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Conformable Fourier Transform on Time Scales

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Abstract. In this paper, we formulate the conformable Fourier transform on time scales, drawing motivation from the structure of the conformable bilateral Laplace transform. Some of the elementary properties are proved, including shifting, transform of derivative, conjugation, transform of Hilger delta function, and transform of integral.

Key Words: Time Scale, Conformable Laplace Transform, Conformable Bilateral Laplace Transform, Conformable Fourier Transform, Generalized Exponential Function
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1 Introduction

Since Hilger's pioneering work [14], the study of time scales and measure chains has drawn considerable attention. Significant progress has been made by researchers, particularly in exploring dynamic equations on time scales, which unify concepts from both differential and difference equations (see, for example, [4]).

In the study of dynamic equations on time scales, the development of the integral transform method is a key focus for solving initial value problems. Thus far, various integral transforms have been generalized on time scales; we refer to [6, 9, 11, 12, 17–24] among the others.

The classical Fourier transform is a mathematical technique that converts a signal from the time domain to the frequency domain, revealing the different frequency components present and their magnitudes. During the initial development of time scale theory, Hilger [15] began exploring Fourier analysis on time scales. The Fourier transform defined by Hilger combines various forms of Fourier analysis into a unified framework. It offers a closed-form expression representing both the Fourier integral and Fourier series, through a simple integral defined on a time scale. Later, he explored Fourier transforms on specific subclasses of time scales that also have a group structure, as discussed in [16]. The concept of discrete Fourier transform was introduced in [7], extending Fourier analysis to discrete time scales. Furthermore, a "generalized Fourier transform" with kernel as a classical exponential function is defined on a time scale.

Researchers have extended the conformable fractional calculus to arbitrary time scales by applying the principles of classical fractional calculus (see, for example, [1–3, 26]). In [25], we introduced conformable Laplace transform on time scales as follows:

$$\mathfrak{L}_\alpha\{f(t)\}(z) = \int_{t_0}^{\infty} \mathbb{E}_{\ominus_\alpha}(\sigma(t), t_0) f(t) \Delta^\alpha t.$$

Subsequently, in [13], the conformable bilateral Laplace transform was defined on time scales, where the region of integration is the entire time scales, as follows:

$$\mathcal{L}^b(f)(z, s) = \int_{-\infty}^{\infty} \mathbb{E}^\sigma_{\ominus_\alpha z}(t, s) f(t) \Delta^\alpha t. \tag{1}$$

Motivated by the bilateral Laplace transform given by equation (1), in the present work, we develop conformable Fourier transform on time scales.

This paper is organized as follows. The foundational concepts and notations of conformable fractional calculus are presented in Section 2. In Section 3, we introduce the conformable Fourier transform on time scales. Some of its basic properties, including transform of conformable fractional derivative, and conformable α-fractional integral, are given. At last, in Section 4, we solve a conformable dynamic equation using conformable Fourier transform.

2 Preliminaries

Assuming a foundational understanding of time scale calculus as elaborated in [4, 5, 8, 10, 26], we present the essential definitions and theorems for our discussion. Throughout this paper, we consider a time scale $\mathbb{T}$ that is unbounded both above and below. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\},$$

and the forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined as

$$\mu(t) = \sigma(t) - t.$$

For a given time scale $\mathbb{T}$, the non-maximal set $\mathbb{T}^\kappa$ is given by

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus \rho(\sup \mathbb{T}, \sup \mathbb{T}] & \text{for} \quad \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{for} \quad \sup \mathbb{T} = \infty. \end{cases}$$

Definition 1 *A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$, and a positively regressive if $1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}^\kappa$. The set of regressive and positively regressive functions are denoted as $\mathcal{R}$ and $\mathcal{R}^+$, respectively.*

Let $h > 0$. The Hilger complex numbers and the Hilger real axis are defined as

$$\mathbb{C}_h = \mathbb{C} \backslash \left\{ \frac{-1}{h} \right\} \quad \text{and} \quad \mathbb{R}_h = (\mathbb{C}_h \cap \mathbb{R}) \backslash \left\{ \frac{-1}{h} \right\},$$

respectively. If $h = 0$, we have $\mathbb{C}_0 := \mathbb{C}$ and $\mathbb{R}_0 = \mathbb{R}$. Further, $\mathbb{Z}_h$ is a strip

$$\mathbb{Z}_h := \left\{ z \in \mathbb{C} : \frac{-\pi}{h} < \mathrm{Im}(z) \leq \frac{\pi}{h} \right\},$$

and for $h = 0$, we have $\mathbb{Z}_0 = \mathbb{C}$.

For $z \in \mathbb{C}_h$, the Hilger real part of z is is given by

$$Re_h(z) := \frac{|hz + 1| - 1}{h}.$$

For $k > 0$, the cylindrical transformation $\xi_k : \mathbb{C}_k \to \mathbb{Z}_k$ is defined as

$$\xi_k(z) = \frac{1}{k} \log(1 + zk),$$

where log is the principal logarithm function.

Definition 2 *Let $g : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^\kappa$, and $\alpha \in (0, 1]$. Suppose there is a number $g^{(\alpha)}(t)$ such that for $\epsilon > 0$, there exists a neighbourhood $U \subset \mathbb{T}$ of t such that*

$$|(g(\sigma(t)) - g(s))|t|^{1-\alpha} - g^{(\alpha)}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U$. Then $g^{(\alpha)}(t)$ is called the conformable fractional derivative of g of order α at t.

If g is a regulated function, its conformable α-fractional integral is given by

$$\int g(t)\Delta^\alpha t = \int g(t)|t|^{\alpha-1}\Delta t.$$

Definition 3 *A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if f is continuous at right-dense points in $\mathbb{T}$ and has a finite limit at the left-dense points in $\mathbb{T}$. We denote the set of all rd-continuous functions by $\mathcal{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.*

Definition 4 *A function $f : \mathbb{T} \to \mathbb{R}$ is said to be 'α-regressive' if*

$$1 + \mu(t)f(t)|t|^{\alpha-1} \neq 0 \qquad \text{for all } t \in \mathbb{T}^{\kappa},$$

and is said to be 'α-positively regressive' provided

$$1 + \mu(t)f(t)|t|^{\alpha-1} > 0, \qquad \text{for all } t \in \mathbb{T}^{\kappa}.$$

We denote the set of all α-regressive and rd-continuous (α-positively regressive and rd-continuous) functions by $\mathcal{R}^{\alpha}(\mathcal{R}^{\alpha+})$.

For a set $\mathcal{R}^{\alpha}$, 'α-circle plus' addition $\oplus_{\alpha}$ is a binary operation defined as

$$(f_1 \oplus_{\alpha} f_2)(t) = f_1(t) + f_2(t) + \mu(t)f_1(t)f_2(t)|t|^{\alpha} \qquad \text{for all } t \in \mathbb{T}^{\kappa}.$$

The set $\mathcal{R}^{\alpha}$ forms an abelian group under $\oplus_{\alpha}$. For $f \in \mathcal{R}^{\alpha}$, the inverse of f is given as

$$\ominus_{\alpha} f(t) = \frac{-f(t)}{1 + \mu(t)f(t)|t|^{\alpha}}, \qquad t \in \mathbb{T}^{\kappa}.$$

Further, $\mathcal{R}^{\alpha}$, 'α-circle minus' substraction $\ominus_{\alpha}$ is defined as

$$(f_1 \ominus_{\alpha} f_2)(t) = \frac{f_1(t) - f_2(t)}{1 + \mu(t)f_2(t)|t|^{\alpha-1}}, \qquad t \in \mathbb{T}^{\kappa}.$$

For $f \in \mathcal{R}^{\alpha}$, the generalized exponential function is defined by

$$\mathbb{E}_f(t, s) = \exp\Big(\int_s^t \xi_{\mu(\tau)}\big(f(\tau)|\tau|^{\alpha-1}\big)\Delta\tau\Big) \qquad \text{for all } t, s \in \mathbb{T}. \tag{2}$$

Applying the concept of the cylindrical transformation [4, Definition 2.21], equation (2) can be written as

$$\mathbb{E}_f(t, s) = \exp\Big(\int_s^t \frac{1}{\mu(\tau)} Log(1 + \mu(\tau)f(\tau)|\tau|^{\alpha-1})\Delta\tau\Big), \qquad t, s \in \mathbb{T}.$$

Some important properties of the generalized exponential function are given in the following theorem.

Theorem 1 *If $g_1, g_2 \in \mathcal{R}^{\alpha}$, then for all $s, t, r \in \mathbb{T}$,*

(1) $\mathbb{E}_0(t, s) = 1$ *and* $\mathbb{E}_{g_1}(t, t) = 1$;

(2) $\mathbb{E}_{g_1}(t, s)\mathbb{E}_{g_1}(s, r) = \mathbb{E}_{g_1}(t, r)$;

(3) $\mathbb{E}_{g_1}(t,s) = \dfrac{1}{\mathbb{E}_{g_1}(s,t)} = \mathbb{E}_{\ominus_\alpha g_1}(s,t)$;

(4) $\mathbb{E}_{g_1}(t,s)\mathbb{E}_{g_2}(t,s) = \mathbb{E}_{g_1 \oplus_\alpha g_2}(t,s)$;

(5) $\dfrac{\mathbb{E}_{g_1}(t,s)}{\mathbb{E}_{g_2}(t,s)} = \mathbb{E}_{g_1 \ominus_\alpha g_2}(t,s)$;

(6) $\mathbb{E}_{g_1}(\sigma(t),s) = (1+\mu(t)g_1(t)|t|^{\alpha-1})\mathbb{E}_{g_1}(t,s)$;

(7) $\mathbb{E}_{\ominus_\alpha g_1}(\sigma(t),s) = \dfrac{\mathbb{E}_{\ominus_\alpha g_1}(t,s)}{1+\mu(t)g_1(t)|t|^{\alpha-1}}$;

(8) $\mathbb{E}^{\Delta}_{g_1}(t,s) = g_1(t)\mathbb{E}_{g_1}(t,s)|t|^{\alpha-1}$;

(9) $\mathbb{E}^{(\alpha)}_{g_1}(t,s) = g_1(t)\mathbb{E}_{g_1}(t,s)$.

Definition 5 *Let $s \in \mathbb{T}$, $\beta \in \mathcal{R}^{\alpha+}([s,\infty))$, $\gamma \in \mathcal{R}^{\alpha+}((-\infty,s])$. We say that (s,β,γ) is an admissible triple if*

$$\mathbb{C}_{s,\beta,\gamma} = \left\{ z \in \mathbb{C} : Re_{\mu^*(s)}(z) < \gamma,\ Re_{\mu_*(s)}(z) > \beta,\ 1+\overline{\overline{\mu}}(s)Re_{\overline{\mu}(s)}(z) \neq 0 \right\} \neq \emptyset.$$

Definition 6 *Let $f : \mathbb{T} \to \mathbb{R}$ be regulated. Then for $s \in \mathbb{T}$, the conformable bilateral Laplace transform of f is defined by*

$$\mathcal{L}^b(f)(z,s) = \int_{-\infty}^{\infty} f(t)\mathbb{E}^{\sigma}_{\ominus_\alpha z}(t,s)\Delta^\alpha t$$

for $z \in \mathbb{C}$ such that $1+\mu(t)z|t|^{\alpha-1} \neq 0$ for any $t \in \mathbb{T}^\kappa$ and the improper integral exists.

Next theorems give the conditions for the absolute and uniform convergence for the conformable bilateral Laplace transform.

Theorem 2 *[13] Let (s,β,γ) is an admissible triple and $f \in \mathcal{C}_{rd}(\mathbb{T})$ is of conformable double exponential order (β,γ). Then $\mathcal{L}^b(f)(\cdot,s)$ exists on $\mathbb{C}_{s,\beta,\gamma}$ and converges absolutely.*

Theorem 3 *[13] Let (s,β,γ) is an admissible triple and $f \in \mathcal{C}_{rd}(\mathbb{T})$ is of conformable double exponential order (β,γ). Then $\mathcal{L}^b(f)(\cdot,s)$ converges uniformly in $\mathbb{C}_{s,\beta,\gamma}$.*

3 The conformable Fourier transform

Suppose that $\mathbb{T}$ is a time scale with forward jump operator σ and delta differentiation operator Δ. Also, let $\inf \mathbb{T} = -\infty$, $\sup \mathbb{T} = \infty$ and $s \in \mathbb{T}$. Denote

$$\mu_*(s) = \inf_{t\in[s,\infty)} \mu(t), \quad \mu^*(s) = \sup_{t\in(-\infty,s]} \mu(t), \quad \overline{\mu}(s) = \inf_{t\in(-\infty,s]} \mu(t),$$

and for $x \in \mathbb{R}$, put

$$\overline{\overline{\mu}}(s) = \begin{cases} \mu^*(s) & \text{if } \operatorname{Re}_{\overline{\mu}(s)}(x) \leq 0, \\ \overline{\mu}(s) & \text{if } \operatorname{Re}_{\overline{\mu}(s)}(x) > 0. \end{cases}$$

Definition 7 *Suppose that $f : \mathbb{T} \to \mathbb{R}$ is regulated. Then the Fourier transform of the function f is defined by*

$$\mathcal{F}(f)(x,s) = \int_{-\infty}^{\infty} f(t)\mathbb{E}^{\sigma}_{\ominus_\alpha ix}(t,s)\Delta^\alpha t$$

for $x \in \mathbb{R}$ such that $1 + ix\mu(t)|t|^{\alpha-1} \neq 0$ for any $t \in \mathbb{T}^\kappa$ and the improper integral exists.

Definition 8 *Let $\beta \in \mathcal{R}^+([s,\infty))$, $\gamma \in \mathcal{R}^+((-\infty,s])$. We say that (s,β,γ) is a real admissible triple if*

$$R_{s,\beta,\gamma} = \Big\{ x \in \mathbb{R} : Re_{\mu^*(s)}(ix) < \gamma,\ Re_{\mu_*(s)}(ix) > \beta, \\ 1 + \overline{\overline{\mu}}(s) Re_{\overline{\mu}(s)}(ix) \neq 0 \Big\} \neq \emptyset.$$

If $f \in \mathcal{C}_{rd}(\mathbb{T})$, then triple (s,β,γ) is a real admissible triple and f is of double exponential order (β,γ). By Theorem 2 and Theorem 3, it follows that $\mathcal{F}(f)(\cdot,s)$ exists on $R_{s,\beta,\gamma}$ and converges absolutely and uniformly there. Below we present some of the properties of the Fourier transform.

Theorem 4 *Let $f,g : \mathbb{T} \to \mathbb{R}$, $\lambda_1,\lambda_2 \in \mathbb{C}$. Then*

$$\mathcal{F}(\lambda_1 f + \lambda_2 g)(x,s) = \lambda_1\mathcal{F}(f)(x,s) + \lambda_2\mathcal{F}(g)(x,s)$$

for those $x \in \mathbb{R}$ for which $1+ix\mu(t)|t|^{\alpha-1} \neq 0$, $t \in \mathbb{T}^\kappa$, and the corresponding integrals exist.

Proof. Indeed,

$$\begin{aligned}\mathcal{F}(\lambda_1 f+\lambda_2 g)(x,s) &= \int_{-\infty}^{\infty}(\lambda_1 f+\lambda_2 g)(t)\mathbb{E}^{\sigma}_{\ominus_\alpha ix}(t,s)\Delta^\alpha t\\ &= \lambda_1\int_{-\infty}^{\infty} f(t)\mathbb{E}^{\sigma}_{\ominus_\alpha ix}(t,s)\Delta^\alpha t+\lambda_2\int_{-\infty}^{\infty} g(t)\mathbb{E}^{\sigma}_{\ominus_\alpha ix}(t,s)\Delta^\alpha t\\ &= \lambda_1\mathcal{F}(f)(x,s)+\lambda_2\mathcal{F}(g)(x,s).\end{aligned}$$

□

Theorem 5 *Let* $f:\mathbb{T}\to\mathbb{R}$. *Then*

$$\mathcal{F}\left(\mathbb{E}^{\sigma}y(\cdot,s)f(\cdot)\right)(x,s)=\mathcal{F}(f)(z,s),$$

where $z=x+iy/1+\mu|t|^{\alpha-1}$, *and* $x,y\in\mathbb{R}$ *are such that* $1+\mu(t)|t|^{\alpha-1}y\neq 0$ *for any* $t\in\mathbb{T}^\kappa$, *and the corresponding integrals exist.*

Proof. Note that

$$iz=i\frac{x+iy}{1+\mu|t|^{\alpha-1}y}=-\frac{x+iy}{i(1+\mu|t|^{\alpha-1}y)}$$

and

$$\begin{aligned}\ominus_\alpha iz &= \frac{-iz}{1+i\mu|t|^{\alpha-1}z}=\frac{\frac{x+iy}{i(1+\mu|t|^{\alpha-1}y)}}{1-\mu|t|^{\alpha-1}\frac{x+iy}{i(1+\mu|t|^{\alpha-1}y)}}\\ &= \frac{x+iy}{i+i\mu|t|^{\alpha-1}y-\mu|t|^{\alpha-1}x-i\mu|t|^{\alpha-1}y}\\ &= \frac{x+iy}{i-\mu|t|^{\alpha-1}x}=\frac{i(x+iy)}{-1-i\mu|t|^{\alpha-1}x}\\ &= \frac{y-ix}{1+i\mu|t|^{\alpha-1}x}=y\ominus_\alpha ix.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{F}\left(\mathbb{E}^{\sigma}y(\cdot,s)f(\cdot)\right)(x,s) &= \int_{-\infty}^{\infty}\mathbb{E}^{\sigma}_{\ominus_\alpha ix}(t,s)\mathbb{E}y^{\sigma}(t,s)f(t)\Delta^\alpha t\\ &= \int_{-\infty}^{\infty}\mathbb{E}^{\sigma}_{y\ominus_\alpha ix}(t,s)f(t)\Delta^\alpha t\\ &= \mathcal{F}(f)(z,s).\end{aligned}$$

□

Theorem 6 *Let $f : \mathbb{T} \to \mathbb{R}$. For any $k \in \mathbb{N}$, we have*

$$\mathcal{F}\left(f^{(\alpha^k)}\right)(x,s) = (ix)^k \mathcal{F}(f)(x,s)$$

for those $x \in \mathbb{R}$ for which $1 + ix\mu(t)|t|^{\alpha-1} \neq 0$, $t \in \mathbb{T}^\kappa$, the corresponding integrals exist, and

$$\lim_{t\to\pm\infty} f^{(\alpha^l)}(t)\mathbb{E}_{\ominus_\alpha ix}(t,s) = 0, \quad l \in \{0,\dots,k-1\}.$$

Proof. We will use the principle of mathematical induction. For $k = 1$, we have

$$\begin{aligned}
\mathcal{F}\left(f^{(\alpha)}\right)(x,s) &= \int_{-\infty}^{\infty} f^{(\alpha)}(t)\mathbb{E}^\sigma_{\ominus_\alpha ix}(t,s)\Delta^\alpha t \\
&= \lim_{t\to\infty} f(t)\mathbb{E}_{\ominus_\alpha ix}(t,s) - \lim_{t\to-\infty} f(t)\mathbb{E}_{\ominus_\alpha ix} \\
&\quad - \int_{-\infty}^{\infty} (\ominus_\alpha ix) f(t)\mathbb{E}_{\ominus_\alpha ix}(t,s)\Delta^\alpha t \\
&= ix\int_{-\infty}^{\infty} f(t)\mathbb{E}^\sigma_{\ominus_\alpha ix}(t,s)\Delta^\alpha t \\
&= ix\mathcal{F}(f)(x,s).
\end{aligned}$$

Further, if

$$\mathcal{F}\left(f^{(\alpha^k)}\right)(x,s) = (ix)^k \mathcal{F}(f)(x,s)$$

for some $k \in \mathbb{N}$, then

$$\mathcal{F}\left(f^{(\alpha^{k+1})}\right)(x,s) = ix\mathcal{F}\left(f^{(\alpha^k)}\right)(x,s) = (ix)^{k+1}\mathcal{F}(f)(x,s).$$

This completes the proof. □

Theorem 7 *Let $f : \mathbb{T} \to \mathbb{R}$. Then*

$$\overline{\mathcal{F}(f)(x,s)} = \mathcal{F}(f)(-x,s)$$

for those $x \in \mathbb{R}$ for which $1 \pm ix\mu(t)|t|^{\alpha-1} \neq 0$, $t \in \mathbb{T}^\kappa$, and the corresponding integrals exist.

Proof. Let $t \in \mathbb{T}^\kappa$ and $x \in \mathbb{R}$ be such that $1 \pm ix\mu(t)|t|^{\alpha-1} \neq 0$. Then we have

$$\begin{aligned}
\left(\ominus_\alpha(ix)\right)(t) &= -\frac{ix}{1 + i\mu(t)|t|^{\alpha-1}x} \\
&= -\frac{ix(1 - i\mu(t)|t|^{\alpha-1}x)}{(1 + i\mu(t)|t|^{\alpha-1}x)(1 - i\mu(t)|t|^{\alpha-1}x)} \\
&= -\frac{ix + \mu(t)|t|^{\alpha-1}x^2}{1 + (\mu(t))^2|t|^{2(\alpha-1)}x^2},
\end{aligned}$$

$$1+\mu(t)|t|^{\alpha-1}\left(\ominus_\alpha(ix)\right)(t) = 1-\mu(t)|t|^{\alpha-1}\frac{ix+\mu(t)|t|^{\alpha-1}x^2}{1+(\mu(t))^2|t|^{2\alpha-2}x^2}$$

$$= \frac{1+(\mu(t))^2|t|^{2\alpha-2}x^2 - i\mu(t)|t|^{\alpha-1}x-(\mu(t))^2|t|^{2\alpha-2}x^2}{1+(\mu(t))^2|t|^{2\alpha-2}x^2}$$

$$= \frac{1-i\mu(t)|t|^{\alpha-1}x}{1+(\mu(t))^2|t|^{2\alpha-2}x^2}$$

$$= \frac{1}{\sqrt{1+(\mu(t))^2|t|^{2\alpha-2}x^2}} \cdot\left(\frac{1}{\sqrt{1+(\mu(t))^2|t|^{2\alpha-2}x^2}} - i\frac{\mu(t)|t|^{\alpha-1}x}{\sqrt{1+(\mu(t))^2|t|^{2\alpha-2}x^2}}\right)$$

and

$$\left(\ominus_\alpha(i(-x))\right)(t) = \frac{1}{\sqrt{1+(\mu(t))^2|t|^{2\alpha-2}x^2}} \cdot\left(\frac{1}{\sqrt{1+(\mu(t))^2|t|^{2\alpha-2}x^2}} + i\frac{\mu(t)|t|^{\alpha-1}x}{\sqrt{1+(\mu(t))^2|t|^{2\alpha-2}x^2}}\right).$$

Let

$$r(t) = \frac{1}{\sqrt{1+(\mu(t))^2|t|^{2\alpha-2}x^2}}, \quad \cos\theta(t) = \frac{1}{\sqrt{1+(\mu(t))^2|t|^{2\alpha-2}x^2}}.$$

Then

$$1+\mu(t)|t|^{\alpha-1}\left(\ominus_\alpha(ix)\right)(t) = r(t)e^{-i\theta(t)}$$

and

$$1+\mu(t)|t|^{\alpha-1}\left(\ominus_\alpha(i(-x))\right)(t) = r(t)e^{i\theta(t)}.$$

Also,

$$\log\left(1+\mu(t)|t|^{\alpha-1}\left(\ominus_\alpha(ix)\right)(t)\right) = \ln(r(t)) - i\left(\theta+2k\pi\right),$$

$$\log\left(1+\mu(t)|t|^{\alpha-1}\left(\ominus_\alpha(i(-x))\right)(t)\right) = \ln(r(t)) + i\left(\theta+2k\pi\right), \quad k\in\mathbb{Z}.$$

From here and from the definition of the Fourier transform, we have

$$\begin{aligned}\mathcal{F}(f)(x,s) &= \int_{-\infty}^{\infty} f(t)\mathbb{E}^\sigma_{\ominus_\alpha(ix)}(t,s)\Delta^\alpha t \\ &= \int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)}\frac{1}{\mu(\tau)|\tau|^{\alpha-1}}\log\left(1+\mu(\tau)|\tau|^{\alpha-1}(\ominus_\alpha(ix))(\tau)\right)\Delta^\alpha\tau} f(\tau)\Delta^\alpha\tau \\ &= \int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)}\frac{1}{\mu(\tau)|\tau|^{\alpha-1}}(\ln r(\tau)-i(\theta(\tau)+2k\pi))\Delta^\alpha\tau} f(t)\Delta^\alpha t\end{aligned}$$

and

$$
\begin{aligned}
\overline{\mathcal{F}(f)(x,s)} &= \overline{\int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)|\tau|^{\alpha-1}}(\ln r(\tau) - i(\theta(\tau)+2k\pi))\Delta^\alpha \tau} f(t)\Delta^\alpha t} \\
&= \int_{-\infty}^{\infty} \overline{e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)|\tau|^{\alpha-1}}(\ln r(\tau) - i(\theta(\tau)+2k\pi))\Delta^\alpha \tau} f(t)}\Delta^\alpha t \\
&= \int_{-\infty}^{\infty} \overline{e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)|\tau|^{\alpha-1}}(\ln r(\tau) - i(\theta(\tau)+2k\pi))\Delta^\alpha \tau}} f(t)\Delta^\alpha t \\
&= \int_{-\infty}^{\infty} e^{\overline{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)|\tau|^{\alpha-1}}(\ln r(\tau) - i(\theta(\tau)+2k\pi))\Delta^\alpha \tau}} f(t)\Delta^\alpha t \\
&= \int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)} \overline{\frac{1}{\mu(\tau)|\tau|^{\alpha-1}}(\ln r(\tau) - i(\theta(\tau)+2k\pi))}\Delta^\alpha \tau} f(t)\Delta^\alpha t \\
&= \int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)|\tau|^{\alpha-1}}(\ln r(\tau) + i(\theta(\tau)+2k\pi))\Delta^\alpha \tau} f(t)\Delta^\alpha t \\
&= \int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)|\tau|^{\alpha-1}} \log\left(1+\mu(\tau)|\tau|^{\alpha-1}(\ominus_\alpha(i(-x)))(\tau)\right)\Delta^\alpha \tau} f(\tau)\Delta^\alpha \tau \\
&= \int_{-\infty}^{\infty} f(t)\mathbb{E}^\sigma_{\ominus_\alpha(i(-x))}(t,s)\Delta^\alpha t \\
&= \mathcal{F}(f)(-x,s).
\end{aligned}
$$

□

Theorem 8 *Let $f : \mathbb{T} \to \mathbb{R}$ be regulated and*

$$F(t) = \int_a^t f(\tau)\Delta^\alpha \tau, \quad t \in \mathbb{T},$$

for some fixed $a \in \mathbb{T}$. Then

$$\mathcal{F}(F)(x,s) = -\frac{i}{x}\mathcal{F}(f)(x,s)$$

for those $x \in \mathbb{R}$, $x \neq 0$, for which

$$\lim_{t\to\pm\infty} F(t)\mathbb{E}_{\ominus_\alpha ix}(t,s) = 0.$$

Proof. Let $x \in \mathbb{R}$ satisfy the conditions of the theorem. Then

$$\begin{aligned}
\mathcal{F}(F)(x,s) &= \int_{-\infty}^{\infty} F(t)\mathbb{E}^{\sigma}_{\ominus_\alpha ix}(t,s)\Delta^\alpha t \\
&= \int_{-\infty}^{\infty} F(t)\left(1+\mu(t)|t|^{\alpha-1}(\ominus_\alpha(ix))(t)\right)\mathbb{E}_{\ominus_\alpha ix}(t,s)\Delta^\alpha t \\
&= \int_{-\infty}^{\infty} F(t)\frac{1}{1+i\mu(t)|t|^{\alpha-1}x}\mathbb{E}_{\ominus_\alpha ix}(t,s)\Delta^\alpha t \\
&= -\frac{1}{ix}\int_{-\infty}^{\infty} F(t)\frac{-ix}{1+i\mu(t)|t|^{\alpha-1}x}\mathbb{E}_{\ominus_\alpha ix}(t,s)\Delta^\alpha t \\
&= \frac{i}{x}\int_{-\infty}^{\infty} F(t)(\ominus_\alpha ix)(t)\mathbb{E}_{\ominus_\alpha ix}(t,s)\Delta^\alpha t \\
&= \frac{i}{x}\int_{-\infty}^{\infty} F(t)\mathbb{E}^{(\alpha)}_{\ominus_\alpha ix}(t,s)\Delta^\alpha t \\
&= \frac{i}{x}\left(\lim_{t\to\infty} F(t)\mathbb{E}_{\ominus_\alpha ix}(t,s) - \lim_{t\to-\infty} F(t)\mathbb{E}_{\ominus_\alpha ix}(t,s)\right) \\
&\quad -\frac{i}{x}\int_{-\infty}^{\infty} f(t)\mathbb{E}^{\sigma}_{\ominus_\alpha ix}(t,s)\Delta^\alpha t \\
&= -\frac{i}{x}\mathcal{F}(f)(x,s).
\end{aligned}$$

□

4 Applications

In this section, we will give some applications of the conformable Fourier transform. Consider the equation

$$u^{(\alpha^n)} + a_1 u^{(\alpha^{n-1})} + \ldots a_{n-1}u^{(\alpha)} + a_n u = f(t), \quad t \in \mathbb{T},$$

where $a_j, j \in \{1,\ldots,n\}$, are given constants, and $f : \mathbb{T} \to \mathbb{R}$ is given function such that $\mathcal{F}(f)(x,s)$ exists. Applying the conformable Fourier transform of both sides of the considered equation, we get

$$\begin{aligned}
\mathcal{F}(f)(x,s) &= \mathcal{F}\left(u^{(\alpha^n)}\right)(x,s) + a_1\mathcal{F}\left(u^{(\alpha^{n-1})}\right)(x,s) + \ldots \\
&\quad + a_{n-1}\mathcal{F}\left(u^{(\alpha)}\right)(x,s) + a_n\mathcal{F}(u)(x,s) \\
&= (ix)^n\mathcal{F}(u)(x,s) + (ix)^{n-1}a_1\mathcal{F}(u)(x,s) + \ldots \\
&\quad + ixa_{n-1}\mathcal{F}(u)(x,s) + a_n\mathcal{F}(x,s) \\
&= \left((ix)^n + (ix)^{n-1}a_1 + \cdots + ixa_{n-1} + a_n\right)\mathcal{F}(u)(x,s),
\end{aligned}$$

whereupon

$$\mathcal{F}(u)(x,s) = \frac{\mathcal{F}(f)(x,s)}{(ix)^n + (ix)^{n-1}a_1 + \cdots + ixa_{n-1} + a_n}.$$

Taking the inverse conformable Fourier transform for the solution u of the considered equation, we obtain

$$u(t) = \mathcal{F}^{-1}\Big(\frac{\mathcal{F}(f)(x,s)}{(ix)^n + (ix)^{n-1}a_1 + \ldots ixa_{n-1} + a_n}\Big)(t), \quad t \in \mathbb{T}.$$

For example let us consider the equation

$$u^{(\alpha)} - u = f(t), \quad t \in \mathbb{T},$$

where

$$f(t) = \begin{cases} E_i(t,s) & \text{for} \quad t \geq s, \\ 0 & \text{for} \quad t < s. \end{cases}$$

We will search a solution of this equation in the form

$$u(t) = \begin{cases} v(t) & \text{for} \quad t \geq s \\ 0 & \text{for} \quad t < s. \end{cases}$$

Applying the conformable Fourier transform of both sides of the considered equation, we get

$$\frac{1}{x-i} = \mathcal{F}(f)(x,s) = \mathcal{F}\left(v^{(\alpha)}\right)(x,s) - \mathcal{F}(v)(x,s)$$

$$= ix\mathcal{F}(v)(x,s) - \mathcal{F}(v)(x,s) = (ix-1)\mathcal{F}(v)(x,s)$$

$$= i(x+1)\mathcal{F}(v)(x,s),$$

whereupon

$$\mathcal{F}(v)(x,s) = -\frac{i}{(x-i)(x+i)} = -\frac{i}{x^2+1}.$$

Now, taking the inverse conformable Fourier transform, we find

$$v(t) = -i\mathcal{F}^{-1}\Big(\frac{1}{x^2+1}\Big)(t) = -i\sin_{1,\alpha}(t,s), \qquad t \geq s.$$

Consequently,

$$u(t) = \begin{cases} -i\sin_{1,\alpha}(t,s) & \text{for} \quad t \geq s, \\ 0 & \text{for} \quad t < s, \end{cases}$$

is a solution of the considered equation.

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