eigenenergies  $E$  as a function of the quasiperiodicity amplitude  $\Delta$  and separately as a function of the tuning parameter  $\alpha$  for given value of  $\Delta$ . Our results are in reasonably good agreement with the experimentally realized family of one-dimensional, nearest-neighbor, tight-binding models with quasiperiodic site energy, which model can be viewed as a generalization of the Aubry-André model.

## References

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