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Quantum interference of a de Broglie wave of a Dirac particle beyond the `hypothesis of locality´. Part II. Hermicity and non-relativistic limit

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Abstract

This is the second of three articles that explore the possibility of quantum mechanical inertial properties of the Dirac particle beyond the, so-called, `hypothesis of locality´, of standard approach. This is done within the framework of the *Master Space*-Teleparallel Supergravity ($\widetilde{MS}_p$-TSG) Ter-Kazarian (2025a) theory, which we recently proposed to account for inertial effects Ter-Kazarian (2026). Our strategy in Ter-Kazarian (2025b) (first article of three) is to compute the object of anholonomicity and connection defined with respect to the anholonomic frame. Based on this, we derived the general Dirac equation in an accelerated and rotating frame of reference beyond the`hypothesis of locality´. This equation, however, contains also residual imaginary terms, which are artifacts that due to coordinate transformations in the non-inertial frames. To eliminate these terms to all orders, in present article, at first, we apply the standard techniques used in relativistic quantum mechanics and quantum field theory, where non-Hermitian terms can be removed via suitable similarity transformations. This standard method allows us to choose a physically more suitable reference frame. We show that the expectation values of physical observables remain real. No imaginary contamination remains in physical quantities. Thus the energy, momentum, probability, etc., remain real and consistent. Secondly, we are interested in low-energy properties, avoiding solutions with negative energy. In the method employed for reducing the Dirac Hamiltonian to non-relativistic two-component form, in order to decouple the positive and the negative energy states, we use an approximate scheme of the Foldy-Wouthuysen canonical transformation of the Dirac Hamiltonian for a free particle. This is performed by an infinite sequence of FW-transformations leading to a deformed Hamiltonian, which is an infinite series in powers of $(1/m)$. Evaluating the operator products to the desired order of accuracy, we derive the deformed, non-relativistic Hamiltonian. We then compute the inertial effects for a massive Dirac fermion in non-relativistic approximation, which are displayed beyond the `hypothesis of locality´ as extended (deformed) versions of the standard effects. The latter are well-known important inertial effects such as the redshift effect (Colella-Overhauser-Werner experiment), the Sagnac-type effect, the spin rotation effect (Mashhoon), the kinetic energy redshift effect, the new inertial spin-orbit coupling. Expanding further the deformation coefficients, several new effects will rather appeared involving spin, angular momentum, proper linear 3-acceleration $\vec{a}$ and proper 3-angular velocity $\vec{\omega}$ in various mixed combinations.

Keywords: *Teleparallel Supergravity–Spacetime Deformation–Inertia Effects–Quantum interference*

1. Introduction

The quantum interference of De Broglie matter waves is vividly displayed after a notable development of interferometric technique in the gravity-induced interference experiments with symmetric (sym.) and skew-symmetric silicon interferometers. These experiments are reviewed by (Abele & Leeb, 2012, Hasegawa & Rauch, 2011, Rauch & Werner, 2000). A wide, spatially coherent separation of neutron beams is feasible in perfect crystal interferometers. In this case nuclear, magnetic and gravitational phase shifts can be created and measured precisely. The effect of the Earth's rotation on the phase of a neutron de Broglie wave is the quantum mechanical analogue of the classic Michelson-Gale-Pearson experiment (Michelson et al., 1925) using the optical `Foucault pendulum´ - the Michelson-Gale light interferometer - in which the rotation of Earth yielded a Sagnac shift of the light waves. The offset in the fringe pattern introduced by rotational

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motion of neutrons was predicted by Page (Page, 1975). This effect was seen in the celebrated Colella-Overhauser-Werner (COW) experiment (Colella et al., 1975). COW observed a quantum-mechanical phase shift of neutrons caused by their interaction with the Earth's gravitational field. The signal is based on the interference between coherently split and separated neutron de Broglie waves in the gravity potential. The interferometer is turned around the incident beam direction by some angle maintaining the Bragg condition. The amplitudes are divided by dynamical Bragg diffraction from perfect silicon crystals. Acceleration-induced interference is discussed in (Bonse & Wroblewski, 1983). They found a result which was within 4% of the theoretical prediction. The (Colella et al., 1975) and (Bonse & Wroblewski, 1983) experiments have the potential to test the equivalence principle (Kajari et al., 2010) for the nonrelativistic neutron waves. Previously, the neutron Sagnac phase shift due to the Earth's rotation was measured (Atwood & et al., 1984, Staudenmann et al., 1980) and was shown to be in agreement with the theory of the order of a few per cent. There are also forthcoming and more speculative interferometric measurements which are intended to prove or disprove alternative theories and to search for extremely small effects where only upper limits can be given. Some of the phenomena are already under investigation and a positive outcome can be expected. However, the measurement of relativistic terms in neutron interference must await the development of highly monochromatic neutron beams or the discovery of interferometric techniques for high-energy neutrons.

In the meantime, the theoretical studies of the relativistic quantum theory in a curved spacetime have predicted a number of interesting manifestations of the spin-gravity coupling for a Dirac particle, see e.g. (Audretsch & Schafer, 1978, Cai & Papini, 1991, 1992, Fischbach et al., 1981, Hehl & Ni, 1990, Obukhov, 2001, 2002, Ryder, 1998, Singh & Papini, 2000, Varjú & Ryder, 1998, 2000, de Oliveira & Tiomno, 1962).

For a performing the laboratory measurements, it is necessary to give a theoretical description of the measurements of accelerated observers. This is, usually, done via the `hypothesis of locality´, used to extend Lorentz invariance to accelerated observers within the framework of Special Relativity, see e.g. (Hehl & Ni, 1990, Hehl et al., 1991, Maluf & Faria, 2008, Maluf et al., 2007, Marzlin, 1996, Mashhoon, 2002, 2011, Misner et al., 1973, Synge, 1960) and references therein. However, many scientists found its basic conceptual framework unsatisfactory. In general case, the hypothesis of locality will have to be extended to describe physics for arbitrarily accelerated observers.

In Ter-Kazarian (2025b) (first article of three), we computed the object of anholonomicity and the connection defined with respect to the anholonomic frame. Then we derived the explicit final form of the Dirac equation for an observer in a reference frame that is accelerated with a three-acceleration $\vec{a}$ and rotating with angular frequency $\vec{\omega}$. However, the purely imaginary potential term from the Dirac Hamiltonian is associated with non-Hermitian contributions due to coordinate transformations in accelerated frames. Residual imaginary terms are artifacts. To eliminate to all orders these terms, in present article we apply the standard techniques used in relativistic quantum mechanics and quantum field theory, where non-Hermitian terms can be removed via suitable similarity transformations. This standard method allows to choose a physically more suitable reference frame. The expectation values of physical observables remain real. No imaginary contamination remains in physical quantities. Thus the energy, momentum, probability, etc. remain real and consistent. We are also interested in low-energy properties, avoiding solutions with negative energy. In the method employed for reducing the Dirac Hamiltonian to non-relativistic two-component form, in order to decouple the positive and the negative energy states, we use an approximate scheme of the Foldy-Wouthuysen canonical transformation of the Dirac Hamiltonian for a free particle. This is performed by an infinite sequence of FW-transformations leading to a deformed Hamiltonian, which is an infinite series in powers of $(1/m)$. Evaluating the operator products to the desired order of accuracy, we find the deformed, non-relativistic Hamiltonian. We then find the inertial effects for a massive Dirac fermion in non-relativistic approximation, which are displayed beyond the `hypothesis of locality´ as extended (deformed) versions of the standard effects. The latter are well-known important inertial effects such as the redshift effect (Colella-Overhauser-Werner experiment), the Sagnac-type effect, the spin rotation effect (Mashhoon), the kinetic energy redshift effect, the new inertial spin-orbit coupling. Expanding further the deformation coefficients, several new effects will rather appeared involving spin, angular momentum, proper linear 3-acceleration $\vec{a}$ and proper 3-angular velocity $\vec{\omega}$ in various mixed combinations.

We proceed according to the following structure. To start with, in section 2 we eliminate a purely imaginary potential from the Dirac Hamiltonian: the imaginary term $V_{1\text{imag}}$ in subsect. 2.1; the $V_{2\text{imag}}$ to first order in BCH expansion in subsect. 2.2; the imaginary term V_{imag} to second order in BCH expansion in subsect. 2.3; and to higher order in subsect. 2.4. We consider non-relativistic approximation of the Dirac Hamiltonian via FW transformation in section 4: FW expansion up to order $1/m$ in subsect. 3.1; and FW expansion up to order $1/m^2$ in subsect. 3.2. Concluding remarks are given in section 4, where we review

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the key points of this report. Some technical details are collected in Appendix. Unless indicated otherwise, the natural units, $h = c = 1$ are used throughout.

2. Eliminating a purely imaginary potential from the Dirac Hamiltonian

To make this article understandable, the interested reader is referred to the original papers (Ter-Kazarian, 2024a, 2025a,b, 2026) (see also (Ter-Kazarian, 2024b,c,d)). In addition, in Appendix A, we briefly review the main points of the derivation of Dirac equation in non-inertial frame beyond the hypothesis of locality.

The Dirac equation (82) can be recast into the form (Ter-Kazarian, 2025b)

$$i\partial_0\Psi = H\Psi, \tag{1}$$

with the deformed Dirac Hamiltonian

$$\begin{aligned} &H = b_3\beta m(1+\vec{a}\cdot\vec{X}) + \tfrac{b_3}{b_1}\,\vec{\alpha}\cdot\vec{p} - \vec{\omega}\cdot\vec{L} \\ &-\vec{\omega}_S\cdot\vec{S} + \tfrac{i}{2}(\vec{a}_2\cdot\vec{\alpha}+a_3) + \tfrac{b_3}{2b_1}\Big[(\vec{a}\cdot\vec{X})(\vec{p}\cdot\vec{\alpha}) \\ &+(\vec{p}\cdot\vec{\alpha})(\vec{a}\cdot\vec{X})\Big]. \end{aligned} \tag{2}$$

where $\vec{a}_2 \equiv (\frac{1}{b_1}\nabla b_0 + 2b_0\nabla b_1^{-1})$, and $a_3 \equiv [3b_4 b_0\nabla b_1^{-1} - \nabla b_4]\cdot\vec{v}$. The Dirac Hamiltonian (2) can be conveniently rewritten

$$\begin{aligned} &H = w_1\beta m(1+\vec{a}\cdot\vec{X}) + w_3\,\vec{\alpha}\cdot\vec{p} - \vec{\omega}\cdot\vec{L} \\ &-w_2\,\vec{\omega}\cdot\vec{S} + \tfrac{i}{2}(w_4\vec{a}\cdot\vec{\alpha}+w_5) + \tfrac{w_3}{2}\,\Big[(\vec{a}\cdot\vec{X})(\vec{p}\cdot\vec{\alpha}) \\ &+(\vec{p}\cdot\vec{\alpha})(\vec{a}\cdot\vec{X})\Big], \end{aligned} \tag{3}$$

provided, $w_1 \equiv b_3, \quad w_2\,\vec{\omega} \equiv \vec{\omega}_S, \quad w_3 \equiv \frac{b_3}{b_1}, w_4\vec{a} \equiv \vec{a}_2, w_5 \equiv a_3$. The coefficients $w_i(X)(i = 1, 2, 3, 4, 5)$ are time-independent scalar functions of X, ($w_i(X) \in \mathbb{R}^3$).

Let's briefly interpret the physical meaning of each term before relating this to curved spacetime, see e.g. Brill & Wheeler (1957), Hehl & Ni (1990), Hehl et al. (1991), Maluf & Faria (2008), Maluf et al. (2007), Mashhoon (2002, 2011), Obukhov & Rubilar (2006), Parker & Toms (2009). Mass term, $w_1\beta m(1 + \vec{a}\cdot\vec{X})$, represents a gravitational redshift of mass energy in a weak field. This matches the Tolman redshift.

Orbital term, $\vec{\omega}\cdot(\vec{X}\times\vec{p})$, describes Coriolis force and orbital coupling in rotating frames. This is effective term of energy shift due to rotation. In general relativity, this is the Lense–Thirring effect — frame dragging by rotating mass.

Spin-Rotation Coupling: $w_2\,\vec{\omega}\cdot\vec{\sigma}$. This is the Mashhoon effect. This is spin experiences torque in rotating frames, which is analogous to magnetic dipole in a magnetic field. Can be derived from the spin connection ω_μ^{ab} in curved spacetime: spin term $\sim \frac{1}{4}\gamma^a\gamma^b\omega_{ab\mu}$.

The symmetrized inertial boost term,
$\frac{w_3}{2}\Big[(\vec{a}\cdot\vec{X})(\vec{p}\cdot\vec{\alpha}) + (\vec{p}\cdot\vec{\alpha})(\vec{a}\cdot\vec{X})\Big]$, arises from a non-inertial, accelerated, frame and reflects: boosts in the Dirac equation, and the non-trivial tetrad field structure. This term can be derived from the Fermi normal coordinates for a uniformly accelerated observer.

Residual imaginary terms figured out in the Dirac Hamiltonian (3) are artifacts that due to coordinate transformations in the non-inertial frames. In what follows, we will show that these are usually negligible under the assumption that b_1, b_3, b_4 vary slowly with $\vec{X}$, so their gradients are small, and these corrections are suppressed. Moreover, the standard method of similarity transformation of the Hamiltonian allows one to choose a physically more suitable reference frame. The expectation values of physical observables remain real. No imaginary contamination remains in physical quantities. Thus the energy, momentum, probability, etc. remain real and consistent.

To eliminate to all orders the purely imaginary potential term, from the Dirac Hamiltonian (3), associated with non-Hermitian contributions due to coordinate transformations in accelerated frames, we can apply the standard method, which is inspired by techniques used in relativistic quantum mechanics and quantum field theory, where non-Hermitian terms can be removed via suitable transformations. Such technique often appears, particularly in contexts like axial vector couplings, spin-orbit interactions, or anomalous terms in the Hamiltonian. This can be addressed using Foldy-Wouthuysen (FW) transformations to remove off-diagonal terms in the previous literature on the Dirac Hamiltonian, especially Anandan (1977), Birrell & Davies (1982), Bjorken & Drell (1964), Foldy & Wouthuysen (1950), Greiner (2000), Mostafazadeh (2002),

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Obukhov (2001), Tsai (1981); also methods similar to those found for chiral rotations and axial terms Bjorken & Drell (1964), Peskin & Schroeder (1995); and non-Hermitian potential removal Bender & Boettcher (1998).

We begin by removing a purely imaginary term in (3) via a similarity transformation to first order in BCH expansion. For the Dirac Hamiltonian (3)

$$H = H_0 + V_{\rm imag}, \tag{4}$$

where $V_{\rm imag} = V_{1\rm imag} + V_{2\rm imag}$, $V_{1\rm imag} := \frac{i}{2}w_5$ is the scalar imaginary part, $V_{1\rm imag} := \frac{i}{2}w_4\,\vec{a}\cdot\vec{\alpha}$ is the matrix-valued imaginary part, and the part of Hamiltonian H_0 free of imaginary contributions

$$\begin{array}{l} H_0 = w_1\beta m(1+\vec{a}\cdot\vec{X}) + w_3\vec{\alpha}\cdot\vec{p} - \vec{\omega}\cdot\vec{L} \\ -w_2\vec{\omega}\cdot\vec{S} + \frac{w_3}{2}\left[(\vec{a}\cdot\vec{X})(\vec{p}\cdot\vec{\alpha}) + (\vec{p}\cdot\vec{\alpha})(\vec{a}\cdot\vec{X})\right], \end{array} \tag{5}$$

we define a local similarity transformation of the spinor

$$\Psi(X,t) = S(X,t)\,\tilde{\Psi}(X,t),$$

where $S(X,t)$ is an invertible matrix (in spinor space). Substitute this into the original Dirac equation (1):

$$i\big[(\partial_0 S)\,\tilde{\Psi} + S\,(\partial_0\tilde{\Psi})\big] = H\,(S\,\tilde{\Psi}).$$

Multiply from the left by S^{-1}, and rearrange to isolate $i\partial_0\tilde{\Psi}$

$$i\,\partial_0\tilde{\Psi} = \big(S^{-1}HS - i\,S^{-1}(\partial_0 S)\big)\tilde{\Psi}.$$

Hence, the similarity-transformed Dirac equation is

$$i\,\partial_0\tilde{\Psi}(X,t) = \tilde{H}(X,t)\,\tilde{\Psi}(X,t),$$

where the transformed Hamiltonian becomes

$$\tilde{H}(X,t) = S^{-1}(X,t)\,H\,S(X,t) - i\,S^{-1}(X,t)\,\partial_0 S(X,t). \tag{6}$$

The final similarity transformation of the full Dirac equation reads

$$i\,\partial_0\tilde{\Psi}(X,t) = [S^{-1}(X,t)HS(X,t) - i\,S^{-1}(X,t)(\partial_0 S(X,t))]\tilde{\Psi}(X,t). \tag{7}$$

We further define the generator $S(X,t)$ in the form

$$S(X,t) = S_1(X,t))\,S_2(X)). \tag{8}$$

This, incorporating with (4), give

$$\tilde{H}(X) = S_2^{-1}(X)\,\left\{\left[S_1^{-1}(X,t))(H_0 + V_{1\rm imag})S_1(X,t))\right] + V_{2\rm imag}\right\}\,S_2(X), \tag{9}$$

which becomes

$$\begin{array}{l} \tilde{H}(X) = S_2^{-1}(X)\,\tilde{H}_1(X)\,S_2(X) + V_{2\rm imag}, \\ \tilde{H}_1(X) = S_1^{-1}(X,t))(H_0 + V_{1\rm imag})S_1(X,t)). \end{array} \tag{10}$$

2.1. Removing an imaginary term $V_{1\rm imag}$

According to (10), our strategy will be to start with the full Dirac equation

$$i\,\partial_0\Psi(X,t) = (H_{02} + V_{1\rm imag})\,\Psi(X,t), \tag{11}$$

where $H_{02} := H_0 + V_{2\rm imag}$, and redefine wave function

$$\Psi(X,t) = S_1(X,t))\,\tilde{\Psi}_1(X,t) = e^{F(X,t)}\,\tilde{\Psi}_1(X,t), \tag{12}$$

to eliminate, at the first step, a purely imaginary potential $V_{1\rm imag}$, where $F(X,t)$ is to be determined. Compute the time derivative $\partial_0\Psi = (\partial_0 F)e^F\tilde{\Psi}_1 + e^F\partial_0\tilde{\Psi}_1$, and substitute into the Dirac equation,

$$i\left[(\partial_0 F)e^F\tilde{\Psi}_1 + e^F\partial_0\tilde{\Psi}_1\right] = (H_{02} + \frac{i}{2}w_5)e^F\tilde{\Psi}_1. \tag{13}$$

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Cancel the common factor e^F (which commutes since it's scalar):

$$i\partial_0\tilde{\Psi}_1 + i(\partial_0 F)\tilde{\Psi}_1 = H_{02}\tilde{\Psi}_1 + \frac{i}{2}w_5\tilde{\Psi}_1. \tag{14}$$

Rearrange

$$i\partial_0\tilde{\Psi}_1 = \tilde{H}_1(X)\tilde{\Psi}_1 = H_{02}\tilde{\Psi}_1 + \frac{i}{2}w_5\tilde{\Psi}_1 - i(\partial_0 F)\tilde{\Psi}_1. \tag{15}$$

We require the term $\frac{i}{2}w_5\tilde{\Psi}$ to disappear. This suggests the following condition for cancellation to be satisfied

$$-i(\partial_0 F) = -\frac{i}{2}w_5 \quad \Rightarrow \quad \partial_0 F = \frac{1}{2}w_5(X). \tag{16}$$

Since $w_5(X)$ is time-independent, we integrate over time $F(X,t) = \frac{1}{2}w_5(X)t + \text{const.}$, and (const.) set to zero. Thus the wave transformation reads

$$\begin{aligned} &\Psi(X,t) = S_1(X,t)\tilde{\Psi}_1(X,t) = e^{\frac{1}{2}w_5(X)t}\,\tilde{\Psi}_1(X,t), \\ &i\partial_0\tilde{\Psi}_1(X,t) = \tilde{H}_1(X)\tilde{\Psi}_1 = H_{02}\,\tilde{\Psi}_1(X,t). \end{aligned} \tag{17}$$

We now turn to momentum-operator consistency check. Namely, starting from

$$i\partial_0\Psi = (H_{02} + \tfrac{i}{2}w_5)\Psi,$$

and using $\Psi = e^{F(X,t)}\tilde{\Psi}_1$, according to the momentum replacement rule, $p \to p - i\nabla F$, we get

$$e^{-F}H_{02}e^F = H_{02}(p \to p - i\nabla F). \tag{18}$$

This expands to $\tilde{H}_{02} = H_{02} - i\triangle H$, with

$$\begin{aligned} &\Delta H = w_3(X)\,\alpha\cdot\nabla F \\ &+\tfrac{w_3(X)}{2}\left[(a\cdot X)(\alpha\cdot\nabla F) + (\alpha\cdot\nabla F)(a\cdot X)\right]. \end{aligned} \tag{19}$$

The transformed equation is then

$$i\partial_0\tilde{\Psi} = \tilde{H}_{02}\tilde{\Psi} - i(\partial_0 F)\tilde{\Psi} + \tfrac{i}{2}w_5(X)\tilde{\Psi}. \tag{20}$$

To eliminate the scalar imaginary part, we now require

$$\Delta H + \partial_0 F - \frac{1}{2}w_5(X) = 0.$$

For scalar cancellation, ∇F must vanish (otherwise $\alpha\cdot\nabla F$ produces non-scalar terms). Hence $\nabla F = \frac{1}{2}t\nabla w_5(X) = 0$ only, so $\Delta H = 0$, leading again to

$$\partial_0 F = \frac{1}{2}w_5(X) \quad \Rightarrow \quad F = \frac{1}{2}w_5(X)t.$$

Final results is

$$\begin{aligned} &\Psi(X,t) = e^{\frac{1}{2}w_5(X)t}\,\tilde{\Psi}(X,t), \\ &i\partial_0\tilde{\Psi} = H_{02}\tilde{\Psi}, \end{aligned} \tag{21}$$

and the momentum replacement rule, $p \to p - i\nabla F$, is inert here because as we noted earlier the position scalar functions, b_1, b_3, b_4 vary slowly with $\vec{X}$, so their gradients are negligible, and these corrections are suppressed:

$$\begin{aligned} &\nabla F = \tfrac{1}{2}t\nabla w_5(X) \\ &= \tfrac{1}{2}t\nabla\left\{\tfrac{b_1}{b_3}[(\nabla b_4 - 3b_4 b_0\nabla b_1^{-1})]\cdot\vec{v}\right\} = \\ &\tfrac{t}{2}\Bigg[\tfrac{b_3\nabla b_1 - b_1\nabla b_3}{b_3^2}\left((\nabla b_4 + 3b_4 b_0 b_1^{-2}\nabla b_1)\cdot\vec{v}\right) \\ &+\tfrac{b_1}{b_3}\Big\{(\nabla^2 b_4)\vec{v} + 3\Big[\nabla(b_4 b_0 b_1^{-2})(\nabla b_1\cdot\vec{v}) \\ &+b_4 b_0 b_1^{-2}(\nabla^2 b_1)\vec{v}\Big]\Big\}\Bigg] \approx 0. \end{aligned} \tag{22}$$

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2.2. Removing the $V_{\mathbf{2imag}}$ to first order in BCH expansion

According to our strategy (10), we next redefine the wave function

$$\tilde{\Psi}_1 = S_2(X))\,\tilde{\Psi}(X,t), \tag{23}$$

and perform the second similarity transformation on the Dirac Hamiltonian

$$H = H_0 + V_{2\mathrm{imag}},$$

provided that we choose S such that the imaginary potential

$$V_{2\mathrm{imag}} = \frac{i}{2}\big(w_4(\vec{X})\,\vec{a}\cdot\vec{\alpha}\big),$$

is eliminated to first order.

Dropping the index 2 at S_2 and $V_{2\mathrm{imag}}$, this equivalents to performing similarity transformation on the Dirac Hamiltonian

$$H' = e^S H e^{-S} \approx H + [S,H],$$

and choose S so that H' becomes Hermitian to first order. Namely,

$$\mathrm{Im}(H') = 0,$$

or

$$V_{\mathrm{imag}} + [S,H_0] + [S,V_{\mathrm{imag}}] = 0.$$

We assume S is of the same general structure as V_{imag},

$$S = \frac{i}{2}\,f(\vec{X}),$$

with $f(\vec{X})$ a real operator-valued function to be determined.

Compute $[S,H_0]$. Since $f(\vec{X})$ depends only on $\vec{X}$, non-trivial commutators come only from momentum terms in H_0, i.e.

$$H_0 \supset w_3(\vec{X})\,(\vec{\alpha}\cdot\vec{p}).$$

All other pieces (mass, spin, rotation couplings) commute or give higher-order effects. We computed

$$[f(\vec{X}),\vec{\alpha}\cdot\vec{p}] = i\,\vec{\alpha}\cdot\nabla f.$$

Thus,

$$[S,H_0] \approx \frac{i}{2}[f,H_0] = -\frac{1}{2}w_3\,\vec{\alpha}\cdot\nabla f, \tag{24}$$

and

$$[S,V_{\mathrm{imag}}] = -\frac{1}{4}[f,w_4\,\vec{a}\cdot\vec{\alpha}].$$

Since f, w_4 are scalar functions of position and commute with $\vec{\alpha}$, $[f,w_4\,\vec{a}\cdot\vec{\alpha}] = 0$. To remove the imaginary term, we impose the condition

$$-\frac{1}{2}w_3\,\vec{\alpha}\cdot\nabla f = -\frac{i}{2}w_4\,\vec{\alpha}\cdot\vec{a}.$$

Hence

$$\nabla f = i\frac{w_4}{w_3}\,\vec{a},$$

and

$$f(\vec{X}) = i\int^{\vec{X}} \frac{w_4(\vec{X}\,')}{w_3(\vec{X}\,')}\,\vec{a}\cdot d\vec{X}\,'.$$

Generator is

$$S = \tfrac{i}{2}\,f(\vec{X}) = -\tfrac{1}{2}\int^{\vec{X}} \tfrac{w_4(\vec{X}\,')}{w_3(\vec{X}\,')}\,\vec{a}\cdot d\vec{X}\,', \tag{25}$$

and transformed Hamiltonian (to first order) becomes

$$H' = H_0 - \tfrac{1}{2}w_3(\vec{X})\,\vec{\alpha}\cdot\nabla f(\vec{X}) = H_0 - \tfrac{i}{2}w_4(\vec{X})\,\vec{\alpha}\cdot\vec{a}. \tag{26}$$

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2.3. Removing the V_{imag} to second order in BCH expansion

Next, for BCH expansion to second order, we use

$$\begin{aligned} H' &= H + [S, H] + \tfrac{1}{2}[S,[S,H]] \\ &= H_0 + V_{\text{imag}} + [S, H_0] + [S, V_{\text{imag}}] + \tfrac{1}{2}\left([S,[S,H_0]] + [S,[S,V_{\text{imag}}]]\right), \end{aligned} \tag{27}$$

where the second-order corrections are

$$\delta H^{(2)} = \frac{1}{2}[S,[S,H_0]] + \frac{1}{2}[S,[S,V_{\text{imag}}]].$$

Taking into account (24) and $S = \frac{i}{2} f(\vec{X})$, we have

$$[S,[S,H_0]] = -\frac{1}{2} w_3 [S, \vec{\alpha}\cdot\nabla f].$$

The $\vec{\alpha}\cdot\nabla f$ commutes with f, therefore, $[S,[S,H_0]] = 0$. So in this approximation, the dominant second-order correction from the kinetic term vanishes. However, let's check whether momentum dependence hidden in H_0 could yield higher-order pieces. More precisely

$$[S,[S,H_0]] \sim \frac{i}{2}[f,[S,H_0]] \propto \partial_i f\, \partial_j f\, [p_i, p_j] = 0,$$

so indeed, it vanishes to all orders in ∇f. Since $[S, V_{\text{imag}}] = 0$, it follows that

$$[S,[S,V_{\text{imag}}]] = 0.$$

Now let's look at the extra commutators that were neglected previously. Since $\vec{L} = \vec{X}\times\vec{p}$, so

$$[S, L_i] = \frac{i}{2}\epsilon_{ijk}[f, X_j p_k],$$

and $[f, X_j] = 0$, we have

$$[S, L_i] = \frac{1}{2}\epsilon_{ijk} X_j \partial_k f.$$

Hence

$$[S, \vec{L}] = \frac{1}{2}\vec{X}\times\nabla f.$$

Therefore,

$$[S, -\vec{\omega}\cdot\vec{L}] = -\frac{1}{2}(\vec{\omega}\times\vec{X})\cdot\nabla f.$$

So, because of $\nabla f = i\frac{w_4}{w_3}\,\vec{a}$, we have

$$[S, -\vec{\omega}\cdot\vec{L}] = -\frac{i}{2}\frac{w_4}{w_3}(\vec{\omega}\times\vec{X})\cdot\vec{a}.$$

This is purely imaginary, so this term must be taken into account if we want to remove all imaginary contributions. Since f and w_2 are both scalar functions of $\vec{X}$, $[f, w_2] = 0$. so spin term commutes:

$$[S, -w_2\,\vec{\omega}\cdot\vec{S}] = 0.$$

Second-order commutator with angular momentum term also is zero:

$$[S,[S,-\vec{\omega}\cdot\vec{L}]] = 0.$$

So higher-order corrections from this channel vanish too. Collecting all second-order results together, we obtain $\delta H^{(2)} = 0$. To remove new imaginary piece induced by rotation coupling, one could adjust S by adding a small correction

$$\delta S \propto (\vec{\omega}\times\vec{X})\cdot\vec{a},$$

but that goes beyond current order. Since all second-order nested commutators vanish, final explicit form (up to second order) reads

$$\begin{aligned} H' &\approx H_0 - \tfrac{1}{2}w_3\,\vec{\alpha}\cdot\nabla f + V_{\text{imag}} + [S, -\vec{\omega}\cdot\vec{L}] \\ &= H_0 - \tfrac{i}{2}w_4\,\vec{\alpha}\cdot\vec{a} - \tfrac{i}{2}\tfrac{w_4}{w_3}(\vec{\omega}\times\vec{X})\cdot\vec{a}. \end{aligned} \tag{28}$$

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As we see, the main imaginary potential $\frac{i}{2}w_4\,\vec{a}\cdot\vec{\alpha}$ is removed by the transformation generated by S. However, in a rotating frame ($\vec{\omega}\neq 0$), an additional imaginary correction term, $-\frac{i}{2}\frac{w_4}{w_3}(\vec{\omega}\times\vec{X})\cdot\vec{a}$, arises. All second-order BCH contributions vanish, so higher corrections are negligible to this order.

Let's now explicitly verify that the modified generator $S = S_0 + S_1$, where

$$S_0 = \tfrac{i}{2}f(\vec{X}), \qquad \text{with} \quad \nabla f = i\,\tfrac{w_4}{w_3}\,\vec{a}, \tag{29}$$

and

$$S_1 = \tfrac{i}{2}g(\vec{X}), \qquad \text{with} \quad (\vec{\omega}\times\vec{X})\cdot\nabla g = -i\,\tfrac{w_4}{w_3}\,(\vec{\omega}\times\vec{X})\cdot\vec{a}, \tag{30}$$

really cancels both imaginary pieces: the original $V_{\rm imag}$ and the rotation–induced imaginary correction $-\frac{i}{2}\frac{w_4}{w_3}(\vec{\omega}\times\vec{X})\cdot\vec{a}$. Equivalently, the gradients of f and g are

$$\nabla f = i\,\frac{w_4}{w_3}\,\vec{a}, \qquad \nabla g = -i\,\frac{w_4}{w_3}\,\frac{(\vec{\omega}\times\vec{X})\times\vec{a}}{|\vec{\omega}\times\vec{X}|^2},$$

up to functions constant along rotation orbits. Computing $[S,H]$ to first order

$$[S,H] = [S_0,H_0] + [S_0,V_{\rm imag}] + [S_1,H_0] + [S_1,V_{\rm imag}], \tag{31}$$

We analyze each term carefully. Previously we found $[S_0,H_0] \approx -\frac{1}{2}w_3\,\vec{\alpha}\cdot\nabla f$. Substitute $\nabla f = i\frac{w_4}{w_3}\vec{a}$: $[S_0,H_0] = -\frac{i}{2}w_4\,\vec{\alpha}\cdot\vec{a}$. This term cancels $V_{\rm imag}$ exactly

$$V_{\rm imag} + [S_0,H_0] = \frac{i}{2}w_4\,\vec{\alpha}\cdot\vec{a} - \frac{i}{2}w_4\,\vec{\alpha}\cdot\vec{a} = 0.$$

We computed earlier $[S_0,-\vec{\omega}\cdot\vec{L}] = -\frac{i}{2}\frac{w_4}{w_3}(\vec{\omega}\times\vec{X})\cdot\vec{a}$. This is the undesired residual imaginary term we want to cancel. By the same rule,

$$[S_1,-\vec{\omega}\cdot\vec{L}] = -\frac{1}{2}(\vec{\omega}\times\vec{X})\cdot\nabla g.$$

Now substitute the defining condition of $g(\vec{X})$ (30)

$$[S_1,-\vec{\omega}\cdot\vec{L}] = \frac{i}{2}\frac{w_4}{w_3}(\vec{\omega}\times\vec{X})\cdot\vec{a}$$

is exactly the opposite of the residual imaginary term from $[S_0,-\vec{\omega}\cdot\vec{L}]$. Thus the two cancel: $[S_0,-\vec{\omega}\cdot\vec{L}] + [S_1,-\vec{\omega}\cdot\vec{L}] = 0$. The S_1 depends only on position, so $[S_1, w_1\beta m(1+\vec{a}\cdot\vec{X})] = 0$, and $[S_1,-w_2\vec{\omega}\cdot\vec{S}] = 0$, hence $[S_1,\vec{\alpha}\cdot\vec{p}] = i\,\vec{\alpha}\cdot\nabla g$. But this yields $-\frac{1}{2}w_3\,\vec{\alpha}\cdot\nabla g$, which is real, so it does not contribute to imaginary terms and can be retained as a Hermitian correction. Finally, the $[S_0,V_{\rm imag}]$ and $[S_1,V_{\rm imag}]$, both vanish since f, g, w_4 commute with $\vec{\alpha}$ and themselves Collect all terms to first order

$$\begin{aligned}[S,H] == &\left(-\tfrac{i}{2}w_4\,\vec{\alpha}\cdot\vec{a}\right) + \left(-\tfrac{i}{2}\tfrac{w_4}{w_3}(\vec{\omega}\times\vec{X})\cdot\vec{a}\right)\\ &+\left(-\tfrac{1}{2}w_3\,\vec{\alpha}\cdot\nabla g\right) + \left(+\tfrac{i}{2}\tfrac{w_4}{w_3}(\vec{\omega}\times\vec{X})\cdot\vec{a}\right).\end{aligned} \tag{32}$$

The second and fourth terms cancel exactly. The first term cancels $V_{\rm imag}$. The remaining piece is real: $-\frac{1}{2}w_3\,\vec{\alpha}\cdot\nabla g$, which is Hermitian. Thus we explicitly verify that the modified generator

$$S = S_0 + S_1,$$

really cancels both imaginary pieces: the original $V_{\rm imag}$ and the rotation–induced imaginary correction. Thus final transformed Hamiltonian (Hermitian to first order) becomes

$$H' = H_0 - \tfrac{1}{2}w_3\,\vec{\alpha}\cdot\nabla(f+g), \tag{33}$$

with all imaginary contributions cancelled.

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2.4. Removing the V_{imag} to higher order in BCH expansion

We now turn to a higher order — up to third order terms (the first genuinely new corrections) — and understand their structure, Hermiticity, and physical relevance. With the full BCH expansion, the similarity-transformed Hamiltonian is written

$$\begin{aligned} &H' = e^S H e^{-S} = H + [S,H] + \tfrac{1}{2!}[S,[S,H]] \\ &+\tfrac{1}{3!}[S,[S,[S,H]]] + \cdots \end{aligned} \tag{34}$$

Define the nested commutators

$$\begin{aligned} &C_1 := [S,H], \\ &C_2 := [S,[S,H]], \\ &C_3 := [S,[S,[S,H]]], \end{aligned}$$

etc. Then

$$H' = H + C_1 + \frac{1}{2}C_2 + \frac{1}{6}C_3 + \cdots$$

We recall that S is purely imaginary, $S = \frac{i}{2}F(\vec{X})$, where $F(\vec{X}) = f(\vec{X}) + g(\vec{X})$ real. Therefore $S^\dagger = -S$ (anti-Hermitian), and e^S is unitary, so $H' = e^S H e^{-S}$ is Hermitian if H is Hermitian. However, H itself contains V_{imag}, which is non-Hermitian, so higher-order commutators will restore Hermiticity by generating real compensating terms. We start from $H = H_0 + V_{\mathrm{imag}}$. We already know from the first-order analysis that $[S, V_{\mathrm{imag}}] = 0$, and $[S,[S,V_{\mathrm{imag}}]] = 0$. The similarly all higher nested commutators with V_{imag} vanish, because S and V_{imag} are both functions of position and Dirac matrices that commute. Thus we can focus only on the H_0 part:

$$C_1 = [S,H_0], \quad C_2 = [S,[S,H_0]], \quad C_3 = [S,[S,[S,H_0]]].$$

We already have the first

$$\begin{aligned} &C_1 = [S,H_0] \approx -\tfrac{1}{2}w_3(\vec{X})\,\vec{\alpha}\cdot\nabla F(\vec{X}) \\ &-\tfrac{1}{2}(\vec{\omega}\times\vec{X})\cdot\nabla F(\vec{X}) + (\text{real spin terms}). \end{aligned} \tag{35}$$

Let's start with $C_1 = -\frac{1}{2}w_3\,\vec{\alpha}\cdot\nabla F$, so

$$C_2 = [S, C_1] = -\frac{1}{2}w_3[S, \vec{\alpha}\cdot\nabla F].$$

Now, $S = \frac{i}{2}F$, and both w_3 and F are scalar functions of position. Then

$$[S, \vec{\alpha}\cdot\nabla F] = \frac{i}{2}\,\vec{\alpha}\cdot[F, \nabla F] = 0,$$

and that $C_2 = 0$. Thus, the second-order correction $\frac{1}{2}C_2 = 0$, consistent with our earlier approximate result. Similarly, since $C_2 = 0$, all higher nested commutators vanish as well $C_3 = [S, C_2] = [S, 0] = 0$. We might worry that $w_3(\vec{X})$ is position dependent and that H_0 includes momentum operators, so higher-order terms could appear via: $[S,[S,w_3(\vec{X})\,\vec{\alpha}\cdot\vec{p}]]$. Let's check that carefully. Using the basic commutation rule: $[p_i, f(\vec{X})] = -i\,\partial_i f(\vec{X})$, and $S = \frac{i}{2}F(\vec{X})$, we can compute explicitly

$$[S, w_3\,\alpha_i p_i] = -\frac{1}{2}w_3\,\vec{\alpha}\cdot\nabla F.$$

Then

$$[S,[S,w_3\,\vec{\alpha}\cdot\vec{p}]] = -\frac{1}{2}w_3[S,\vec{\alpha}\cdot\nabla F].$$

As above, $[S, \nabla F] = 0$. Hence, $C_2 = 0$. And recursively all higher nested commutators vanish. Therefore, the only potentially nonzero higher-order contributions would arise if the operator ordering between $w_3(\vec{X})$ and $\vec{p}$ were not fully symmetric, or if the gradient acts nontrivially on w_3. Let's include that subtlety now. Correcting for noncommuting $w_3(\vec{X})$ and $\vec{p}$, we compute exactly

$$[S, H_0] = \frac{i}{2}[F, w_3\,\vec{\alpha}\cdot\vec{p}] = \frac{i}{2}w_3[F, \vec{\alpha}\cdot\vec{p}] + \frac{i}{2}[F, w_3]\,\vec{\alpha}\cdot\vec{p}.$$

The first term gives $-\frac{1}{2}w_3\,\vec{\alpha}\cdot\nabla F$, as before, The second term gives zero. Thus, even accounting for operator ordering, no new terms appear at higher order. Clearing the general BCH pattern for scalar $S = \frac{i}{2}F(\vec{X})$,

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note that if S depends only on position and H_0 is at most linear in $\vec{p}$, the BCH expansion terminates after the first commutator $e^S He^{-S} = H + [S, H]$, because every further commutator introduces an additional factor of $[F, [F, p_i]]$, and since $[F, p_i]$ is a c-number function (derivative), the higher double commutators vanish: $[F, [F, p_i]] = 0$. So for all such cases

$$[S, [S, H]] = 0, \quad [S, [S, [S, H]]] = 0,$$

etc. This means that the BCH expansion terminates exactly for this class of generators and Hamiltonians. Hence final full transformed Hamiltonian to all orders becomes $H' = e^S He^{-S} = H + [S, H]$, with

$$\begin{aligned} &S = \tfrac{i}{2}(f + g), \quad \nabla f = i\,\tfrac{w_4}{w_3}\,\vec{a}, \\ &(\vec{\omega} \times \vec{X}) \cdot \nabla g = -i\,\tfrac{w_4}{w_3}(\vec{\omega} \times \vec{X}) \cdot \vec{a}. \end{aligned} \tag{36}$$

Substituting $[S, H]$ explicitly, all imaginary terms cancel and we obtain a purely Hermitian effective Hamiltonian

$$H' = H_0 - \frac{1}{2} w_3(\vec{X})\,\vec{\alpha} \cdot \nabla(f + g).$$

Since higher-order commutators vanish identically, this result is exact (not just approximate) for all orders of the BCH expansion in this case. From definition of g, we need ∇g itself, not just its projection. To satisfy this relation, the minimal choice for ∇g that preserves the symmetry and dimensional structure is $\nabla g = -i\,\frac{w_4}{w_3}\,\vec{a}_\perp$, where $\vec{a}_\perp$ is the projection of $\vec{a}$ perpendicular to $\vec{\omega} \times \vec{X}$, i.e.

$$\vec{a}_\perp = \frac{(\vec{\omega} \times \vec{X})[(\vec{\omega} \times \vec{X}) \cdot \vec{a}]}{|\vec{\omega} \times \vec{X}|^2}.$$

However, it is more illuminating to express the total contribution directly in operator form, keeping the vector structure $(\vec{\omega} \times \vec{X})$ explicit. Then $-\frac{1}{2} w_3\,\vec{\alpha} \cdot \nabla g = \frac{i}{2} w_4\,\vec{\alpha} \cdot \vec{a}_\perp$, whose projection along $(\vec{\omega} \times \vec{X})$ cancels the residual imaginary term coming from $[S, -\vec{\omega} \cdot \vec{L}]$. Collecting all terms together, the full Hermitian transformed Hamiltonian becomes

$$\begin{aligned} H' &= w_1(\vec{X})\,\beta m(1 + \vec{a} \cdot \vec{X}) + w_3(\vec{X})\,\vec{\alpha} \cdot \vec{p} - \vec{\omega} \cdot \vec{L} - w_2(\vec{X})\,\vec{\omega} \cdot \vec{S} \\ &+ \tfrac{w_3(\vec{X})}{2} \Big[(\vec{a} \cdot \vec{X})(\vec{p} \cdot \vec{\alpha}) + (\vec{p} \cdot \vec{\alpha})(\vec{a} \cdot \vec{X})\Big] - \tfrac{1}{2} w_3(\vec{X})\,\vec{\alpha} \cdot \nabla f(\vec{X}) - \tfrac{1}{2} w_3(\vec{X})\,\vec{\alpha} \cdot \nabla g(\vec{X}). \end{aligned} \tag{37}$$

Substituting the explicit gradients

$$\begin{aligned} H' &= w_1(\vec{X})\,\beta m(1 + \vec{a} \cdot \vec{X}) + w_3(\vec{X})\,\vec{\alpha} \cdot \vec{p} - \vec{\omega} \cdot \vec{L} - w_2(\vec{X})\,\vec{\omega} \cdot \vec{S} \\ &+ \tfrac{w_3(\vec{X})}{2} \Big[(\vec{a} \cdot \vec{X})(\vec{p} \cdot \vec{\alpha}) + (\vec{p} \cdot \vec{\alpha})(\vec{a} \cdot \vec{X})\Big] - \tfrac{i}{2} w_4(\vec{X})\,\vec{\alpha} \cdot \vec{a} + \tfrac{i}{2} w_4(\vec{X})\,\tfrac{(\vec{\omega} \times \vec{X})[(\vec{\omega} \times \vec{X}) \cdot \vec{a}]}{|\vec{\omega} \times \vec{X}|^2} \cdot \vec{\alpha}. \end{aligned} \tag{38}$$

The last two correction terms (those proportional to w_4) come from the similarity transformation and can be grouped as:

$$H'_{\text{corr}} = -\tfrac{i}{2} w_4(\vec{X})\,\vec{\alpha} \cdot \left[\vec{a} - \tfrac{(\vec{\omega} \times \vec{X})[(\vec{\omega} \times \vec{X}) \cdot \vec{a}]}{|\vec{\omega} \times \vec{X}|^2}\right]. \tag{39}$$

The quantity in brackets is the component of acceleration perpendicular to the rotation plane $\vec{a}_\parallel$, $\vec{a}_\perp = \vec{a} - \vec{a}_\parallel$. thus $H'_{\text{corr}} = -\frac{i}{2} w_4(\vec{X})\,\vec{\alpha} \cdot \vec{a}_\perp$, i.e. only the component of acceleration orthogonal to the local rotation plane survives in the Hermitianized Hamiltonian. The final explicit Hermitian Hamiltonian reads

$$\begin{aligned} H' &= w_1(\vec{X})\,\beta m(1 + \vec{a} \cdot \vec{X}) + w_3(\vec{X})\,\vec{\alpha} \cdot \vec{p} - \vec{\omega} \cdot \vec{L} - w_2(\vec{X})\,\vec{\omega} \cdot \vec{S} \\ &+ \tfrac{w_3(\vec{X})}{2} \Big[(\vec{a} \cdot \vec{X})(\vec{p} \cdot \vec{\alpha}) + (\vec{p} \cdot \vec{\alpha})(\vec{a} \cdot \vec{X})\Big] - \tfrac{i}{2} w_4(\vec{X})\,\vec{\alpha} \cdot \left[\vec{a} - \tfrac{(\vec{\omega} \times \vec{X})[(\vec{\omega} \times \vec{X}) \cdot \vec{a}]}{|\vec{\omega} \times \vec{X}|^2}\right]. \end{aligned} \tag{40}$$

All coefficients are real, and all imaginary potentials have been systematically removed by the similarity transformation. Higher BCH terms vanish identically, so this H' is exact to all orders in S.

Recall that the key properties of the similarity transformation are as follows: $H' = e^S He^{-S}$, where $S = \frac{i}{2}\big(f(\vec{X}) + g(\vec{X})\big)$, with f, g real scalar operators (functions of position). Then $S^\dagger = -S$, e^S is unitary. Thus, the transformation $\Psi = e^S \Psi$ is a unitary change of representation. This implies

$$< \Psi'|\Psi' > = < \Psi|e^{S^\dagger} e^S|\Psi > = < \Psi|\Psi > .$$

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Therefore, probability is conserved. All expectation values of properly transformed observables are real and physically consistent. Actually, let O be any physical observable (Hermitian operator) in the original frame. Then under the transformation $O' = e^S O e^{-S}$, $\quad \Psi' = e^S \Psi$, the expectation value of O in the physical state is invariant

$$< O >=< \Psi|O|\Psi >=< \Psi'|e^S O e^{-S}|\Psi' > \; =< \Psi'|O'|\Psi' > .$$

Thus, although H was non-Hermitian, the similarity transformation makes it unitarily equivalent to a Hermitian H'. Therefore, expectation values of all physical observables remain real. Let's check Hermiticity of $H' = H_0 + H'_{\rm corr}$ directly: H_0 is Hermitian by construction (it contains only real coefficients and Hermitian operators: β, $\vec{\alpha}$, $\vec{p}$, $\vec{L}$, $\vec{S}$). The correction term is also Hermitian (the overall minus from i and the Hermiticity of $\vec{\alpha}$ cancel). Hence $H'^{\dagger} = H'$. So the transformed Hamiltonian is exactly Hermitian. Then for any normalized spinor Ψ', $< H' >=< \Psi'|H'|\Psi' >$ is guaranteed to be real because H' is Hermitian. In particular, we have for energy eigenvalues

$$H'|\Psi'_n >= E_n|\Psi'_n > E_n \in \mathbb{R}.$$

Thus we provide a rationale for the fact that all physical observable quantities, including energy, momentum, and probability, remain real and physically consistent. Since the key properties of the similarity transformation we used are merely unitary representation transformations, probability density is, therefore, conserved:

$$\rho' = \Psi'^{\dagger}\Psi' = \Psi^{\dagger}\Psi.$$

The corresponding transformed Hamiltonian H' is Hermitian, and therefore the physical theory described by H' is completely real and consistent.

3. Non-relativistic Approximation via Foldy-Wouthuysen Transformation

The derivation of the Dirac Hamiltonian in non-relativistic approximation is usually accomplished by following a sequence of FW-canonical transformations Foldy & Wouthuysen (1950). The corresponding unitary operator can be easily obtained for the free Dirac particle (64). But in most cases for a fermion interacting with an electromagnetic field, or in the case of going beyond the locality hypothesis that we pursue through deformations (**??**), there is no room for exact FW-transformation. An approximate scheme can be used instead. The FW-transformation must now be made by an infinite sequence of FW-transformations leading to a deformed Hamiltonian, which is an infinite series in powers of $(1/m)$. In this way one can obtain the proper non-relativistic limit for the Hamiltonian representing a free Dirac particle beyond the hypothesis of locality. The essential reason why four components are in general necessary to describe a state of positive or negative energy in the representation of the Dirac theory corresponding to (65) is that the Hamiltonian in this equation contains odd operators, specifically the components of the operator $\vec{\alpha}$. Observe that an odd operator is a Dirac matrix which has only matrix elements connecting upper and lower components of the wave function, while an even operator is one having no such matrix elements. It should be noted that although the CT transformation follows the same mathematical principles as the previous one, it is far from obvious that it can do the same thing to achieve the high energy limit for the Dirac Hamiltonian. This is because it is unclear how to classify the Dirac Hamiltonian into a high-energy analogue of odd and even operators, as found in the FW approach, in order to systematically remove unwanted terms. An undesirable consequence of this impasse is that it becomes impossible to analyze the motion and properties of fast moving massive particles in the presence of fields without arbitrarily setting their mass equal to zero in the Hamiltonian. Obviously, this excludes any possibility of comparing the behavior of these particles with the behavior of strictly massless particles.

To start the explicit calculations beyond the locality hypothesis, where a Dirac particle is subject to interactions (deformations), we can go over to the Hamiltonian (40) and apply the FW-transformation method up to the third-order in $(1/m)$, assuming the particle's momentum is small compared to its rest mass energy (i.e., non-relativistic limit). Therewith we drop the last term because it is of higher order term with respect to non-relativistic limit we are going to deal. The reduced Hamiltonian can be rewritten in more convenient form for dealing with FW-transformations to derive its non-relativistic limit:

$$H = \beta m + \mathcal{E} + \mathcal{O}, \tag{41}$$

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where we split Hamiltonian into even and odd parts

$$\begin{aligned}\mathcal{E} &= (w_1-1)\beta m + w_1\beta m\vec{a}\cdot\vec{X} - \vec{\omega}\cdot\vec{L} - w_2\,\vec{\omega}\cdot\vec{\sigma},\\ \mathcal{O} &= w_3\vec{\alpha}\cdot\vec{p} + \tfrac{w_3}{2}\left[(\vec{a}\cdot\vec{X})(\vec{p}\cdot\vec{\alpha}) + (\vec{p}\cdot\vec{\alpha})(\vec{a}\cdot\vec{X})\right],\end{aligned}\tag{42}$$

where, $\mathcal{E}$: even operators (commute with β,) diagonal, $\mathcal{O}$: odd operators (anticommute with β), off-diagonal,

$$\left.\begin{aligned}&1. w_1\beta m\vec{a}\cdot\vec{X},\\ &2. \tfrac{w_3}{2}[(\vec{a}\cdot\vec{X})(\vec{p}\cdot\vec{\alpha}) + (\vec{p}\cdot\vec{\alpha})(\vec{a}\cdot\vec{X})],\end{aligned}\right\}$$

are the extended energy-momentum redshift effects, the $-\vec{\omega}\cdot\vec{L}$ is a Sagnac-type effect, the $-w_2\,\vec{\omega}\cdot\vec{\sigma}$ is an extended rotation-spin coupling.

We will perform three canonical FW - transformations to eliminate the odd (off-diagonal) components from the Dirac Hamiltonian order-by-order in $(1/m)$. In the method employed for reducing the Hamiltonian (41) to non-relativistic two-component form, following Bjorken & Drell (1964), we introduce the FW-transformation by $\Psi' = e^{iS}\Psi$. The BCH expansion gives

$$H' = e^{iS}(H - i\partial_0)e^{-iS} = H + i[S,H] - i\partial_0 + O(1/m^2).$$

Since we neglect time-dependence in fields (i.e., inertial fields are time-independent), we ignore the $-i\partial_0$ term for now. Consider the first FW-transformation with just the terms through order unity:

$$H' = e^{iS}(H - i\partial_0)e^{-iS} = H + i[S,\beta]m,$$

and require that the odd term in Hamiltonian H' vanish. This gives the FW canonical transformation generated by the Hermitian operator $S := -\frac{i\beta}{2m}\mathcal{O}$, such that

$$i[S,\beta m] = i\left[-\frac{i\beta}{2m}\mathcal{O},\beta m\right] = -\mathcal{O},$$

and only keeping terms of order $1/m$, we get

$$H' = \beta m + \mathcal{E} + \underbrace{\mathcal{O} + i[S,\beta m]}_{=0} + i[S,\mathcal{E}].\tag{43}$$

We apply a second FW-transformation using the same prescription but now with $S' := -\frac{i\beta}{2m}\mathcal{O}'$, where $\mathcal{O}'$ defines new odd term

$$\mathcal{O}' := i[S,\mathcal{E}] = -\tfrac{i\beta}{2m}[\mathcal{O},\mathcal{E}] - \tfrac{\mathcal{O}^3}{3m^2}.\tag{44}$$

Taking into account that: $i[S',\beta m] \sim -\mathcal{O}'$; $i[S',\mathcal{E}] \sim \frac{\beta}{2m}[\mathcal{O}',\mathcal{E}]$ (even); and $i[S',\mathcal{O}']$ is higher order (odd); the transformation

$$H'' = H' + i[S',H'] + \cdots$$

gives

$$H'' = \beta m + \mathcal{E} + \text{even terms} + \mathcal{O}'',$$

where the new odd operator (which we will ignore beyond this order) will be

$$\mathcal{O}'' = \frac{\beta}{2m}[\mathcal{O}',\mathcal{E}] - \frac{(\mathcal{O}')^3}{3m^2} - \frac{i}{8m^2}[\mathcal{O},\dot{\mathcal{O}}] + \cdots$$

At this point we restore also the time-dependent ($\dot{\mathcal{O}} \neq 0$) term just for illustration purposes. Finally, we apply a third canonical transformation $S'' := -\frac{i\beta}{2m}\mathcal{O}''$. After three successive FW transformations, the FW method yields the block-diagonal standard FW transformed Hamiltonian up to order $1/m^2$ (dropping triple primes):

$$H = \beta\left(m + \tfrac{\mathcal{O}^2}{2m} - \tfrac{\mathcal{O}^4}{8m^3}\right) + \mathcal{E} - \tfrac{1}{8m^2}\left[\mathcal{O},[\mathcal{O},\mathcal{E}]\right] - \tfrac{i}{8m^2}\left[\mathcal{O},\dot{\mathcal{O}}\right] + O(1/m^2).\tag{45}$$

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3.1. FW expansion up to order $1/m$

Assuming all fields are time-independent ($\dot{\mathcal{O}} = 0$), our goal is to compute the FW-transformed Hamiltonian to $\mathcal{O}(1/m^2)$. The expansion of FW Hamiltonian up to order $1/m$ is

$$H_{\rm FW} = \beta\left(m + \frac{\mathcal{O}^2}{2m}\right) + \mathcal{E} - \frac{1}{8m^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] + \dots$$

Define

$$\mathcal{O} := w_3\left(\vec{\alpha}\cdot\vec{p} + A\right), \tag{46}$$

with

$$A := \frac{1}{2}\left((\vec{a}\cdot\vec{X})(\vec{p}\cdot\vec{\alpha}) + (\vec{p}\cdot\vec{\alpha})(\vec{a}\cdot\vec{X})\right). \tag{47}$$

Computing $\mathcal{O}^2$, we use the identity $(\vec{\alpha}\cdot\vec{p})^2 = \vec{p}^2\cdot I_{4\times4}$. Rather than computing this fully now, observe that the $\{\vec{\alpha}\cdot\vec{p}, A\}$ term and A^2 are both second order in momentum, and produce contributions proportional to $\vec{p}^2$, $(\vec{a}\cdot\vec{X})^2$, etc. But we're interested only in the even part — so we retain scalar terms and drop odd contributions (off-diagonal), which will vanish at first- order correction. Thus $\mathcal{O}^2 = w_3^2\left[\vec{p}^2 + \mathcal{A}(X,\vec{p})\right]$. where $\mathcal{A}(X,\vec{p})$ denotes acceleration-coupled terms like: $(\vec{a}\cdot\vec{X})^2(\vec{p}\cdot\vec{\alpha})^2$; -$(\vec{a}\cdot\vec{X})\vec{p}^2$,; and spin-independent.

To evaluate the $\delta H^{(2)} = -\frac{1}{8m^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]]$, note that

$$\mathcal{E} = \underbrace{(w_1-1)\beta m}_{\text{scalar}} + \underbrace{w_1\beta m\vec{a}\cdot\vec{X}}_{\text{scalar}} - \vec{\omega}\cdot\vec{L} - w_2\vec{\omega}\cdot\vec{\sigma},$$

and split this into two parts: scalar parts $\to$ commute with $\mathcal{O}$ and spin-orbit parts $\to$ contribute nontrivial commutators.

For computation of the double commutator $[\mathcal{O},[\mathcal{O},\vec{\omega}\cdot\vec{L}]]$, we first write the first commutator

$$[\mathcal{O},\vec{\omega}\cdot\vec{L}] = w_3\left[\vec{\alpha}\cdot\vec{p} + A, \vec{\omega}\cdot\vec{L}\right].$$

Since A is a symmetrized operator involving $\vec{\alpha}\cdot\vec{p}$ multiplied by a scalar function, the total first commutator becomes

$$[\mathcal{O},\vec{\omega}\cdot\vec{L}] = iw_3\vec{\alpha}\cdot(\vec{\omega}\times\vec{p})\left[1 + (\vec{a}\cdot\vec{X})\right].$$

Computing the second commutator

$$[\mathcal{O},[\mathcal{O},\vec{\omega}\cdot\vec{L}]] = [\mathcal{O}, iw_3\vec{\alpha}\cdot(\vec{\omega}\times\vec{p})(1+\vec{a},\cdot\vec{X})]$$

focus on the operator inside $Q := \vec{\alpha}\cdot\vec{Q}'$, where $\vec{Q}' = (\vec{\omega}\times\vec{p})(1 + \vec{a}\cdot\vec{X})$. Then,

$$[\mathcal{O},Q] = w_3[\vec{\alpha}\cdot\vec{p} + A, \vec{\alpha}\cdot\vec{Q}'].$$

Therefore, we obtain

$$[\vec{\alpha}\cdot\vec{p}, \vec{\alpha}\cdot\vec{Q}'] = 2i\epsilon^{ijk}\Sigma^k Q'^j p^i + \alpha^i\alpha^j[p^i, Q'^j].$$

Now use

$$[p^i, Q'^j] = -i\frac{\partial Q'^j}{\partial x^i},$$

so that

$$[\vec{\alpha}\cdot\vec{p}, \vec{\alpha}\cdot\vec{Q}'] = 2i(\vec{\Sigma}\cdot(\vec{Q}'\times\vec{p})) - i\alpha^i\alpha^j\partial_i Q'^j.$$

Similarly, the commutator with A will produce higher-order terms in $\vec{a}$ and momenta, negligible at second order in $1/m$ if $\vec{a}$ is small. Thus, dominant term is

$$[\mathcal{O},[\mathcal{O},\vec{\omega}\cdot\vec{L}]] \approx 2iw_3^2\vec{\Sigma}\cdot\left(\vec{Q}'\times\vec{p}\right) + \text{gradient terms}.$$

Finally we obtain

$$[\mathcal{O},[\mathcal{O},\vec{\omega}\cdot\vec{L}]] \approx 2iw_3^2\vec{\Sigma}\cdot(1 + \vec{a}\cdot\vec{X})\left[\vec{p}(\vec{\omega}\cdot\vec{p}) - \vec{\omega}p^2\right]. \tag{48}$$

Next, evaluating $[\mathcal{O},[\mathcal{O}, w_2\vec{\omega}\cdot\vec{\sigma}]]$, the key commutator will come from

$$[\mathcal{O},\vec{\omega}\cdot\vec{\sigma}] \approx w_3[\vec{\alpha}\cdot\vec{p}, \vec{\omega}\cdot\vec{\sigma}] + \dots$$

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and as $[\vec{\alpha}\cdot\vec{p},\vec{\sigma}]=0$, hence $[\mathcal{O},w_2\vec{\omega}\cdot\vec{\sigma}]\approx 0$, implying for the double commutator $[\mathcal{O},[\mathcal{O},w_2\vec{\omega}\cdot\vec{\sigma}]]\approx 0$. Finally we assemble the FW Hamiltonian to order $1/m^2$:

$$H_{\rm FW}=\beta m+\mathcal{E}+\frac{\beta}{2m}\mathcal{O}^2-\frac{1}{8m^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]],$$

and putting everything together, it reads

$$\begin{array}{l}H_{\rm FW}=\beta m w_1+\beta m w_1\vec{a}\cdot\vec{X}-\vec{\omega}\cdot\vec{L}-w_2\vec{\omega}\cdot\vec{\sigma}\\ +\frac{\beta w_3^2}{2m}\vec{p}^{\,2}+\frac{i\beta w_3^2}{4m^2}\vec{\Sigma}\cdot(1+\vec{a}\cdot\vec{X})\left[\vec{\omega}p^2-\vec{p}(\vec{\omega}\cdot\vec{p})\right]+\ldots,\end{array}\tag{49}$$

where $\ldots$ denote smaller correction terms involving gradients of the position-dependent scalar functions $w_i(\vec{X})$, as well as higher order in $\vec{a}$. We complete the derivation of the full FW Hamiltonian (49), by clarifying the acceleration-dependent terms and carefully writing out the gradient corrections that arise due to the position dependence of $w_i(\vec{X})$ and the acceleration vector $\vec{a}$. In the Hamiltonian, acceleration enters via: the scalar potential term $\beta m w_1\vec{a}\cdot\vec{X}$; the modified odd operator $\mathcal{O}$ contains symmetrized terms proportional to $(\vec{a}\cdot\vec{X})(\vec{p}\cdot\vec{\alpha})$; and gradient terms come from commutators of momentum with position-dependent scalar functions $w_i(\vec{X})$. The FW Hamiltonian contains the following acceleration-dependent terms:

$$\begin{array}{l}H_{\rm FW}\supset\beta m w_1+\beta m w_1\vec{a}\cdot\vec{X}+\frac{\beta w_3^2}{2m}\vec{p}^{\,2}(1+\vec{a}\cdot\vec{X})\\ -\vec{\omega}\cdot\vec{L}-w_2\vec{\omega}\cdot\vec{\sigma}+\frac{\beta w_3^2}{2m}\vec{\Sigma}\cdot(\vec{a}\times\vec{p})+\frac{i\beta w_3^2}{4m^2}\vec{\Sigma}\cdot(1+\vec{a}\cdot\vec{X})\left[\vec{\omega}p^2-\vec{p}(\vec{\omega}\cdot\vec{p})\right].\end{array}\tag{50}$$

Now calculate $[\mathcal{O},[\mathcal{O},\mathcal{E}]]$. Since $\mathcal{E}$ contains position-dependent scalar terms and spin couplings, focus on gradient terms; operators like $\vec{p}$ act non-trivially on w_1,w_2,w_3 and $\vec{a}\cdot\vec{X}$. The double commutator generates spin-orbit-like terms, e.g.

$$[\mathcal{O},[\mathcal{O},\mathcal{E}]]\sim-2imw_3^2\beta\vec{\Sigma}\cdot\left[\nabla(w_1\vec{a}\cdot\vec{X})\times\vec{p}\right]+\ldots,$$

plus corrections involving ∇w_2 and $\nabla\vec{\omega}$, which couple spin and momentum.

Putting all together, the full FW Hamiltonian for the generalized Dirac Hamiltonian with position-dependent scalar coefficients $w_i(\vec{X})$ and acceleration $\vec{a}$, rotation $\vec{\omega}$ becomes

$$\begin{array}{l}H_{\rm FW}=\beta m w_1+\beta m w_1\vec{a}\cdot\vec{X}+\frac{\beta}{2m}w_3^2\left((1+\vec{a}\cdot\vec{X})\vec{p}^{\,2}-i\vec{\Sigma}\cdot(\vec{a}\times\vec{p})\right)-\vec{\omega}\cdot\vec{L}-w_2\vec{\omega}\cdot\vec{\sigma}\\ +\frac{\beta}{4m^2}w_3^2\vec{\Sigma}\cdot\left[\nabla(w_1\vec{a}\cdot\vec{X})\times\vec{p}\right]+\ldots\end{array}\tag{51}$$

The ellipsis $\ldots$ stands for additional gradient terms from w_i, higher order terms in $1/m^2$, and corrections due to the rotation $\vec{\omega}$. This Hamiltonian explicitly incorporates the acceleration $\vec{a}$, rotation $\vec{\omega}$, and the spatial variation of the scalar functions $w_i(\vec{X})$, capturing the main physics at order $1/m$.

3.2. FW expansion up to order $1/m^2$

The expansion of FW Hamiltonian up to order $1/m^2$ can be written

$$H_{\rm FW}=\beta\left(m+\frac{\mathcal{O}^2}{2m}-\frac{\mathcal{O}^4}{8m^3}\right)+\mathcal{E}-\frac{1}{8m^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]]+\ldots\tag{52}$$

Thus we aimed to evaluate $\mathcal{O}^2,\mathcal{O}^4$ up to terms with gradients of $w_i(\vec{X})$ and $\vec{a}$, compute the double commutator $[\mathcal{O},[\mathcal{O},\mathcal{E}]]$, and extract gradient and rotation corrections explicitly.

Write $w_3(\vec{X})$ explicitly in the product $\mathcal{O}=w_3(\vec{X})\left[\vec{\alpha}\cdot\vec{p}+\frac{1}{2}\{\vec{a}\cdot\vec{X},\vec{\alpha}\cdot\vec{p}\}\right]$. After some calculation this yields

$$\begin{array}{l}\mathcal{O}^2=w_3^2\left((1+\vec{a}\cdot\vec{X})\vec{p}^{\,2}-i\vec{\Sigma}\cdot(\vec{a}\times\vec{p})\right)\\ -iw_3\left(\vec{\alpha}\cdot\nabla w_3+\frac{1}{2}\{\vec{a}\cdot\vec{X},\vec{\alpha}\cdot\nabla w_3\}\right)\\ \times\left(\vec{\alpha}\cdot\vec{p}+\frac{1}{2}\{\vec{a}\cdot\vec{X},\vec{\alpha}\cdot\vec{p}\}\right),\end{array}\tag{53}$$

where we use the identity, $\vec{\alpha}\cdot\vec{p}\,\vec{\alpha}\cdot\vec{q}=\vec{p}\cdot\vec{q}+i\vec{\Sigma}\cdot(\vec{p}\times\vec{q})$, to evaluate the second term.

First commutator in $[\mathcal{O},[\mathcal{O},\mathcal{E}]]$ reads

$$[\mathcal{O},\mathcal{E}]=[w_3\vec{\alpha}\cdot\vec{p}+\ldots,\mathcal{E}]\approx w_3[\vec{\alpha}\cdot\vec{p},\mathcal{E}]+\ldots$$

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The $\vec{\alpha}\cdot\vec{p}$ does not commute with position-dependent scalars w_1, w_2, e.g. $[p_i, w_1] = -i\partial_i w_1$, also the $\vec{L} = \vec{X}\times\vec{p}$ introduces further commutators, therefore explicit terms are $[\vec{\alpha}\cdot\vec{p}, w_1\beta m\vec{a}\cdot\vec{X}] = -i\beta m\vec{\alpha}\cdot\nabla\left(w_1\vec{a}\cdot\vec{X}\right)$. Similarly, $[\vec{\alpha}\cdot\vec{p}, w_2\vec{\omega}\cdot\vec{\sigma}] = -i\vec{\alpha}\cdot\nabla(w_2\vec{\omega}\cdot\vec{\sigma})$. Then the double commutator is equal to

$$[\mathcal{O}, [\mathcal{O}, \mathcal{E}]] \approx w_3^2[\vec{\alpha}\cdot\vec{p}, [\vec{\alpha}\cdot\vec{p}, \mathcal{E}]] + \ldots$$

Using the identity for double commutators and the Pauli matrix algebra, this generates spin-orbit coupling terms

$$[\vec{\alpha}\cdot\vec{p}, [\vec{\alpha}\cdot\vec{p}, \mathcal{E}]] \sim -2i\beta m\vec{\Sigma}\cdot\left(\nabla\left(w_1\vec{a}\cdot\vec{X}\right)\times\vec{p}\right) + \ldots$$

For $\frac{1}{2}\{\vec{a}\cdot\vec{X}, \vec{\alpha}\cdot\nabla w_3\}$, write symmetrized operator $\frac{1}{2}\left((\vec{a}\cdot\vec{X})(\vec{\alpha}\cdot\nabla w_3) + (\vec{\alpha}\cdot\nabla w_3)(\vec{a}\cdot\vec{X})\right)$. Replacing $\vec{\alpha}$ as before and applying the identity, after some algebra this yields terms like

$$(\vec{a}\cdot\vec{X})\nabla w_3\cdot\vec{p} + i\vec{\sigma}\cdot\left((\vec{a}\cdot\vec{X})\nabla w_3\times\vec{p}\right)$$

plus additional symmetrized corrections. Also recall the spin-orbit-like terms $T_2 = \frac{1}{4m^2}w_3^2\vec{\sigma}$ $\cdot\left(\nabla(w_1\vec{a}\cdot\vec{X})\times\vec{p}\right)$, and $T_3 = \frac{1}{4m^2}w_3^2\vec{\sigma}$ $\cdot\left(\nabla(w_2\vec{\omega})\times\vec{p}\right)$. These are standard spin-orbit coupling terms generalized by scalar functions w_1, w_2 and position dependence. Hence, the gradient terms given in Pauli 2-spinor form become

$$\begin{aligned}
&H_{\rm FW}^{(\rm grad)} = \tfrac{1}{2m}w_3\,\vec{\sigma}\cdot(\nabla w_3\times\vec{p})\\
&-\tfrac{i}{4m}w_3\,(\nabla w_3\cdot\vec{p} + \vec{p}\cdot\nabla w_3)\\
&+\tfrac{1}{2m}w_3\,\vec{\sigma}\cdot\left((\vec{a}\cdot\vec{X})\nabla w_3\times\vec{p}\right)\\
&-\tfrac{i}{4m}w_3\left((\vec{a}\cdot\vec{X})\nabla w_3\cdot\vec{p} + \vec{p}\cdot(\vec{a}\cdot\vec{X})\nabla w_3\right)\\
&+\tfrac{1}{4m^2}w_3^2\,\vec{\sigma}\cdot\left(\nabla(w_1\vec{a}\cdot\vec{X})\times\vec{p}\right)\\
&+\tfrac{1}{4m^2}w_3^2\,\vec{\sigma}\cdot(\nabla(w_2\vec{\omega})\times\vec{p}) + \ldots
\end{aligned} \tag{54}$$

Notes: The terms proportional to $\vec{\sigma}\cdot(\nabla f\times\vec{p})$ describe spin-orbit coupling with space-dependent coefficients. The $-i(\nabla f\cdot\vec{p} + \vec{p}\cdot\nabla f)$ terms correspond to Hermitian symmetrized momentum-dependent potentials (like generalized Darwin terms). Terms with $(\vec{a}\cdot\vec{X})$ multiply the gradients and represent acceleration-dependent corrections to these couplings.

Putting together all the terms derived above, a final full FW Hamiltonian including the gradient terms $H_{\rm FW}^{\rm grad}$ up to order $1/m^2$, with all β factors correctly restored, is written in the two-component Pauli spinor form:

$$\begin{aligned}
&H_{\rm FW} = \beta m w_1(1 + \vec{a}\cdot\vec{X}) + \beta\tfrac{1}{2m}w_3^2\left(\vec{p}^2\right.\\
&\left.+2(\vec{a}\cdot\vec{X})\vec{p}^2\right) - \vec{\omega}\cdot\vec{L} - w_2\vec{\omega}\cdot\vec{\sigma}\\
&+\beta\tfrac{1}{2m}w_3\vec{\sigma}\cdot(\nabla w_3\times\vec{p})\\
&-\beta\tfrac{i}{4m}w_3\,(\nabla w_3\cdot\vec{p} + \vec{p}\cdot\nabla w_3)\\
&+\beta\tfrac{1}{2m}w_3\vec{\sigma}\cdot\left((\vec{a}\cdot\vec{X})\nabla w_3\times\vec{p}\right)\\
&-\beta\tfrac{i}{4m}w_3\left((\vec{a}\cdot\vec{X})\nabla w_3\cdot\vec{p} + \vec{p}\cdot(\vec{a}\cdot\vec{X})\nabla w_3\right)\\
&+\beta\tfrac{1}{4m^2}w_3^2\vec{\sigma}\cdot\left(\nabla(w_1\vec{a}\cdot\vec{X})\times\vec{p}\right)\\
&+\beta\tfrac{1}{4m^2}w_3^2\vec{\sigma}\cdot(\nabla(w_2\vec{\omega})\times\vec{p}) + \cdots
\end{aligned} \tag{55}$$

The standard case of a rotating/accelerated frame corresponds to removing all position dependence of the metric-like scalar functions $w_i(\vec{X})$. Namely, to obtain low-energy FW Hamiltonian in the two-component (Pauli) limit, we simplify all terms in the full FW Hamiltonian (55): Zeroth-order term becomes standard rest mass energy + inertial potential, First-order kinetic and inertial/spin effects match known inertial and spin-rotation couplings in low-energy relativistic quantum mechanics. Similarly, we obtain standard second-order relativistic and spin-orbit-like corrections: Kinetic energy correction, Spin–acceleration–gradient term, Spin–rotation gradient. Gradient corrections vanish in the $w_i \to 1$ limit. Putting together all these terms, we obtain the low-energy FW Hamiltonian in the two-component (Pauli) limit:

$$\begin{aligned}
&H_{\rm FW}^{\rm flat} = \beta m(1 + \vec{a}\cdot\vec{X}) + \beta\tfrac{1}{2m}(1 + \vec{a}\cdot\vec{X})\,\vec{p}^2\\
&-\vec{\omega}\cdot\vec{L} - \vec{\omega}\cdot\vec{\Sigma} + \beta\tfrac{1}{2m}\vec{\Sigma}\cdot(\vec{a}\times\vec{p})\\
&-\beta\tfrac{1}{8m^3}\vec{p}^4 + \beta\tfrac{1}{4m^2}\vec{\Sigma}\cdot(\vec{a}\times\vec{p}) - \beta\tfrac{1}{4m^2}\vec{\Sigma}\cdot(\vec{\omega}\times\vec{p})
\end{aligned} \tag{56}$$

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Now combine similar terms $\beta\vec{\Sigma}\cdot\left[\left(\frac{1}{2m}+\frac{1}{4m^2}\right)(\vec{a}\times\vec{p})\right]$. So we can also write

$$\begin{array}{l} H^{\rm flat}_{\rm FW}=\beta m(1+\vec{a}\cdot\vec{X})+\beta\frac{1}{2m}(1+\vec{a}\cdot\vec{X})\,\vec{p}^{\,2} \\ -\vec{\omega}\cdot\vec{L}-\vec{\omega}\cdot\vec{\Sigma}+\beta\vec{\Sigma}\cdot\left[\left(\frac{1}{2m}+\frac{1}{4m^2}\right)(\vec{a}\times\vec{p})\right] \\ -\beta\frac{1}{8m^3}\vec{p}^{\,4}-\beta\frac{1}{4m^2}\vec{\Sigma}\cdot(\vec{\omega}\times\vec{p}). \end{array} \tag{57}$$

This matches the known low-energy FW Hamiltonian

$$\begin{array}{l} H_{\rm FW}=\beta m+\beta\frac{\vec{p}^{\,2}}{2m}-\beta\frac{\vec{p}^{\,4}}{8m^3}-\vec{\omega}\cdot\vec{L}-\vec{\omega}\cdot\vec{\Sigma} \\ +\beta\frac{1}{2m}\vec{\Sigma}\cdot(\vec{a}\times\vec{p})+\cdots \end{array} \tag{58}$$

Including higher-order spin–inertial terms,

$$+\beta\frac{1}{4m^2}\vec{\Sigma}\cdot(\vec{a}\times\vec{p})-\beta\frac{1}{4m^2}\vec{\Sigma}\cdot(\vec{\omega}\times\vec{p}),$$

we get perfect agreement. Final conclusion is as follows: taking the flat-space limit of full FW Hamiltonian (55), this reduces precisely to the known low-energy limit of the standard Dirac Hamiltonian in an accelerated and rotating frame. This includes all known inertial, spin–inertial, and relativistic corrections to $\mathcal{O}(1/m^2)$, and all gradient corrections vanish, as expected in flat backgrounds. Thus the important inertial effects in the non-relativistic approximation are now derived by the full Hamiltonian (55) beyond the `hypothesis of locality´. These effects are displayed as extended (deformed) versions of the standard ones: namely, extended redshift (Bonse-Wroblewski → COW); extended Sagnac type effect (Page-Werner et al.); extended spin-rotation effect (Mashhoon); extended redshift effect of kin. energy; extended new inertial spin-orbit coupling.

Expanding further the deformation coefficients $w_i(\vec{X})$ into the series in powers of the $\varrho(\tilde{s})$ (**??**), several new effects will rather appeared involving spin, angular momentum, proper linear 3-acceleration $\vec{a}$ and proper 3-angular velocity $\vec{\omega}$ in various mixed combinations.

Actually, expansions for the functions b_0,b_1,b_2,b_3,b_4 to second order in (ϱ), read

$$\begin{array}{l} b_1(\varrho)\approx 1-\frac{\sqrt{2}}{2}\varrho v^{\underline{1}}-\frac{1}{4}\varrho^2(1+(v^{\underline{1}})^2), \\ b_2(\varrho)\approx -\sqrt{2}\varrho-\varrho^2 v^{\underline{1}}, \\ b_0(\varrho)\approx 1+\frac{\sqrt{2}}{2}\varrho v^{\underline{1}}+\frac{1}{4}\varrho^2(3+(v^{\underline{1}})^2), \\ b_3(\varrho)\approx 1-\vec{a}\cdot\vec{X}+\frac{\sqrt{2}}{2}\varrho v^{\underline{1}}+\frac{1}{4}\varrho^2\left(3+(v^{\underline{1}})^2\right), \\ b_4(\varrho)\approx\sqrt{2}\frac{\varrho}{v^{\underline{j}}}+\frac{3}{2}\varrho^2\frac{v^{\underline{1}}}{v^{\underline{j}}}. \end{array}$$

Based on these premises, we compute the expansions of the position scalar functions w_1,w_2,w_3:

$$\begin{array}{l} w_1(\varrho)\approx 1-\vec{a}\cdot\vec{X}+\frac{\sqrt{2}}{2}\,\varrho v^{\underline{1}}+\frac{1}{4}\,\varrho^2(3+(v^{\underline{1}})^2), \\ w_2(\varrho)\approx\frac{\sqrt{2}\,\varrho}{v^{\underline{j}}}+\frac{3}{2}\frac{\varrho^2 v^{\underline{1}}}{v^{\underline{j}}} \\ +\frac{\sqrt{2}}{2}\frac{\varrho}{(v^{\underline{j}})^2}\,\vec{\omega}\cdot(\nabla v^{\underline{j}}\times\vec{v}), \\ w_3(\varrho)\approx 1-\vec{a}\cdot\vec{X}+\sqrt{2}\,\varrho v^{\underline{1}}+\frac{1}{2}\,\varrho^2(2+(v^{\underline{1}})^2). \end{array} \tag{59}$$

The $w_2(\varrho)$ can be simplified for locally linear $v^{\underline{j}}$, and rewritten fully in vector form in terms of $\vec{\omega}$ and $\vec{X}$:

$$\begin{array}{l} w_2(\vec{X})\approx\frac{\sqrt{2}\,\varrho}{|\vec{\omega}\times\vec{X}|}\left(1-\frac{1}{2}\omega^2\right)+\frac{3}{2}\frac{\varrho^2\,v^1}{|\vec{\omega}\times\vec{X}|}\left(1-\frac{1}{3}\omega^2\right) \\ +\mathcal{O}(\varrho^3). \end{array} \tag{60}$$

The further vector form of the set of expansions $w_i(\vec{X})$ $(i=1,2,3)$ is given

$$\begin{array}{l} w_1(\vec{X})\approx 1-\vec{a}\cdot\vec{X}+\frac{\sqrt{2}}{2}\,\varrho\,v^1+\frac{1}{4}\varrho^2\left(3+(v^1)^2\right), \\ w_2(\vec{X})\approx\frac{\sqrt{2}\,\varrho}{|\vec{\omega}\times\vec{X}|}\left(1-\frac{1}{2}\omega^2\right)+\frac{3}{2}\frac{\varrho^2\,v^1}{|\vec{\omega}\times\vec{X}|}\left(1-\frac{1}{3}\omega^2\right), \\ w_3(\vec{X})\approx 1-\vec{a}\cdot\vec{X}+\sqrt{2}\,\varrho\,v^1+\frac{1}{2}\varrho^2\left(2+(v^1)^2\right). \end{array} \tag{61}$$

of position dependence scalar functions $w_i(\vec{X})$, which makes clear the centrifugal-type corrections (proportional to ω^2) in the local rotational frame. Using this set, we may further obtain compact operator form of

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the FW Hamiltonian (55) to second order in ϱ beyond the `hypothesis of locality´. Particularly, keeping only the linear terms ($\vec{a}\cdot\vec{X}$) and up to $O(\varrho^2)$, the FW Hamiltonian finally becomes

$$\begin{array}{l} H_{\rm FW} \approx \beta m_{\rm eff} + \beta \frac{Z_p}{2m}\vec{p}^2 - \vec{\omega}\cdot\vec{L} - V_{\sigma\omega} \\ +V_{(\sigma a,p)} + V_{(\sigma\nabla v_1,p)} + \mathcal{O}(\varrho^3), \end{array} \tag{62}$$

provided,

$$\begin{array}{l} m_{\rm eff} = m\left[1+\frac{\sqrt{2}}{2}\varrho v_1 + \frac{1}{4}\varrho^2(3+v_1^2)\right], \\ Z_p = 1 + 2\sqrt{2}\varrho v_1 + \varrho^2(4+2v_1^2), \\ V_{\sigma\omega} = \frac{\vec{\omega}\cdot\vec{\sigma}}{|\vec{\omega}\times\vec{X}|}\left[\sqrt{2}\varrho(1-\frac{1}{2}\omega^2) + \frac{3}{2}\varrho^2 v_1(1-\frac{1}{3}\omega^2)\right], \\ V_{(\sigma a,p)} = -\frac{\beta}{2m}\left[1+\varrho^2(2+v_1^2)\right]\vec{\sigma}\cdot(\vec{a}\times\vec{p}), \\ V_{(\sigma\nabla v_1,p)} = \frac{\beta\sqrt{2}\varrho}{2m}\,\vec{\sigma}\cdot(\nabla v_1\times\vec{p}), \end{array} \tag{63}$$

where $m_{\rm eff}$ is the mass correction ($\mathcal{O}(\varrho^2)$)- gravitational - type energy shift; $Z_p\vec{p}^2/2m$ is the kinetic term renormalization ($\mathcal{O}(\varrho^2)$) - curvature/acceleration dependence; $-\vec{\omega}\cdot\vec{L}$ is the rotational coupling - $\mathcal{O}(\varrho^0)$ - inertial frame rotation; $-V_{\sigma\omega}$ is the spin–rotation ($\mathcal{O}(\varrho)$–$\mathcal{O}(\varrho^2)$) - coupling to rotating frame; $V_{(\sigma a,p)}$ is the spin–acceleration ($\mathcal{O}(\varrho^0)$–$\mathcal{O}(\varrho^2)$) - generalized Thomas precession; $V_{(\sigma\nabla v_1,p)}$ is the gradient/spin–momentum ($\mathcal{O}(\varrho)$) - inhomogeneous field correction.

4. Concluding remarks

In this section we briefly highlight the key points of present report. This is the second of three papers that explore the possibility of quantum mechanical inertial properties of the Dirac particle beyond the `hypothesis of locality´. This is done within the framework of the *Master Space*-Teleparallel Supergravity ($\widetilde{MS}_p$-TSG) (Ter-Kazarian, 2025a) theory, which we recently proposed taking into account inertial effects (Ter-Kazarian, 2026). In Ter-Kazarian (2025b) (first article of three), we derived the general Dirac equation in an accelerated and rotating frame of reference beyond the`hypothesis of locality´. This equation, however, contains also residual imaginary terms, which are artifacts that due to coordinate transformations in the non-inertial frames. To eliminate to all orders these terms, in present article we apply the standard techniques used in relativistic quantum mechanics and quantum field theory, where non-Hermitian terms can be removed via suitable similarity transformations. We begin by removing a purely imaginary term in (3) via a similarity transformation to first order in BCH expansion. According to our strategy (10), we next redefine the wave function and perform the second similarity transformation on the Dirac Hamiltonian, provided that we choose S such that the imaginary potential is eliminated to first order. This standard method allows to choose a physically more suitable reference frame. The expectation values of physical observables remain real. No imaginary contamination remains in physical quantities. Thus the energy, momentum, probability, etc. remain real and consistent. We are also interested in low-energy properties, avoiding solutions with negative energy. In the method employed for reducing the Hamiltonian (41) to non-relativistic two-component form, in order to decouple the positive and the negative energy states, we used an approximate scheme of the FW canonical transformation of the Dirac Hamiltonian for a free particle. This is made by an infinite sequence of FW-transformations leading to a deformed Hamiltonian, which is an infinite series in powers of $(1/m)$. Hence the reduced deformed Hamiltonian can be recast into the form (dropping triple primes) (45). Evaluating the operator products to the desired order of accuracy, we find the deformed, non-relativistic Hamiltonian (55). We then find the inertial effects for a massive Dirac fermion in non-relativistic approximation, which are displayed beyond the `hypothesis of locality´ as extended (deformed) versions of the standard effects: extended redshift (Bonse-Wroblewski → COW); extended Sagnac type effect (Page-Werner et al.); extended spin-rotation effect (Mashhoon); extended redshift effect of kin. energy; extended new inertial spin-orbit coupling. Taking the flat-space limit of full FW Hamiltonian (55), we obtain the low-energy FW Hamiltonian in the two-component (Pauli) limit (57), which matches the known low-energy FW Hamiltonian (58). Including higher-order spin–inertial terms, we get perfect agreement with the known low-energy limit of the standard Dirac Hamiltonian in an accelerated and rotating frame. Expanding further the deformation coefficients into the series in powers of the $\varrho(\tilde{s})$, several new effects will rather appeared involving spin, angular momentum, proper linear 3-acceleration $\vec{a}$ and proper 3-angular velocity $\vec{\omega}$ in various mixed combinations.

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References

Abele H., Leeb H., 2012, New J. Phys., 14, 055010

Anandan J., 1977, Phys. Rev. D., 15, 1448

Atwood D., et al. 1984, Phys. Lett., 52, 1673

Audretsch J., Schafer G., 1978, Gen. Rel. Grav., 9, 243

Bender C., Boettcher S., 1998, Phys. Rev. Lett., 80, 5243

Birrell N., Davies P., 1982, Quantum Fields in Curved Space. Cambridge University Press

Bjorken J., Drell S., 1964, Relativistic Quantum Mechanic. McGraw-Hill, New York

Bonse U., Wroblewski T., 1983, Phys. Rev. Lett., 51, 1401

Brill D., Wheeler J., 1957, Rev. of Mod. Phys., 29, 465

Cai Y., Papini G., 1991, Phys. Rev. Lett., 66, 1259

Cai Y., Papini G., 1992, Phys. Rev. Lett., 68, 3811

Colella R., Overhauser A., Werner S., 1975, Phys. Rev. Lett., 34, 1472

Fischbach E., Freeman B., Cheng W.-K., 1981, Phys. Rev. D, 23, 2157

Foldy L., Wouthuysen S., 1950, Phys. Rev., 78, 29

Greiner W., 2000, Relativistic Quantum Mechanics: Wave Equations. Springer

Hasegawa Y., Rauch H., 2011, New J. Phys., 13, 115010

Hehl F., Ni W.-T., 1990, Phys. Rev. D, 42, 2045

Hehl F., Lemke J., Mielke E., 1991, Two lectures on fermions and gravity". In: Geometry and Theoretical Physics, p.56. J.Debrus and A.C. Hirshfeld (eds.) Springer, Berlin Heidelberg

Kajari E., Harshman N., Rasel E., et al. 2010, Appl. Phys. B, 100, 43

Maluf J., Faria F., 2008, Ann. der Physik, 520, 326

Maluf J., Faria F., Ulhoa S., 2007, Class. Quantum Grav., 24, 2743

Marzlin K.-P., 1996, Phys. Lett. A, 215, 1

Mashhoon B., 2002, Ann. der Physik, 514, 532

Mashhoon B., 2011, Ann. der Physik, 523, 226

Michelson A., Gale H., Pearson F., 1925, Astrophys. J., 61, 140

Misner C., Thorne K., Wheeler J., 1973, Gravitation. Freeman, San Francisco

Mostafazadeh A., 2002, J. Math. Phys., 43, 205

Obukhov Y. N., 2001, Phys. Rev. Lett., 86, 192

Obukhov Y. N., 2002, Fortschr. Phys., 50, issue 5, 711

Obukhov Y. N., Rubilar G., 2006, Phys. Rev. D, 73, 124017

Page L., 1975, Phys. Rev. Lett., 35, 543

Parker L., Toms D., 2009, Quantum Field Theory in Curved Spacetime. Cambridge University Pres

Peskin M., Schroeder D., 1995, An Introduction to Quantum Field Theory. Addison-Wesley

Rauch H., Werner S., 2000, Neutron Interferometry, Lessons in Experimental Quantum Mechanics. Oxford: Oxford Science Publications, Clarendon

Ryder L., 1998, J. Phys. A: Math. Gen. A, 31, 2465

Schwartz E., Meitav N., 2013, Phys. Educ., 48, 203

Singh D., Papini G., 2000, Nuovo Cim. B, 115, 223

Staudenmann J., Werner S., Colella R., Overhauser A., 1980, Phys. Rev. A, 21, 1419

Synge J., 1960, Relativity: The General Theory. North-Holland, Amsterdam

Ter-Kazarian G., 2024a, Grav. Cosmol., 30, issue 1, 8

doi:https://doi.org/10.52526/25792776-25.72.2-226

Ter-Kazarian G., 2024b, Communications of BAO, 71, Issue 2,

Ter-Kazarian G., 2024c, Communications of BAO, 71, Issue 2, 249

Ter-Kazarian G., 2024d, Communications of BAO, 71, Issue 2, 289

Ter-Kazarian G., 2025a, Grav. Cosmol., 31, issue 2, 145

Ter-Kazarian G., 2025b, Communications of BAO, 72, Issue 2

Ter-Kazarian G., 2026, Grav. Cosmol., 32, Issue 1

Tsai W., 1981, Phys. Rev. D, 24, 1375

Varjú K., Ryder L., 1998, Phys. Lett. A, 250, 263

Varjú K., Ryder L., 2000, Phys. Rev. D, 62, 024016

Werner S., Staudenmann J.-L., Colella R., 1979, Phys. Rev. Lett., 42, 1103

de Oliveira C., Tiomno J., 1962, Nuovo Cim., 24, 672

Appendices

Appendix A Dirac equation in non-inertial frame beyond the hypothesis of locality: Revisited

We consider only mass points, then the non-inertial frame of reference in the Minkowski space of SR is represented by a curvilinear coordinate system, since it is conventionally accepted to use the names `curvilinear coordinate system´ and `non-inertial system´ interchangeably. In the Minkowski spacetime of SR in Cartesian coordinates $\overline{x}^{\mu'} = (\overline{x}^0, \overline{x}^1, \overline{x}^2, \overline{x}^3)$, the Dirac equation for a massive fermion reads (Ter-Kazarian, 2025b)

$$i\gamma^{\mu'}\overline{\partial}_{\mu'}\Psi' = m\Psi', \tag{64}$$

with the Dirac matrices $\gamma^{\mu'}$. Here we will essentially follow Bjorken-Drell Bjorken & Drell (1964), in particular we use its conventions for the Dirac matrices: $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha^i$. In its most fundamental form, the Dirac equation in a locally accelerated and rotating frame of reference of the observer, obtained from first principles, is a generalization of the equation (64):

$$i\gamma^{\hat{\mu}} D_{(\hat{\mu})}\Psi = m\Psi, \tag{65}$$

where the anholonomic Dirac matrices are defined by $\gamma^{\hat{\mu}} := e^{\hat{\mu}}{}_{\nu}\gamma^{\nu}$, and $\gamma^{\hat{\mu}}\gamma^{\hat{\nu}} + \gamma^{\hat{\nu}}\gamma^{\hat{\mu}} = 2o^{\hat{\mu}\hat{\nu}}$. The partial derivative in the Dirac equation is simply replaced by the covariant derivative

$$D_{(\hat{\mu})} := \partial_{(\hat{\mu})} + \Gamma_{(\hat{\mu})}, \tag{66}$$

where the quantities $\Gamma_{(\hat{\mu})}$ are related to the connection coefficients. The latter can be written in terms of the object of anholonomicity of the commutation table for the anholonomic frame. The components of anholonomicity (the structure-constants) read

$$C^{(\hat{\lambda})}{}_{(\hat{\mu})(\hat{\nu})} = e_{(\hat{\mu})}{}^{\mu} e_{(\hat{\nu})}{}^{\nu}(\partial_\mu e^{(\hat{\lambda})}{}_{\nu} - \partial_\nu e^{(\hat{\lambda})}{}_{\mu}). \tag{67}$$

Rewrite the anholonomic frame vectors in terms of coordinate basis

$$\begin{aligned} e_{(\hat{0})} &= b_0^{-1}\left(e_0 - \bar{v}^k e_k\right) = b_0^{-1}\left(\partial_0 - \bar{v}^k\partial_k\right), \\ e_{(\hat{i})} &= b_1^{-1}e_i = b_1^{-1}\partial_i, \end{aligned} \tag{68}$$

provided, denote $\frac{b_2^j}{b_1} \equiv \bar{v}^j = -b_4 v^j, \quad v^j = (\vec{\omega}\times\vec{X})^j$. To lower the upper index by a metric $o_{(\hat{\rho})(\hat{\lambda})}$, using an orthogonal basis $o = (diag{+1}, -1, -1, -1)$, the structure-constants become $C_{(\hat{\mu})(\hat{\nu})(\hat{\rho})}$ $= o_{(\hat{\rho})(\hat{\lambda})}C^{(\hat{\lambda})}{}_{(\hat{\mu})(\hat{\nu})}$. Hence, by virtue of (68) and (67), we summarize all non vanishing components $C_{(\hat{\mu})(\hat{\nu})(\hat{\lambda})}$ of the anholonomicity as follows:

$$\begin{aligned} &C_{(\hat{0})(\hat{\imath})(\hat{0})} = -C_{(\hat{\imath})(\hat{0})(\hat{0})} = \tfrac{1}{b_1^{-1}}\partial_i \ln b_0, \\ &C_{(\hat{0})(\hat{\imath})(\hat{\jmath})} = -C_{(\hat{\imath})(\hat{0})(\hat{\jmath})} = -\tfrac{\partial_i v_j}{b_0} + \vec{\bar{v}}\cdot(\nabla b_1^{-1})\delta_{ij}, \\ &C_{(\hat{\imath})(\hat{\jmath})(\hat{k})} = -C_{(\hat{\jmath})(\hat{\imath})(\hat{k})} = -(\partial_i b_1^{-1})\delta_{jk} \\ &+(\partial_j b_1^{-1})\delta_{ik}, \quad C_{(\hat{\imath})(\hat{\jmath})(\hat{0})} = C_{(\hat{\mu})(\hat{\nu})(\hat{\lambda})} = 0. \end{aligned} \tag{69}$$

The connection components defined with respect to the anholonomic frame (68) read

$$\begin{aligned} &\Gamma_{(\hat{\lambda})(\hat{\nu})(\hat{\mu})} \\ &= \tfrac{1}{2}(C_{(\hat{\lambda})(\hat{\nu})(\hat{\mu})} + C_{((\hat{\lambda})\hat{\mu})(\hat{\nu})} - C_{(\hat{\nu})(\hat{\mu})(\hat{\lambda})}). \end{aligned} \tag{70}$$

Based on (69), it is straightforward to calculate the connection coefficients:

$$\begin{aligned} &\Gamma_{(\hat{0})(\hat{\imath})(\hat{0})} = -\Gamma_{(\hat{\imath})(\hat{0})(\hat{0})} = \tfrac{1}{b_1^{-1}}\partial_i \ln b_0, \\ &\Gamma_{(\hat{\imath})(\hat{\jmath})(\hat{0})} = \tfrac{1}{2b_0}(\partial_i\bar{v}_j - \partial_j\bar{v}_i) \\ &= \tfrac{b_4}{b_0}\partial_i v_j + \tfrac{1}{2b_0}\left[(\partial_i b_4)v_j - (\partial_j b_4)v_i\right], \\ &\Gamma_{(\hat{\imath})(\hat{\jmath})(\hat{k})} = -(\partial_i b_1^{-1})\delta_{jk} + (\partial_j b_1^{-1})\delta_{ik}, \\ &\Gamma_{(\hat{0})(\hat{\imath})(\hat{\jmath})} = -\Gamma_{(\hat{\imath})(\hat{0})(\hat{\jmath})} = -\tfrac{1}{2b_0}(\partial_i\bar{v}_j + \partial_j\bar{v}_i) \\ &+\vec{\bar{v}}\cdot(\nabla b_1^{-1})\delta_{ij} = -\tfrac{1}{2b_0}\left[(\partial_i b_4)v_j + (\partial_j b_4)v_i\right] \\ &+\vec{\bar{v}}\cdot(\nabla b_1^{-1})\delta_{ij}, \quad \Gamma_{(\hat{0})(\hat{0})(\hat{0})} = \Gamma_{(\hat{\mu})(\hat{\nu})(\hat{\imath})} = 0. \end{aligned} \tag{71}$$

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In the limit of the hypothesis of locality: $(\pi) \to 1$, the deformation coefficients $b_1, b_3 \equiv b_0/(1+\vec{a}\cdot\vec{X}), b_4$ tend to 1, and hence the (69) and (71) restore the standard contributions by Hehl et al. (1991). Actually,

$$\begin{array}{l}\Gamma_{(\hat{0})(\hat{\imath})(\hat{0})} = C_{(\hat{0})(\hat{\imath})(\hat{0})} \\ \to\ \Gamma_{\hat{0}\,\hat{\imath}\,\hat{0}} = C_{\hat{0}\,\hat{\imath}\,\hat{0}} = \frac{a_i}{(1+\vec{a}\cdot\vec{X})}.\end{array} \tag{72}$$

Since $[a_i = g_{ij}a^j = -\delta_{ij}a^j = -a^i$, then

$$\Gamma_{\hat{0}\,\hat{\imath}\,\hat{0}} = -\Gamma_{\hat{\imath}\,\hat{0}\,\hat{0}} = -\frac{a^i}{(1+\vec{a}\cdot\vec{X})}. \tag{73}$$

Next,

$$\begin{array}{l}\Gamma_{(\hat{0})(\hat{\imath})(\hat{\jmath})} \to \Gamma_{\hat{0}\,\hat{\imath}\,\hat{\jmath}} = \frac{\partial_i v_j}{(1+\vec{a}\cdot\vec{X})} = o_{jk}\frac{\partial_i v^k}{(1+\vec{a}\cdot\vec{X})} \\ = o_{jk}\frac{\varepsilon^k{}_{mi}\omega^m}{(1+\vec{a}\cdot\vec{X})} = -\frac{\varepsilon_{mij}\omega^m}{(1+\vec{a}\cdot\vec{X})} = -\frac{\varepsilon_{ijm}\omega^m}{(1+\vec{a}\cdot\vec{X})}.\end{array} \tag{74}$$

All other components vanish $\Gamma_{\hat{\mu}\,\hat{\nu}\,\hat{\imath}} = \Gamma_{\hat{0}\,\hat{0}\,\hat{0}} = 0$. The quantities

$$\Gamma_{(\hat{\mu})} := -\frac{i}{4}\sigma^{\hat{\lambda}\hat{\nu}}\Gamma_{(\hat{\lambda})(\hat{\nu})(\hat{\mu})}, \tag{75}$$

that appear in the Dirac equation can now be calculated by means of (71) and the six matrices $\sigma^{\hat{\lambda}\hat{\nu}}$ of the infinitesimal generators of the Lorentz group

$$\sigma^{\hat{\lambda}\hat{\nu}} := \frac{i}{2}[\gamma^{\hat{\lambda}}, \gamma^{\hat{\nu}}]. \tag{76}$$

After calculation and simplification of (75), we find

$$\begin{array}{l}\Gamma_{(\hat{0})} = -\frac{1}{2}\vec{a}_1\cdot\alpha - \frac{i}{2}\vec{\omega}_1\cdot\vec{\sigma}, \\ \Gamma_{(\hat{k})} = \vec{a}_k\cdot\vec{\alpha} + i\,\vec{\omega}_k\cdot\vec{\sigma},\end{array} \tag{77}$$

provided,

$$\begin{array}{l}\vec{a}_1 \equiv \frac{1}{b_1}(\nabla\ln b_0), \\ \vec{\omega}_1 \equiv \frac{1}{b_0}\left[b_4\vec{\omega} - \frac{1}{2}(\nabla\ln b_4)\times\vec{v}\right], \\ a_{ik} = a_{ki} \equiv \frac{1}{4b_0}\left[(\partial_i b_4)v_k + (\partial_k b_4)v_i\right] \\ -\frac{1}{2}(\vec{v}\cdot\nabla b_1^{-1})\delta_{ik}, \\ \omega_{ik} = -\omega_{ki} \equiv \frac{1}{2}\varepsilon_{kli}(\partial_i b_1^{-1})),\end{array} \tag{78}$$

where ε_{ijk} is the three-dimensiona1 Levi-Civita symbol with $\varepsilon_{123} = 1$. Collecting (66) and (77) together, we find for the deformed spinor covariant derivatives

$$\begin{array}{l}D_{(\hat{0})} = \frac{1}{b_0}\left\{\frac{\partial}{\partial X^0} - \frac{1}{2b_1}\nabla b_0\cdot\vec{\alpha} - i\,\vec{\omega}\cdot\vec{L} - i\,\vec{\omega}_S\cdot\vec{S}\right\}, \quad \omega_S \equiv b_4\vec{\omega} - \frac{1}{2}\nabla(b_4)\times\vec{v}, \\ D_{(\hat{\imath})} = b_1^{-1}\frac{\partial}{\partial X^i} + \vec{a}_i\cdot\vec{\alpha} + i\,\vec{\omega}_i\cdot\vec{\sigma},\end{array} \tag{79}$$

where the orbital ($\vec{L}$) and spin ($\vec{S}$) operators respectively have the form

$$\vec{L} \equiv (\vec{X}\times\frac{\partial}{i\partial\vec{X}}) = (\vec{X}\times\vec{p}), \quad \vec{S} \equiv \frac{1}{2}\vec{\sigma}. \tag{80}$$

In the standard limit, $(\pi) \to 1$, of the hypothesis of locality, the deformation coefficients b_1, b_3, b_4, tend to 1, so that the (79) and (80) restore the results of Hehl & Ni (1990):

$$\begin{array}{l}\vec{J} \to \vec{J} = \vec{L} + \vec{S} \equiv (\vec{X}\times\frac{\partial}{i\partial\vec{X}}) + \frac{1}{2}\vec{\sigma}, \\ D_{(\hat{0})} \to D_{\hat{0}} = \frac{1}{(1+\vec{a}\cdot\vec{X})}\left(\frac{\partial}{\partial X^0} - \vec{v}\cdot\frac{\partial}{\partial\vec{X}} + \frac{1}{2}\vec{a}\cdot\vec{\alpha} - i\,\vec{\omega}\cdot\vec{J}\right), \quad D_{(\hat{\imath})} \to D_{\hat{\imath}} = \frac{\partial}{\partial X^i}.\end{array} \tag{81}$$

Substituting (79) into (65) and multiplying it by $\gamma^0\beta b_0$, we obtain the explicit form of the Dirac equation beyond the hypothesis of locality for an observer in a reference frame that is accelerated with a proper linear 3-acceleration $\vec{a}$ and rotating with proper 3-angular velocity $\vec{\omega}$:

$$\begin{array}{l}\left\{i\partial_0 - i\frac{1}{2b_1}(\nabla b_0\cdot\vec{\alpha}) + \vec{\omega}\cdot\vec{L} + \vec{\omega}_S\cdot\vec{S} - b_0 b_1^{-1}(\vec{\alpha}\cdot\vec{p}) + \frac{i}{2}(\nabla b_4 - 3b_4 b_0\nabla b_1^{-1})\cdot\vec{v}\right. \\ \left.-ib_0(\vec{\alpha}\cdot\nabla b_1^{-1})\right\}\Psi = b_0\beta m\Psi.\end{array} \tag{82}$$

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