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Aggrandization of spaces: a new general approach and application

Alexey Karapetyants[1,*]

[1] Southern Federal University, Rostov-on-Don, Russia

[*]Correspondence: karapetyants@gmail.com

Abstract
Our study is based on recent results concerning the so-called local aggrandization of Lebesgue spaces. We extend this framework to the case of arbitrary Banach spaces of functions on metric spaces. Furthermore, we demonstrate that grand spaces of holomorphic functions can be equivalently defined through aggrandization associated exclusively with the boundary. This paper review presents recent results obtained in collaboration with Stefan Samko and provides a concise introduction to the classical theory of grand spaces for comparative analysis. We also discuss the underlying motivation for investigating these spaces.

Keywords: Grand spaces, Holomorphic functions, Aggrandization
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1. Introduction

Drawing on recent results concerning the so-called local aggrandization of Lebesgue spaces (see e.g., [7,8,9]), we present a new approach to the construction of grand spaces. This framework was recently introduced and developed in [5] in the case of arbitrary Banach spaces of functions on metric spaces. By examining specific classes of analytic functions, we demonstrate that grand spaces of holomorphic functions can be equivalently defined via an aggrandization associated solely with the boundary. Although the problem of aggrandizing analytic spaces was the initial objective for this research, it became immediately apparent that this concept could be applied to a wide range of Banach spaces of functions on metric spaces.

To better illustrate the underlying concept, we consider the specific case of p-integrable holomorphic functions on the unit disc in the complex plane. By applying the general method of local aggrandization to such spaces, we establish the following equivalence of norms:

$$\sup_{0<\varepsilon<p-1} \varepsilon^{\theta} \left(\int_{\mathbb{D}} |f(z)|^{p-\varepsilon} \, dA(z) \right)^{\frac{1}{p-\varepsilon}}$$

$$\approx \sup_{0<\varepsilon<l} \varepsilon^{\theta} \left(\int_{\mathbb{D}} |f(z)|^{p} \, (1-|z|)^{\lambda\varepsilon} \, dA(z) \right)^{\frac{1}{p}}$$

for all $\lambda > 0$ and $l > 0$. In fact, we prove more general statement with $(1-|z|)^{\lambda\varepsilon}$ replaced by $a(1-|z|)^{\varepsilon}$, under some assumptions on the function a, called aggrandizer.

Naturally, this theory allows for further development, particularly through the selection of specific function spaces. Our objective here, however, is not to pursue these extensions, but rather to illustrate the core idea using a fundamental case, while simultaneously presenting the definition in the most general setting to date.

As previously noted, the main definition of the aggrandization and proofs of the theorems presented in Sections 3, 4 originally appeared in [5]; we refer to this source for more details and generalizations.

2. Classical approach. Definitions and some results

We do not aim to provide a comprehensive survey of the vast literature on classical grand spaces; we mention only a few key works: [1,2,3,4].

2.1. What is Grand Lebesgue space over finite measure domain Ω

Given an open set $\Omega \subset \mathbb{R}^n$, $|\Omega| = 1$, $1 < p < \infty$, and $\theta > 0$, the grand Lebesgue space $L^{p),\theta}(\Omega)$ consists of all measurable on Ω functions f such that

$$\| f \|_{p),\theta} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^{\theta} \int_{\Omega} | f(x) |^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}. \tag{2.1}$$

Let us outline basic relations with Lebesgue spaces. For a finite measure domain Ω we have:

$$L^p \subset L^{p)} \subset L^{p-\varepsilon}, \quad L^{p+\varepsilon} \subset L^{(p} \subset L^p, \quad L^p \subset L^{p,\infty} \subset L^{p)}$$

$$L^p \subset L^p \log^{-1} L \subset L^{p)} \subset \cap_{\alpha>1} L^p \log^{-\alpha} L$$

The fundamental function provides insight into how grand and small Lebesgue spaces differ from the classical Lebesgue space. The following result can be inferred from the work of Alberto Fiorenza; however, its formulation in terms of the Lambert function first appeared in a paper by S. Umarkhadzhiev.

Lemma 2.1 ([6]). Let $1 < p < \infty$, $\theta > 0$, Ω be an open set in $\mathbb{R}^n$, $|\Omega| = 1$, and E be an open set in Ω with $|E| = \eta$. Then

$$\| \chi_E \|_{p),\theta} = \begin{cases} e^{\frac{\theta}{p} W_{-}\left(-\frac{p\eta^{1/\theta}}{e}\right)}, & 0 < \eta^{\frac{1}{\theta}} < \frac{1}{p-1} e^{-\frac{1}{p-1}}, \\ (p-1)^{\theta} \eta, & \frac{1}{p-1} e^{-\frac{1}{p-1}} \leqslant \eta^{\frac{1}{\theta}} < 1, \end{cases} \tag{2.2}$$

where W_{-} is the branch of Lambert function W, which yields

$$\| \chi_E \|_{p),\theta} \simeq \eta^{\frac{1}{p}} \ln^{-\frac{\theta}{p}} \frac{e}{\eta}, \qquad \| \chi_E \|_{(p,\theta} \simeq \eta^{\frac{1}{p}} \ln^{\frac{\theta}{p'}} \frac{e}{\eta}.$$

2.2. Whether there is a difference between a space of p - integrable holomorphic functions and its grand counterpart

The answer is affirmative for $\mathcal{A}^p(\mathbb{D})$ and $\mathcal{A}^{p),\theta}(\mathbb{D})$. More precisely,

$$\frac{1}{(1-z)^{\lambda}} \in \mathcal{A}^p(\mathbb{D}) \Leftrightarrow \lambda < \frac{2}{p},$$

see Duren and Schuster book (2004), while it is easily checked that

$$\frac{1}{(1-z)^{\frac{2}{p}}} \in \mathcal{A}^{p),\theta}(\mathbb{D}), \quad \theta = 1.$$

Moreover, we know that (see [10]):

$$g_\theta(z) = \frac{1}{(1-z)^{\frac{2}{p}}} \ln^{\frac{\theta-1}{p}} \frac{e}{1-z} \in \mathcal{A}^{p),\theta}(\mathbb{D})$$

under the proper choice of the branch of logarithmic function.

3. Local aggrandization of function spaces over metric spaces: a general approach

Let (Λ, d) be a metric space and $\mathcal{D} = \operatorname{diam}\Lambda$, $0 < \mathcal{D} \leqslant \infty$. Given a closed non empty subset $F \subset \Lambda$, denote

$$\delta_F(x) = \inf_{y\in F} d(x,y), \quad x \in \Lambda.$$

Let $X = X(\Lambda)$ be an arbitrary normed space of functions $f:\Lambda \to \mathbb{C}$ and $\|\cdot\|_X$ be its norm.

With the space X we associate the normed space $X_w = X(\Lambda, w)$ depending on a functional parameter $w:\Lambda \to \mathbb{R}_+$, and assume that $X_w = X$, for $w = 1$. By $\|\cdot\|_{X,w}$ we denote the norm in X_w. We assume that w is nonnegative. Also, in what follows we assume that the space X_w possesses the following property (lattice property with respect to weights): for any two weights u, v such that $u \leqslant v$, the following inequality holds

$$\| f \|_{X,u} \leqslant \| f \|_{X,v}\ .$$

Definition 3.1. Let $F \subset \Lambda$ be a closed non-empty set. For a positive almost increasing function $a \in L^\infty(0,\mathcal{D})$, $a(0+) = 0$, we define the local grand space

$$X^g_{F,a,\theta} = X^g_{F,a,\theta}(\Lambda)$$

related to the space X, by the norm

$$\| f \|_{X^g_{F,a,\theta}} = \sup_{0<\varepsilon<l} \left(\varepsilon^\theta \| f \|_{X,(a\circ\delta_F)^\varepsilon} \right), \ l > 0. \tag{3.3}$$

The function a used in definition of $X^g_{F,a,\theta}$ will be referred to as aggrandizer. The embedding

$$X \subset X^g_{F,a,\theta} \tag{3.4}$$

holds because $a \in L^\infty(0,\mathcal{D})$ by definition.

4. Application to the space of holomorphic functions
4.1. Let us recall classical definitions

Let $\Lambda = \mathbb{D}$ be the unit disc in $\mathbb{C}$. We identify $\mathbb{R}^2 \equiv \mathbb{C}$ so that $(x,y) = z$. Let $\mathrm{dA}(z) = \frac{1}{\pi} dxdy$. Let $\rho = \rho(t)$. In the sequel we always assume that for arbitrary $\delta \in (0,1)$

$$0 < \inf_{t\in(\delta,1)} \rho(t) \le \sup_{t\in(\delta,1)} \rho(t) < \infty.$$

The weighted Lebesgue space $L^p(\mathbb{D},\rho)$ is defined by the norm

$$\| f \|_{L^p(\mathbb{D},\rho)} = \left(\int_{\mathbb{D}} | f(z) |^p \rho(1-|z|) \mathrm{dA}(z) \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty.$$

Let $\mathcal{A}^p(\mathbb{D},\rho)$ stand for the subspace of $L^p(\mathbb{D},\rho)$, which consist of functions holomorphic in $\mathbb{D}$.

For $1<p<\infty$ and $\theta>0$, the (classical weighted) grand Lebesgue space $L^{p),\theta}(\mathbb{D},\rho)$ consists of all functions f measurable on $\mathbb{D}$ such that

$$\|f\|_{L^{p),\theta}(\mathbb{D},\rho)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\theta} \|f\|_{L^{p-\varepsilon}(\mathbb{D},\rho)} < \infty. \tag{4.5}$$

The corresponding subspace of holomorphic functions will be denoted by $\mathcal{A}^{p),\theta}(\mathbb{D},\rho)$.

4.2. Local aggrandization of the space $L^p(\mathbb{D},\rho)$

We take $F=\mathbb{T}$, $\delta_{\mathbb{T}}(z)=1-|z|$. We find it convenient to use the notation $L^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho)$ for the aggrandized space for $L^p(\mathbb{D},\rho)$. Same for $\mathcal{A}^{p)}_{\mathbb{T},a\theta}(\mathbb{D},\rho)$. Hence,

$$\|f\|_{L^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho)} = \sup_{0<\varepsilon<l}\left(\varepsilon^{\theta}\|f\|_{L^p(\mathbb{D},\rho),(a\circ\delta_{\mathbb{T}})^{\varepsilon}}\right)$$

$$= \sup_{0<\varepsilon<l}\left(\varepsilon^{\theta}\int_{\mathbb{D}}|f(z)|^p\,(a\circ\delta_{\mathbb{T}})(z)^{\varepsilon}\,\rho(1-|w|)\,\mathrm{dA}(z)\right)^{\frac{1}{p}},$$

where $l>0$ (we will assume that $l<p-1$).

4.3. The inclusions between classical grand spaces and spaces constructed via local aggrandization

Here, we illustrate our approach by comparing classical grand spaces with those constructed via the local method, specifically for the case of weighted spaces of holomorphic functions. In fact, this comparison demonstrates that under minimal assumptions, the spaces coincide, up to equivalence of norms. This allows for the interchangeability of norms in applications and further research, enabling the choice of the most suitable or productive norm for each specific case.

Recall, that a function φ satisfies the reverse doubling condition if there exists $c_{(2)}>0$ such that $\varphi(t)\leqslant c_{(2)}\varphi(2t)$, $t>0$.

Theorem 4.3. The following statements hold true.

(1) Let there exist $\delta>0$ such that

$$\int_0^1 \frac{\rho(t)}{a(t)^{\delta}}\,dt<\infty.$$

Then

$$\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho)\hookrightarrow\mathcal{A}^{p),\theta}(\mathbb{D},\rho).$$

(2) Let any of the two following conditions be satisfied:

(a) the function $\rho=\rho(t)$ is almost decreasing and satisfying the reverse doubling condition and there exist $\eta>0$, $c_a>0$ and $\varepsilon_0\in(0,p-1)$ such that

$$a(t)\leqslant c_a\left(t^{\frac{2}{p}}\rho(t)^{\frac{1}{p-\varepsilon_0}}\right)^{\eta}, \qquad t\in(0,1),$$

(b) the function $\rho=\rho(t)$ is almost increasing and $M(\rho)<p-1$, and there exist $\eta>0$ and $c_a>0$, such that

$$a(t) \leqslant c_a \left(t^{\frac{2}{p}} \rho(t)^{\frac{1}{p}} \right)^{\eta}, \qquad t \in (0,1).$$

Then

$$\mathcal{A}^{p),\theta}(\mathbb{D},\rho) \hookrightarrow \mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho).$$

The case of a power weight is of particular interest.

Theorem 4.4. Let $\rho(t) = t^{\gamma}$, $-1 < \gamma < p-1$, and there exists $\delta > 0$ such that

$$\int_0^1 \frac{t^{\gamma}\, dt}{a(t)^{\delta}} < \infty,$$

and there exist $C > 0$ and $\eta > 0$ such that

$$a(t) \leqslant C t^{\eta}.$$

Then the space $\mathcal{A}^{p),\theta}(\mathbb{D},\rho)$ coincides with the space $\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho)$, up to equivalence of norms.

We single out particular corollary for the unweighed case: $\rho = 1$. We use symbols $\mathcal{A}^{p),\theta}(\mathbb{D})$ and $\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D})$ for the unweighed spaces.

Theorem 4.5. The space $\mathcal{A}^{p),\theta}(\mathbb{D})$ coincides with the space $\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D})$ with any aggrandizer a satisfying $m(a) > 0$, up to equivalence of norms.

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