The Problem of Diffuse Reflection in the Case of Redistribution of Radiation by Frequencies and Directions: Complete Simplification of the Solution by means of Decomposition of Resultant Field (DRF) Method

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Abstract

The method of decomposition of the resulting field (DRF) developed by the author is used to further simplify the solution of the diffuse reflection problem (DRP) from a semi-infinite medium in the case of redistribution of radiation by frequencies and directions. Taking into account the presence of a natural combination of frequency and angle of absorption of a quantum makes it possible to bring the solution of the problem to its complete simplification. The search for eigenfunctions and eigenvalues of DRP that depend on three independent variables and satisfy the joint system of integral equations with nonlinearity of the fifth degree and algebraic equations of the second degree is reduced to the determination of two types of new auxiliary functions from the system of nonlinear integral equations. The first type depends on only one argument with the second degree of nonlinearity of the equations, and the second one on three arguments with the third degree of nonlinearity of integral equations. The accompanying systems of algebraic equations that determine the corresponding eigenvalues are also simplified: instead of nonlinearity of the second degree, linear systems are obtained.

Keywords: *radiative transfer, diffuse reflection problem, redistribution of radiation by frequencies and directions, Ambartsumian's nonlinear functional equation, eigenfunctions in radiative transfer, decomposition of the resulting field*

1. Initial task

In a recent paper by the author (Pikichyan, 2023c) , the method of decomposition of the resultant field (DRF) was applied to the solution of classical diffuse reflection problem (DRP) of radiation from a semi-infinite medium in the case when there is a redistribution of radiation by frequencies and directions during the elementary act of scattering. The following notations are adopted: r - the redistribution function of radiation by frequencies and directions in a single act of scattering; ρ - the function of diffusely reflected radiation from a semi-infinite scattering-absorbing medium in the probabilistic representation; $\mu,\ \mu'$ - the cosines of the angles of the quanta reflected from the medium and incident on its boundary, respectively, relative to the outer normal of boundary; $\varphi,\ \varphi'$ - the corresponding azimuths, and x, x' - the corresponding dimensionless frequencies. In the theory of radiant energy transfer the standard relations are well known for this general case:

$$r\left(x,x';\gamma\right)=r\left(x,\mu;x',\mu';\varphi-\varphi'\right),\quad \gamma=\vec{n}\cdot\vec{n}',\quad \vec{n}\equiv(\mu,\varphi\ ),\quad \mu\in[0,1]\,,\quad \varphi\in[0,2\pi]$$

$$\begin{gathered}r\left(x,-\mu;x',-\mu';\varphi-\varphi'\right)=r\left(x,+\mu;x',+\mu';\varphi-\varphi'\right),\\ r\left(x,+\mu;x',-\mu';\varphi-\varphi'\right)=r\left(x',+\mu';x,-\mu;\varphi'-\varphi\right),\\ \rho\left(x,\mu;x'\mu';\varphi-\varphi'\right)\mu'=\rho\left(x'\mu';x,\mu;\varphi'-\varphi\right)\mu.\end{gathered}\tag{1}$$

The analogue of Ambartsumian's nonlinear functional integral equation in this case is written in the form

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doi: https://doi.org/10.52526/25792776-25.72.1-113

$$\frac{4\pi}{\lambda}\mu'\left(\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}\right)\rho\left(x,\mu;x'\mu';\varphi-\varphi'\right)=r\left(x,\mu;x'-\mu';\varphi-\varphi'\right)+$$

$$\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 r\left(x,\mu;x''',\mu''';\varphi-\varphi'''\right)\rho\left(x',\mu';x''',\mu''';\varphi'-\varphi'''\right)d\mu'''d\varphi'''dx'''+$$

$$\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 \rho\left(x,\mu;x'',\mu'';\varphi-\varphi''\right)r\left(x'',\mu'';x',\mu';\varphi''-\varphi'\right)d\mu''d\varphi''dx'' \; + \tag{2}$$

$$\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 \rho\left(x,\mu;x''',\mu''';\varphi-\varphi'''\right)r\left(x''',-\mu''';x'',\mu'';\varphi'''-\varphi''\right)\times$$

$$\rho\left(x',\mu';x''\mu'';\varphi'-\varphi''\right)d\mu''d\varphi''dx''d\mu'''d\varphi'''dx''' \; ,$$

Here the value $\alpha(x)$ is an absorption profile.

2. Initial solution

To solve equation (2), by means of the DRF method the following value was introduced:

$$K\left(x,\mu;x',\mu';\varphi-\varphi'\right)\equiv\mu'\left(\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}\right)\rho\left(x,\mu;x'\mu';\varphi-\varphi'\right),$$

$$K\left(x,\mu;x',\mu';\varphi-\varphi'\right)=K\left(x',\mu';x,\mu;\varphi-\varphi'\right), \tag{3}$$

and then equation (2) was rewritten for the introduced symmetric kernel – K. For a more compact presentation of further formulas, it is advisable to introduce shortened notations below:

$$\pm M\equiv\{x,\pm\mu,\varphi\}, \quad A=A\{x,\mu\}\equiv\frac{\alpha(x)}{\mu}, \quad \alpha(+x)=\alpha(-x),$$

$$K\left(M,M'\right)\equiv\mu'\left(A+A'\right)\rho\left(M,M'\right)), \; K\left(M,M'\right)=K\left(M',M\right),$$

$$\begin{gathered}\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 F_1\left(x,\mu;x'',\mu'';\varphi-\varphi''\right)F_2\left(x'',\mu'';x',\mu';\varphi''-\varphi'\right)d\mu''d\varphi''dx'' \\ \equiv F_1\left(x,\mu;x'',\mu'';\varphi-\varphi''\right)\circ F_2\left(x'',\mu'';x',\mu';\varphi''-\varphi'\right)\equiv F_1\left(M,M''\right)\circ F_2\left(M'';M'\right).\end{gathered} \tag{4}$$

Then the equation mentioned above for the quantity $K(M,M')$ can be recorded compactly - in the operator form with the help of the relations (2), (3) and (4)

$$\frac{4\pi}{\lambda}K\left(M,M'\right)=r\left(M,-M'\right)+r\left(M,M''\right)\frac{1}{\mu''}\circ\frac{K\left(M'',M'\right)}{A''+A'}+\frac{K\left(M,M''\right)}{A+A''}\frac{1}{\mu''}\circ r\left(M'',M'\right)+$$

$$\frac{K\left(M,M'''\right)}{A+A'''}\frac{1}{\mu'''}\circ r\left(-M''',M''\right)\circ\frac{1}{\mu''}\frac{K\left(M'',M'\right)}{A''+A'}. \tag{5}$$

In the DRF method, in contrast to the standard method of decomposition of the characteristic of a single act of scattering (DSA method), which is widespread in the theory of radiative transfer, is used the opportunity of approximate representation of the introduced symmetric (3) and positive $K>0$ nucleus $K(M,M')$ directly through its eigenfunctions and eigenvalues. There is no need for a preliminary expansion or any special representation of the value $r(M,M'')$ describing the elementary act of scattering. To apply the DRF method, you first set the problem on eigenfunctions and eigenvalues

$$\nu_i\beta_i(M)=K\left(M,M'\right)\circ\beta_i\left(M'\right),\beta_i(M)\circ\beta_j(M)=\delta_{ij}, \tag{6}$$

doi: https://doi.org/10.52526/25792776-25.72.1-113

and then the desired kernel in the Nth approximation was represented in terms of its eigenfunctions $\beta_j(M)$ and eigenvalues ν_j by bilinear decomposition

$$K\left(M, M'\right) \sim K_N\left(M, M'\right) = \sum_{j=1}^{N} \nu_j \beta_j\left(M\right) \beta_j\left(M'\right) \ . \tag{7}$$

From expressions (5)-(7) follows a system of nonlinear integral functional equations to determine the eigenfunctions of the nucleus $K_N(M, M')$.

$$\frac{4\pi}{\lambda}\nu_i\beta_i\left(M\right) = \left[r\left(M, -M'\right) + \ r\left(M, M''\right) \circ \frac{1}{\mu''}\frac{\sum_{j=1}^{N}\nu_j\beta_j\left(M''\right)\beta_j\left(M'\right)}{A'' + A'}\right] \circ \beta_i\left(M'\right) +$$
$$+\sum_{j=1}^{N}\nu_j\beta_j\left(M\right)\frac{\beta_j\left(M'''\right)}{A + A'''}\frac{1}{\mu'''} \circ \left[r\left(M''', M'\right) + \ r\left(-M''', M''\right) \circ \frac{1}{\mu''}\frac{\sum_{k=1}^{N}\nu_k\beta_k\left(M''\right)\beta_k\left(M'\right)}{A'' + A'}\right] \circ \beta_i\left(M'\right) \tag{8}$$

The latter system is not difficult to imagine in the form of

$$\frac{4\pi}{\lambda}\nu_i\beta_i\left(M\right) = Z_i\left(M, \beta\right) + \sum_{j=1}^{N}\nu_j D_{ji}\left(M, \beta^3\right) + \sum_{j=1}^{N}\sum_{k=1}^{N}\nu_j\nu_k V_{jki}\left(M, \beta^5\right) \tag{9}$$

where are the notations taken into account:

$$Z_i\left(M, \beta\right) \equiv r\left(M, -M'\right) \circ \beta_i\left(M'\right),$$

$$D_{ji}\left(M, \beta^3\right) \equiv \beta_j\left(M\right)\frac{\beta_j\left(M''\right)}{A + A''}\frac{1}{\mu''} \circ r\left(M'', M'\right) \circ \beta_i\left(M'\right) + r\left(M, M''\right) \circ \frac{1}{\mu''}\frac{\beta_j\left(M''\right)\beta_j\left(M'\right)}{A'' + A'} \circ \beta_i\left(M'\right),$$

$$V_{jki}\left(M, \beta^5\right) \equiv \frac{\beta_j\left(M\right)\beta_j\left(M'''\right)}{A + A'''}\frac{1}{\mu'''} \circ r\left(-M''', M''\right) \circ \frac{1}{\mu''}\frac{\beta_k\left(M''\right)\beta_k\left(M'\right)}{A'' + A'} \circ \beta_i\left(M'\right). \tag{10}$$

In order to determine the corresponding eigenvalues, a system of non-linear, but already algebraic equations follow from expressions (9)

$$\frac{4\pi}{\lambda}\nu_i = b_i + \sum_{k=1}^{N}\nu_k c_{ki} + \sum_{m=1}^{N}\sum_{n=1}^{N}\nu_m\nu_n f_{mni} \ , \tag{11}$$

where indicated:

$$b_i \equiv \beta_i\left(M\right) \circ r\left(M, -M'\right) \circ \beta_i\left(M'\right),$$

$$c_{ki} \equiv 2 \ \ \beta_i\left(M\right) \circ \beta_k\left(M\right)\frac{1}{A + A''} \circ \frac{1}{\mu''}\beta_k\left(M''\right) r\left(M'', M'\right) \circ \beta_i\left(M'\right),$$

$$f_{mni} \equiv \beta_i\left(M\right) \circ \frac{\beta_m\left(M\right)\beta_m\left(M'''\right)}{A + A'''}\frac{1}{\mu'''} \circ r\left(-M''', M''\right) \circ \frac{1}{\mu''}\frac{\beta_n\left(M''\right)\beta_n\left(M'\right)}{A'' + A'} \circ \beta_i\left(M'\right). \tag{12}$$

The final joint system of integral and algebraic equations (9) and (12) in expanded notation was obtained in the above work (Pikichyan, 2023c). Here, the value of the five independent variables is expressed in terms of their eigenfunctions that depend on only three independent variables

$$K_N\left(x, \mu; x', \mu'; \varphi - \varphi'\right) \longrightarrow \left\{\beta_j\left(x, \mu, \varphi\right), \ \beta_j\left(x', \mu', \varphi'\right)\right\}_{j=1 \div N}$$

However, the presence in equations (8) of a combination of frequency and angle, i.e., quantities of the type $A = \alpha(x)/\mu$ is likely to further simplify the solution of the DRP. Indeed, in the more particular

doi: https://doi.org/10.52526/25792776-25.72.1-113

case of the directionally averaged redistribution function $r(x,x')$ of radiation by frequencies after the first simplification step (Pikichyan, 2023b, 2024a)

$$K_N(x,\mu;x',\mu') \longrightarrow \{\beta_j(x,\mu),\ \beta_j(x',\mu')\}_{j=1\div N},$$

It was possible to introduce a new variable $z=\mu/\alpha(x)$ take another step of simplification (Pikichyan, 2024b) and achieve complete simplification of the solution of the DRP. The required eigenfunctions of the problem, which depend on two independent variables, were explicitly represented through two new auxiliary functions by one independent variable

$$\widetilde{\beta}_j(x,z) \longrightarrow \{Q_m(x),G_{mj}(z)\}_{m=1\div N}.$$

The purpose of this note is to implement a similar simplification also in the more general case considered here, when in the elementary act of scattering there is a redistribution of radiation, both in frequencies and directions. Below, based on the format of this publication, the results of the complete simplification of the solution of the DRP are presented briefly - without derivations of the formulas and the corresponding cumbersome calculations.

3. Complete simplification of the task

3.1. Final system of equations

It is not difficult to show that the eigenfunctions $\widetilde{\beta}_i(M)$ from (8) can be explicitly expressed in terms of the simpler auxiliary functions $\widetilde{Q}_m(M)$ and $\widetilde{G}_{mj}(z)$ in the form

$$\widetilde{\beta}_j(M)=\sum_{m=1}^{N}\widetilde{Q}_m(M)\,\widetilde{G}_{mj}(z)\,, \tag{13}$$

where a single variable z is introduced, combining frequency and direction. The auxiliary functions $\widetilde{Q}_m(M)$, $\widetilde{G}_{mj}(z)$ appearing in (13) and the eigenvalues ν_m necessary for the construction of (7), are determined together from a unified system of nonlinear integral equations and linear algebraic equations. Indeed, let us introduce new variables first

$$z\equiv\frac{1}{A}=\frac{\mu}{\alpha(x)}\,,\qquad \pm M\equiv\{x,\pm\mu,\varphi\}\to\{x,\pm\, z\alpha(x),\varphi\}\to\{x,\pm z,\varphi\}\equiv\pm\widetilde{M},$$

as well as designations:

$$F(M,M')=F(x,\mu;x',\mu';\varphi-\varphi')=F(x,z\alpha(x);x',z'\alpha(x');\varphi-\varphi')$$
$$\equiv\widetilde{F}(x,z;x',z';\varphi-\varphi')=\widetilde{F}\left(\widetilde{M},\widetilde{M}'\right),\quad f(A,A')=f\left(\frac{\alpha(x)}{\mu},\frac{\alpha(x')}{\mu'}\right)=\widetilde{f}(z,z'),$$

$$F_1(M,M'')\circ F_2(M'';M')=\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 F_1(x,\mu;x'',\mu'';\varphi-\varphi'')\,F_2(x'',\mu'';x',\mu';\varphi''-\varphi')\,d\mu''d\varphi''dx''=$$
$$\int_0^{2\pi}d\varphi''\int_{-\infty}^{+\infty}\alpha(x'')\,dx''\int_0^{\frac{1}{\alpha(x'')}}\widetilde{F}_1(x,z;x'',z'';\varphi-\varphi'')\,\widetilde{F}_2(x'',z'';x',z';\varphi''-\varphi')\,dz'=$$
$$2\int_0^{2\pi}d\varphi''\int_0^{\infty}dz''\int_{E(z'')}\widetilde{F}_1(x,z;x'',z'';\varphi-\varphi'')\,\widetilde{F}_2(x'',z'';x',z';\varphi''-\varphi')\,\alpha(x'')\,dx''$$
$$\equiv\widetilde{F}_1\left(\widetilde{M};\widetilde{M}''\right)\circ\widetilde{F}_2\left(\widetilde{M}'';\widetilde{M}'\right)\ ,$$

$$\widetilde{F}_1(z;z')\circ\widetilde{F}_2(M')=\int_0^{\infty}\widetilde{F}_1(z,z')\,dz'\ \ 2\int_{E(z')}\alpha(x')\,dx'\int_0^{2\pi}\widetilde{F}_2(x',z',\varphi')\,d\varphi'$$
$$\equiv\int_0^{\infty}\widetilde{F}_1(z,z')\,f(z')\,dz'\ \ \equiv\widetilde{F}_1(z,z')\cdot f(z')\ ,$$

doi: https://doi.org/10.52526/25792776-25.72.1-113

$$f\left(z'\right) \equiv 2\int_{E(z')} \alpha\left(x'\right) dx' \int_0^{2\pi} \widetilde{F}_2\left(x', z', \varphi'\right) d\varphi', \qquad \int_0^{\infty} F_1\left(z\right) F_2\left(z\right) dz \ \equiv F_1\left(z\right) \cdot F_2\left(z\right)\ , \tag{14}$$

where the procedure for reprioritizing integration was used

$$\int_{-\infty}^{+\infty} \ldots dx \int_0^{\frac{1}{\alpha(x)}} \ldots dz = 2\int_0^{\infty} \ldots dz \int_{E(z)} \ldots dx\ , \qquad E\left(z\right) = \left\{x : \alpha\left(x\right) \leq \frac{1}{z}\right\}\ . \tag{15}$$

Then, after some simple but rather cumbersome calculations, we can come to the final system of joint definition of auxiliary functions and eigenvalues

$$\begin{cases} \frac{4\pi}{\lambda}\nu_i\widetilde{G}_{hi}\left(z\right) = \delta_{hi} + z\sum_{s=1}^{N}\nu_s\widetilde{G}_{hs}\left(z\right)\sum_{l=1}^{N}\frac{\widetilde{G}_{ls}(z''')}{z+z'''}\cdot\widetilde{\varrho}_{li}\left(z''', \widetilde{Q}^2\right) \\ \widetilde{Q}_i\left(\widetilde{M}\right) = \sum_{p=1}^{N}\left[\underbrace{\alpha\widetilde{r}_-\widetilde{Q}_p}\left(\widetilde{M}; z''\right)\cdot\widetilde{G}_{pi}\left(z''\right) + \underbrace{\widetilde{r}_+\widetilde{Q}_p}\left(\widetilde{M}, z''\right)\cdot\widetilde{H}_{pi}\left(z'', \widetilde{Q}^2\right)\right] \\ \frac{4\pi}{\lambda}\nu_i = \widetilde{C}_i + \sum_{s=1}^{N}\nu_s\widetilde{D}_{si} \qquad \mathit{or} \qquad \nu_i = \widetilde{A}_i + \sum_{j=1}^{N}\nu_j\widetilde{B}_{ji} \end{cases} \tag{16}$$

In expression (16) there are notations:

$$\widetilde{\varrho}_{li}\left(z''', \widetilde{Q}^2\right) = \underbrace{\widetilde{Q}_l^+\widetilde{Q}_i^-}\left(z'''\right) \equiv 2\int_{E(z''')} dx''' \int_0^{2\pi} \widetilde{Q}_m\left(x''', z''', \varphi'\right)\widetilde{Q}_i\left(x''', -z''', \varphi'''\right) d\varphi'''$$

$$\underbrace{\alpha\widetilde{r}_-\widetilde{Q}_p}\left(\widetilde{M}; z''\right) \equiv 2\int_{E(z'')} \alpha\left(x''\right) dx'' \int_0^{2\pi} \widetilde{r}\left(x, z; x'', -z''; \varphi - \varphi''\right)\widetilde{Q}_p\left(x'', z''; \varphi''\right) d\varphi'',$$

$$\underbrace{\widetilde{r}_+\widetilde{Q}_p}\left(\widetilde{M}, z''\right) \equiv 2\int_{E(z'')} dx'' \int_0^{2\pi} \widetilde{r}\left(x, z; x'', z''; \varphi - \varphi''\right)\widetilde{Q}_p\left(x'', z''; \varphi''\right) d\varphi'',$$

$$\widetilde{H}_{pi}\left(z'', \widetilde{Q}^2\right) = \sum_{h=1}^{N}\sum_{m=1}^{N}\widetilde{R}_{ph}\left(z'', z'\right) z'\widetilde{G}_{mi}\left(z'\right)\cdot\underbrace{\alpha\widetilde{Q}_h\widetilde{Q}_m}\left(z'\right)\ ,$$

$$\widetilde{R}_{hl}\left(z, z'\right) = \frac{\sum_{j=1}^{N}\nu_j\widetilde{G}_{hj}\left(z\right)\widetilde{G}_{lj}\left(z'\right)}{z+z'}\ .$$

$$\underbrace{\alpha\widetilde{Q}_h\widetilde{Q}_m}\left(z'\right) \equiv 2\int_{E(z')} \alpha\left(x'\right) dx' \int_0^{2\pi} \widetilde{Q}_h\left(x', z', \varphi'\right)\widetilde{Q}_m\left(x', z', \varphi'\right) d\varphi'\ ,$$

$$\widetilde{C}_i \equiv \int_0^{\infty}\frac{dz}{\omega_i\left(z\right)}\ , \quad \widetilde{D}_{si} \equiv \sum_{l=1}^{N}\int_0^{\infty}\widetilde{G}_{ls}\left(z'''\right)\varrho_{li}\left(z''', \widetilde{Q}^2\right)\widetilde{\omega}_{si}\left(z'''\right) dz'''\ ,$$

doi: https://doi.org/10.52526/25792776-25.72.1-113

$$\omega_i(z) \equiv \sum_{h=1}^{N} \widetilde{G}_{hi}(z), \qquad \widetilde{\omega}_{si}(z''') \equiv \int_0^\infty \frac{z\omega_s(z)\,dz}{(z'''+z)\,\omega_i(z)} \ ,$$

$$\widetilde{A}_i \equiv \frac{\int \widetilde{Q}_i\left(\widetilde{M}\right) d\widetilde{M} - \sum_{p=1}^{N} \widetilde{G}_{pi}(z'') \cdot \psi_p(z'')}{\sum_{p=1}^{N} \widetilde{G}_{pi}(z'') \cdot \chi_p(z'')} \ , \qquad \psi_p(z'') \equiv \int \underbrace{\alpha\widetilde{r}_{-}\widetilde{Q}_p}\left(\widetilde{M}; z''\right) d\widetilde{M},$$

$$\widetilde{B}_{ji} = \frac{h_{ji}\left(z', \widetilde{Q}^2\right) \cdot \frac{1}{z'+z''} \cdot z'' \sum_{p=1}^{N} \widetilde{G}_{pj}(z'')\,\chi_p(z'')}{\sum_{p=1}^{N} \widetilde{G}_{pi}(z'') \cdot \chi_p(z'')} \ ,$$

$$h_{ji}\left(z', Q^2\right) \equiv \sum_{l=1}^{N} \sum_{m=1}^{N} \widetilde{G}_{lj}(z')\,\widetilde{G}_{mi}(z')\,\underbrace{\alpha\widetilde{Q}_l\widetilde{Q}_m}(z') \ , \qquad \chi_p(z'') \equiv \int \underbrace{\widetilde{r}_{+}\widetilde{Q}_p}\left(\widetilde{M}, z''\right) d\widetilde{M}.$$

3.2. The final solution to the problem of DRP

After defining the auxiliary functions $\widetilde{G}_{hi}(z)$ and $\widetilde{Q}_h(M)$ with accounting (13), the final solution of the initial DRP is given explicitly

$$\rho\left(M, M'\right) = \frac{z}{\alpha(x')} \sum_{h=1}^{N} \sum_{l=1}^{N} \widetilde{Q}_h\left(\widetilde{M}\right) \widetilde{R}_{hl}(z, z')\,\widetilde{Q}_l\left(\widetilde{M}'\right) \ , \qquad \widetilde{R}_{hl}(z, z') = \frac{\sum_{j=1}^{N} \nu_j \widetilde{G}_{hj}(z)\,\widetilde{G}_{lj}(z')}{z+z'}. \tag{17}$$

In a particular case of a DRP with a directionally averaged redistribution function of radiation by frequencies, the above results are transferred to those obtained earlier in the work (Pikichyan, 2024b).

4. On the algorithm of numerical calculation of the system

To implement the calculations of the system (16), it seems expedient to propose an algorithm of successive approximations in the form:

$$\begin{bmatrix} \widetilde{G}^{(0)} \\ \widetilde{Q}^{(0)} \end{bmatrix} \longrightarrow \begin{bmatrix} \widetilde{C}^{(0)} \\ D^{(0)} \\ \nu^{(0)} \approx \widetilde{C}^{(0)} \end{bmatrix} \longrightarrow \nu^{(1)} \longrightarrow \begin{bmatrix} \nu^{(1)} \\ \widetilde{G}^{(0)} \\ \widetilde{Q}^{(0)} \end{bmatrix} \longrightarrow \widetilde{G}^{(1)} \longrightarrow \begin{bmatrix} \nu^{(1)} \\ \widetilde{G}^{(1)} \\ \widetilde{Q}^{(0)} \end{bmatrix} \longrightarrow \widetilde{Q}^{(1)} \longrightarrow \begin{bmatrix} \nu^{(1)} \\ \widetilde{G}^{(1)} \\ \widetilde{Q}^{(1)} \end{bmatrix} \longrightarrow \nu^{(2)} \cdots ,$$

where first in some suitable way some zero approximation $\widetilde{G}^{(0)}$, $\widetilde{Q}^{(0)}$ must be chosen which satisfies to the orthonormalization condition in the form of

$$\sum_{h=1}^{N} \sum_{m=1}^{N} \widetilde{G}_{hi}(z)\,\widetilde{G}_{mj}(z) \cdot \underbrace{\alpha\widetilde{Q}_h\widetilde{Q}_m}(z) = \delta_{ij} \tag{18}$$

arising from (6) and (13).

doi: https://doi.org/10.52526/25792776-25.72.1-113

5. Conclusion

In conclusion, we will only emphasize that the introduction of the DRF method (Pikichyan, 2023a, Pikichyan, 2023b, Pikichyan, 2023c, Pikichyan, 2024a,b) for the solution of DRP is due to an obvious physical fact. After each successive act of scattering in the medium, the diffuse field of radiation becomes more and more smooth. It is clear that the naturally "smoothed" resulting field of diffuse radiation (formed by multiple scatterings of the quantum) is much easier to imagine directly in the decomposed form than for the same final purpose the use of the procedure of preliminary decomposition of the "unsmoothed" value of a single act of scattering. However, the DSA method is still widely used both in the theory of radiation transfer and in practical applications, i.e., the preliminary decomposition of the primary field of the unit scattering act is traditionally assumed (see, e.g., (Ambartsumian, 1943, 1944, Chandrasekhar, 1950, Sobolev, 1963, 1975). In this work, the DRF method was used for further and complete simplification of the solution of the DRP, when the elementary act of scattering takes into account the redistribution of radiation both in frequencies and directions. DRP Solution $\rho\left(x,\mu;x'\mu';\varphi-\varphi'\right)$ which depends on the five independent variables, reduced to finding two types of auxiliary functions $\widetilde{G}_{hi}\left(z\right)$ and $\widetilde{Q}_{h}\left(x,z,\varphi\right)$ $\left(h,\ i=1\div N\right)$ which depend on one and three independent variables, respectively (see formulas (6)-(8)). Moreover, if the joint system of integral and algebraic equations of eigenfunctions $\beta_{j}\left(x,\mu,\varphi\right)$ and eigenvalues ν_{j} had nonlinearities of the fifth and second degree, respectively, then the complete solution system presented above with respect to auxiliary functions $\widetilde{G}_{hi}\left(z\right)$ and $\widetilde{Q}_{i}\left(\widetilde{M}\right)$ has nonlinearity of only the second and third degrees, respectively. The algebraic equations for determining eigenvalues are completely linear here. This simplification was achieved by taking into account the physical fact of the natural combination of frequency and angle at each elementary act of absorption. It is well known that such a phenomenon appeared in both the case of isotropic scattering with a complete redistribution of radiation by frequencies (see, for example, (Ivanov, 1973, Sobolev, 1963)), and in the case of an averaged by directions function of radiation redistribution by frequencies. In the case of DSA method, the latter is shown in the paper Engibaryan & Nikogosyan (1972a), and in the case of DRF in the work Pikichyan (2024b). In the general case of redistribution of radiation by frequencies and directions using the DSA method, this is shown in the paper Engibaryan & Nikogosyan (1972b), and by the DRF method in this note.

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doi: https://doi.org/10.52526/25792776-25.72.1-113