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Local Fractional Hilbert-Type Inequalities with a Non-Homogeneous Kernel

P. Vuković

Abstract. The main objective of this paper is to study some new local fractional Hilbert-type inequalities with a general kernel. We apply our main results to non-homogeneous kernels. In addition, we obtain the best possible constants.

Key Words: Hilbert Inequality, Conjugate Parameters, Homogeneous Function, Local Fractional Calculus
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Introduction

Let p and q be a pair of non-negative conjugate parameters, i.e., $1/p+1/q=1$, $p>1$. The well-known Hilbert integral inequality (see, for example, [3], [5] and [21]) essentially states that

$$\int_{\mathbb{R}_+^2} \frac{f(x)g(y)}{x+y}dxdy \leq \frac{\pi}{\sin\frac{\pi}{p}}\|f\|_p\|g\|_q, \tag{1}$$

where $f \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$ are non-negative functions. The constant $\pi/(\sin(\pi/p))$, appearing on the right-hand side of (1), is the best possible. Inequality (1) is significant in mathematical analysis and its applications. A large number of generalizations and extensions of the Hilbert inequality have been established, covering various aspects such as different weighted functions, integration domains, types and dimensionality of integrals, etc. For example, through the application of Hermite-Hadamard inequality, Rassias et al. in [11] developed some higher-accuracy multidimensional half-discrete Hilbert-type inequalities. On the other hand, B. Yang et al. in [20] claimed results concerning Hilbert-type inequalities with non-homogeneous kernels (for additional results, see also [9, 10, 14, 15, 19]). In the works [13] and [12], Rassias et al. utilized weight coefficients derive a new, more accurate Hilbert-type inequalities in the whole plane. This paper establishes new Hilbert-type inequalities through the application of local fractional integrals.

Let us briefly remind some fundamental definitions and outcomes of local fractional calculus (see also [1,2,4,6,17,18]). Let $\mathbb{R}^\alpha$, $0 < \alpha \le 1$, be an α-type fractal set of real line numbers. Addition and multiplication operations on $\mathbb{R}^\alpha$ are defined by $a^\alpha + b^\alpha := (a+b)^\alpha$ and $a^\alpha \cdot b^\alpha = a^\alpha b^\alpha := (ab)^\alpha$, $a^\alpha, b^\alpha \in \mathbb{R}^\alpha$. Naturally, with these two operations, $\mathbb{R}^\alpha$ is a field with additive identity 0^α and multiplicative identity 1^α.

If a function f is local continuous in the interval (a, b), we denote $f \in C_\alpha(a, b)$. Recently, Q. Liu (see [7]) provided a unified approach to the 2-dimensional Hilbert-type inequality via fractal integrals. In particular, he proved that if $p > 1$, $1/p + 1/q = 1$, $0 < \alpha \le 1$, $\beta, \lambda > 0$, $f, g\ (\ge 0) \in C_\alpha(0, \infty)$, then the next inequality holds:

$$\begin{aligned}
&\frac{1}{\Gamma^2(1+\alpha)} \int_0^\infty \int_0^\infty E_\alpha(-\lambda^\alpha (xy)^\alpha) f(x) g(y) (dx)^\alpha (dy)^\alpha \\
&\le \frac{\Gamma_\alpha(\beta)}{\lambda^{\beta\alpha}} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty x^{[p(1-\beta)-1]\alpha} f^p(x) (dx)^\alpha \right)^{\frac{1}{p}} \\
&\quad \times \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty y^{[q(1-\beta)-1]\alpha} g^q(x) (dy)^\alpha \right)^{\frac{1}{q}},
\end{aligned}$$

where $\Gamma_\alpha(\cdot)$, $0 < \alpha \le 1$, is a local fractal Gamma function defined by

$$\Gamma_\alpha(x) = \frac{1}{\Gamma(1+\alpha)} \int_{\mathbb{R}_+} E_\alpha(-t^\alpha) t^{\alpha(x-1)} (dt)^\alpha,$$

and $E_\alpha(\cdot)$ denotes the Mittag-Leffler function given by

$$E_\alpha(x^\alpha) = \sum_{k=0}^\infty \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}.$$

Moreover, it has been shown that the constant appearing on the right-hand side of the above inequality is the best possible. The famous Hölder inequality is used to establish Hilbert-type inequalities. A fractal version of the Hölder inequality is proved in [1]: let $\sum_{i=1}^m (1/p_i) = 1$, $p_i > 1$, $i = 1, 2, \ldots, m$, and let Ω be a fractal surface. If $F_i \in C_\alpha(\Omega^n)$, $i = 1, 2, \ldots, m$, then the following inequality holds:

$$\begin{aligned}
&\frac{1}{\Gamma^n(1+\alpha)} \int_{\Omega^n} \prod_{j=1}^m F_j(x_1, \ldots, x_n) (dx_1)^\alpha \ldots (dx_n)^\alpha \\
&\le \prod_{j=1}^m \left(\frac{1}{\Gamma^n(1+\alpha)} \int_{\Omega^n} F_j^{p_j}(x_1, \ldots, x_n) (dx_1)^\alpha \ldots (dx_n)^\alpha \right)^{\frac{1}{p_j}}. \qquad (2)
\end{aligned}$$

This research constructs a new Hilbert-type inequality including a non-homogeneous kernel and an optimal constant, employing the approach of weight functions and local fractional calculus techniques. The equivalent form and several specific examples are derived as applications.

1 Main results

The choice of non-conjugate parameters additionally helps in the investigation of Hilbert-type inequalities. Let p and q be real numbers such that

$$p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} \geq 1, \tag{3}$$

and let $p' = p/(p-1)$ and $q' = q/(q-1)$, respectively, be their conjugate exponents, that is, $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Moreover, define

$$\lambda = \frac{1}{p'} + \frac{1}{q'} \tag{4}$$

and observe that $0 < \lambda \leq 1$ holds for all p and q as in (3). Especially, the equality $\lambda = 1$ holds if and only if $q = p'$, that is, only if p and q are mutually conjugate. Alternatively, we have $0 < \lambda < 1$, and such parameters p and q will be referred to as non-conjugate exponents.

This section begins a discussion of general Hilbert-type and Hardy-Hilbert-type inequalities including non-conjugate exponents. The following theorem formulates and illustrates the equivalent relations.

Theorem 1 *Let p, q and λ be real parameters satisfying* (3) *and* (4)*. If φ, ψ, $f, g \in C_\alpha(\mathbb{R}_+)$ and $K \in C_\alpha(\mathbb{R}_+)^2$ are non-negative functions, then the following inequalities hold and are equivalent:*

$$\begin{aligned} &\frac{1}{\Gamma^2(1+\alpha)} \int_0^\infty \int_0^\infty K^\lambda(x,y) f(x) g(y) (dx)^\alpha (dy)^\alpha \\ &\leq \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty (\varphi\omega_1 f)^p(x)(dx)^\alpha \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty (\psi\omega_2 g)^q(y)(dy)^\alpha \right)^{\frac{1}{q}} \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty (\psi\omega_2)^{-q'}(y) \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty K^\lambda(x,y) f(x)(dx)^\alpha \right)^{q'} \right)^{\frac{1}{q'}} \\ &\leq \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty (\varphi\omega_1 f)^p(x)(dx)^\alpha \right)^{\frac{1}{p}}, \end{aligned} \tag{6}$$

where

$$\omega_1^{q'}(x) := \frac{1}{\Gamma(1+\alpha)} \int_0^\infty K(x,y)\, \psi^{-q'}(y)(dy)^\alpha$$

and

$$\omega_2^{p'}(y) := \frac{1}{\Gamma(1+\alpha)} \int_0^\infty K(x,y)\, \varphi^{-p'}(x)(dx)^\alpha.$$

Proof. In inequality (5), the left-hand side can be rewritten as follows:

$$\begin{aligned}L &:= \frac{1}{\Gamma^2(1+\alpha)}\int_0^\infty\int_0^\infty K^\lambda(x,y)f(x)\,g(y)(dx)^\alpha(dy)^\alpha\\ &= \frac{1}{\Gamma^2(1+\alpha)}\int_0^\infty\int_0^\infty \left[K(x,y)\psi^{-q'}(y)(\varphi^p\omega_1^{p-q'}f^p)(x)\right]^{\frac{1}{q'}}\\ &\qquad\times\left[K(x,y)\varphi^{-p'}(x)(\psi^q\omega_2^{q-p'})(y)\,a_n^{\alpha q}\right]^{\frac{1}{p'}}\\ &\qquad\times\left[(\varphi\omega_1 f)^p(x)(\psi\omega_2 g)^q(y)\right]^{1-\lambda}(dx)^\alpha(dy)^\alpha.\end{aligned}$$

With the conjugate parameters q', p', $1/(1-\lambda)>1$, the local fractional Hölder's inequality (2) applied to the above relation produces

$$\begin{aligned}L &\le \left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty K(x,y)\psi^{-q'}(y)(dy)^\alpha\right)(\varphi^p\omega_1^{p-q'}f^p)(x)(dx)^\alpha\right)^{\frac{1}{q'}}\\ &\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty K(x,y)\varphi^{-p'}(x)(dx)^\alpha\right)(\psi\omega_2^{q-p'}g^q)^q(y)(dy)^\alpha\right)^{\frac{1}{p'}}\\ &\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty(\varphi\omega_1 f)^p(x)(dx)^\alpha\right)^{1-\lambda}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty(\psi\omega_2 g)^q(y)(dy)^\alpha\right)^{1-\lambda}.\end{aligned}$$

Finally, we obtain (5) by using the definitions of the functions ω_1, ω_2, through the Fubini theorem.

The next part will demonstrate the equivalence of inequalities (5) and (6). Assume that inequality (5) is valid. Defining the function

$$g(y)=(\psi\omega_2)^{-q'}(y)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty K(x,y)f(x)(dx)^\alpha\right)^{q'-1}$$

and using (5), we can write

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_0^\infty(\psi\omega_2)^{-q'}(y)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty K(x,y)f(x)(dx)^\alpha\right)^{q'}\\ &=\frac{1}{\Gamma^2(1+\alpha)}\int_0^\infty\int_0^\infty K(x,y)f(x)(dx)^\alpha(dy)^\alpha\\ &\le\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty(\varphi\omega_1 f)^p(x)(dx)^\alpha\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty(\psi\omega_2 g)^q(y)\right)^{\frac{1}{q}}\\ &=\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty(\varphi\omega_1 f)^p(x)(dx)^\alpha\right)^{\frac{1}{p}}\\ &\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty(\psi\omega_2)^{-q'}(y)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty K^\lambda(x,y)f(x)(dx)^\alpha\right)^{q'}(dy)^{q'}\right)^{\frac{1}{q}},\end{aligned}$$

that is, we get (6).

Now assume, that inequality (6) holds true. Then, the application of the local fractional Hölder inequality yields

$$\begin{aligned}
&\frac{1}{\Gamma^2(1+\alpha)}\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)(dx)^\alpha(dy)^\alpha\\
&=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty(\psi\omega_2)^{-1}(y)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty K(x,y)f(x)(dx)^\alpha\right)(\psi\omega_2 g)(y)(dy)^\alpha\\
&\le\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty(\psi\omega_2)^{-q'}(y)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty K(x,y)f(x)(dx)^\alpha\right)^{q'}(dy)^\alpha\right)^{\frac{1}{q'}}\\
&\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty(\psi\omega_2 g)^q(y)(dy)^\alpha\right)^{\frac{1}{q}}\\
&\le\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty(\varphi\omega_1 f)^p(x)(dx)^\alpha\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty(\psi\omega_2 g)^q(y)(dy)^\alpha\right)^{\frac{1}{q}},
\end{aligned}$$

which implies (5). Hence, inequalities (5) and (6) are equivalent. □

Further, we will consider weight function that involve real differentiable functions $v(x)$. More precisely, we assume that $v(x)\ge 0$, $v'(x)\ge 0$ $(x\in\mathbb{R}_+)$, with $v(0^+)=0$, $v(\infty)=\infty$.

In the following, we assume that $h\in C_\alpha(\mathbb{R}_+)$ is a non-negative function. Additionally, we define

$$k(\eta)=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty h(t)t^{-\alpha\eta}(dt)^\alpha$$

under assumption $k(\eta)<\infty$.

We will now provide our results using a general kernel.

Theorem 2 *Let p, q and λ be real parameters satisfying* (3) *and* (4), *and let $A_1,A_2\in\mathbb{R}_+$. If $f,g\in C_\alpha(\mathbb{R}_+)$ are non-negative functions, then the following inequalities hold and are equivalent:*

$$\begin{aligned}
&\frac{1}{\Gamma^2(1+\alpha)}\int_0^\infty\int_0^\infty h^\lambda(xv(y))f(x)g(y)(dx)^\alpha(dy)^\alpha\\
&\le k^{\frac{1}{p'}}(p'A_1)k^{\frac{1}{q'}}(q'A_2)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha p(A_1+A_2)-\frac{\alpha p}{q'}}f^p(x)(dx)^\alpha\right)^{\frac{1}{p}}\\
&\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty[v(y)]^{\alpha q(A_1+A_2)-\frac{\alpha q}{p'}}[v'(y)]^{\alpha(1-q)}g^q(y)(dy)^\alpha\right)^{\frac{1}{q}} \qquad (7)
\end{aligned}$$

and

$$\Big(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty [v(y)]^{-\alpha q'(A_1+A_2)+\frac{\alpha q'}{p'}}[v'(y)]^\alpha \times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty h^\lambda(xv(y))f(x)(dx)^\alpha\right)^{q'}(dy)^\alpha\Big)^{\frac{1}{q'}}$$
$$\leq k^{\frac{1}{p'}}(p'A_1)k^{\frac{1}{q'}}(q'A_2)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha p(A_1+A_2)-\frac{\alpha p}{q'}}f^p(x)(dx)^\alpha\right)^{\frac{1}{p}}. \quad (8)$$

Proof. Let us substitute functions $K(x,y)=h(xv(y))$, $\varphi(x)=x^{\alpha A_1}$ and $(\psi\circ v)(y)=[v(y)]^{\alpha A_2}[v'(y)]^{-\frac{\alpha}{q'}}$ in inequality (5). Evidently, these substitution is well defined since v is injective. Then

$$\frac{1}{\Gamma^2(1+\alpha)}\int_0^\infty\int_0^\infty h^\lambda(xv(y))f(x)g(y)(dx)^\alpha(dy)^\alpha$$
$$\leq\left[\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha pA_1}\omega_1^p(x)f(x)(dx)^\alpha\right]^{\frac{1}{p}}$$
$$\times\left[\frac{1}{\Gamma(1+\alpha)}\int_0^\infty [v(y)]^{\alpha qA_2}[v'(y)]^{\alpha(1-q)}(\omega_2\circ v)(y)g^q(y)(dy)^\alpha\right]^{\frac{1}{q}},$$

where

$$\omega_1(x)=\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty h(xv(y))[v(y)]^{-\alpha q'A_2}[v'(y)]^\alpha(dy)^\alpha\right)^{\frac{1}{q'}}$$

and

$$(\omega_2\circ v)(y)=\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty h(xv(y))x^{-\alpha p'A_1}(dx)^\alpha\right)^{\frac{1}{p'}}.$$

Now, by substitution $t=xv(y)$, we obtain

$$\begin{aligned}\omega_1^{q'}(x) &= x^{\alpha q'A_2-\alpha}\frac{1}{\Gamma(1+\alpha)}\int_0^\infty h(t)t^{-\alpha q'A_2}(dt)^\alpha \\ &= k(q'A_2)x^{\alpha q'A_2-\alpha},\end{aligned} \quad (9)$$

where we used the definition of function $k(\cdot)$. Similarly, we obtain

$$(\omega_2\circ v)^{p'}(y)=[v(y)]^{\alpha p'A_1-\alpha}k(p'A_1). \quad (10)$$

Finally, from relations (9) and (10), we get (7). Due to the equivalence of (5) and (6), we can prove (8) in a similar way. □

In the case of conjugate exponents ($\lambda=1$), setting $p'=q$ and $q'=p$, from Theorem 2, we obtain the following result.

Theorem 3 *Let $1/p + 1/q = 1$, $p > 1$, and let $A_1, A_2 \in \mathbb{R}_+$. If $f, g \in C_\alpha(\mathbb{R}_+)$ are non-negative functions, then the following inequalities hold and are equivalent:*

$$\frac{1}{\Gamma^2(1+\alpha)} \int_0^\infty \int_0^\infty h(xv(y)) f(x) g(y) (dx)^\alpha (dy)^\alpha$$
$$\leq L \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty x^{\alpha p(A_1+A_2)-\alpha} f^p(x) (dx)^\alpha \right)^{\frac{1}{p}}$$
$$\times \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty [v(y)]^{\alpha q(A_1+A_2)-\alpha} [v'(y)]^{\alpha(1-q)} g^q(y) (dy)^\alpha \right)^{\frac{1}{q}} \tag{11}$$

and

$$\Bigg(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty [v(y)]^{-\alpha p(A_1+A_2)+\alpha(p-1)} [v'(y)]^\alpha$$
$$\times \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty h(xv(y)) f(x) (dx)^\alpha \right)^{q'} (dy)^\alpha \Bigg)^{\frac{1}{q'}}$$
$$\leq L \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty x^{\alpha p(A_1+A_2)-\alpha} f^p(x) (dx)^\alpha \right)^{\frac{1}{p}}, \tag{12}$$

where

$$L = k^{\frac{1}{q}}(qA_1) k^{\frac{1}{p}}(pA_2). \tag{13}$$

The primary objective of achieving the optimal constant is to minimize constant L defined by (13), in a form devoid of exponents, by appropriate selection of parameters A_1 and A_2. Therefore, it is logical to establish the following condition

$$pA_2 = qA_1, \tag{14}$$

since in this setting $k(pA_2) = k(qA_1)$. Consequently, constant L from Theorem 3 is transformed into

$$L^* = k(pA_2).$$

Additionally, under assumption (14), inequalities (11) and (12) can be revised as follows:

$$\frac{1}{\Gamma^2(1+\alpha)} \int_0^\infty \int_0^\infty h(xv(y)) f(x) g(y) (dx)^\alpha (dy)^\alpha$$
$$\leq L^* \left[\frac{1}{\Gamma(1+\alpha)} \int_0^\infty x^{\alpha pqA_1-\alpha} f^p(x) (dx)^\alpha \right]^{\frac{1}{p}}$$
$$\times \left[\frac{1}{\Gamma(1+\alpha)} \int_0^\infty [v(y)]^{\alpha pqA_2-\alpha} [v'(y)]^{\alpha(1-q)} g^q(y) (dy)^\alpha \right]^{\frac{1}{q}} \tag{15}$$

and

$$\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty [v(y)]^{\alpha(p-1)(1-pqA_2)}[v'(y)]^\alpha \left[\frac{1}{\Gamma(1+\alpha)}\int_0^\infty h(xv(y))f(x)(dx)^\alpha\right]^p\right)^{\frac{1}{p}}$$
$$\leq L^*\left[\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha pqA_1-\alpha}f^p(x)(dx)^\alpha\right]^{\frac{1}{p}}. \tag{16}$$

Our main aim is to demonstrate that the constants on the right-hand sides of inequalities (15) and (16) are best possible. This is the essence of the following theorem.

Theorem 4 *Let $1/p+1/q=1$, $p>1$, and let $A_1, A_2 \in \mathbb{R}_+$. If the parameters A_1 and A_2 satisfy condition* (14)*, then the constant factor L^* is the best possible in inequalities* (15) *and* (16).

Proof. It suffices to prove that constant L^* is the best possible in inequality (15) since (15) and (16) are equivalent. For this purpose, put $\widetilde{f}(x) = x^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}} \cdot \chi_{[1,+\infty)}$ and $\widetilde{g}(y) = [v(y)]^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}} \times [v'(y)]^\alpha \cdot \chi_{[0,1)}$. Now, let us suppose that inequality (15) holds for the functions $\widetilde{f}(x)$ and $\widetilde{g}(y)$. According to local fractional calculus, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha pqA_1-\alpha}\widetilde{f}^p(x)(dx)^\alpha$$
$$=\frac{1}{\Gamma(1+\alpha)}\int_1^\infty x^{-\alpha-\alpha\varepsilon}(dx)^\alpha = \frac{1}{\varepsilon^\alpha\Gamma(1+\alpha)}. \tag{17}$$

The substitution $t=v(y)$ yields

$$\frac{1}{\Gamma(1+\alpha)}\int_0^\infty [v(y)]^{\alpha pqA_2-\alpha}[v'(y)]^{\alpha(1-q)}\widetilde{g}^q(y)(dy)^\alpha$$
$$=\frac{1}{\Gamma(1+\alpha)}\int_0^1 [v(y)]^{-\alpha+\alpha\varepsilon}[v'(y)]^\alpha(dy)^\alpha = \frac{[v(1)]^{\alpha\varepsilon}}{\varepsilon^\alpha\Gamma(1+\alpha)}. \tag{18}$$

Now assume that there is a positive constant $M<L^*$ such that inequality (15) is still valid if we replace L^* with M. Hence, if we include relations (17) and (18) in inequality (15) with the constant M instead of L^*, we have

$$\widetilde{I} := \frac{1}{\Gamma^2(1+\alpha)}\int_0^\infty\int_0^\infty h(xv(y))\widetilde{f}(x)\widetilde{g}(y)(dx)^\alpha(dy)^\alpha$$
$$\leq \frac{M}{\varepsilon^\alpha\Gamma(1+\alpha)}[v(1)]^{\frac{\alpha\varepsilon}{q}}. \tag{19}$$

Further, let us we estimate the left-hand side of inequality (15). By using the Fubini theorem and substitution $t = xv(y)$ (see also [8]), we obtain

$$\begin{aligned}
\widetilde{I} &= \frac{1}{\Gamma(1+\alpha)}\int_0^1 [v(y)]^{-\alpha pA_2+\frac{\alpha\varepsilon}{q}}[v'(y)]^{\alpha} \\
&\quad \times\left[\frac{1}{\Gamma(1+\alpha)}\int_1^{\infty} h(xv(y))x^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}(dy)^{\alpha}\right](dx)^{\alpha} \\
&= \frac{1}{\Gamma(1+\alpha)}\int_0^1 [v(y)]^{-\alpha-\alpha\varepsilon}[v'(y)]^{\alpha} \\
&\quad \times\left[\frac{1}{\Gamma(1+\alpha)}\int_{v(y)}^{\infty} h(t)t^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}(dt)^{\alpha}\right](dy)^{\alpha} \\
&= \frac{1}{\Gamma(1+\alpha)}\int_0^1 [v(y)]^{-\alpha-\alpha\varepsilon}[v'(y)]^{\alpha} \\
&\quad \times\Big[\frac{1}{\Gamma(1+\alpha)}\int_{v(y)}^{v(1)} h(t)t^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}(dt)^{\alpha} \\
&\quad +\frac{1}{\Gamma(1+\alpha)}\int_{v(1)}^{\infty} h(t)t^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}(dt)^{\alpha}\Big](dy)^{\alpha} \\
&= \frac{1}{\Gamma(1+\alpha)}\int_0^{v(1)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{v^{-1}(t)}[v(y)]^{-\alpha pA_2+\frac{\alpha\varepsilon}{q}}[v'(y)]^{\alpha}(dy)^{\alpha}\right) \\
&\quad \times h(t)t^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}(dt)^{\alpha} \\
&\quad +\frac{[v(1)]^{\alpha\varepsilon}}{\varepsilon^{\alpha}\Gamma(1+\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_{v(1)}^{\infty} h(t)t^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}(dt)^{\alpha} \\
&= \frac{1}{\varepsilon^{\alpha}\Gamma(1+\alpha)}\Big(\frac{1}{\Gamma(1+\alpha)}\int_0^{v(1)} h(t)t^{-\alpha qA_1+\frac{\alpha\varepsilon}{q}}(dt)^{\alpha} \\
&\quad +\frac{[v(1)]^{\alpha\varepsilon}}{\Gamma(1+\alpha)}\int_{v(1)}^{\infty} h(t)t^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}(dt)^{\alpha}\Big). \qquad (20)
\end{aligned}$$

Now, relations (19) and (20) yield

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^{v(1)} h(t)t^{-\alpha qA_1+\frac{\alpha\varepsilon}{q}}(dt)^{\alpha}+\frac{[v(1)]^{\alpha\varepsilon}}{\Gamma(1+\alpha)}\int_{v(1)}^{\infty} h(t)t^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}(dt)^{\alpha} \\
&= \varepsilon^{\alpha}\Gamma(1+\alpha)\widetilde{I} \le M[v(1)]^{\frac{\alpha\varepsilon}{q}}.
\end{aligned}$$

For $\varepsilon \longrightarrow 0^+$, by local fractional Fatou lemma (cf. [18]), we have

$$L^* = k(qA_1) \le M,$$

which contradicts with the assumption that the constant M is less than L^*. □

Below we provide some collaries of Theorem 4. We used kernel $h_1(t) = (1+t)^{-\alpha s}$, where $s > 0$. In the following, we assume that

$$A_1 = \frac{2-s}{2q}, \quad A_2 = \frac{2-s}{2p}. \tag{21}$$

Then constant L^* from Theorem 3 becomes

$$\begin{aligned} L^* &= k(pA_2) = k\left(1 - \frac{s}{2}\right) \\ &= \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{u^{-\alpha - \frac{\alpha s}{2}}}{(1+u)^{\alpha s}} (du)^\alpha = B_\alpha\left(\frac{s}{2}, \frac{s}{2}\right), \end{aligned}$$

and we obtain the following consequence from Theorem 4.

Corollary 1 *Let $1/p + 1/q = 1$, $p > 1$, and let $s > 0$. If $f, g \in C_\alpha(\mathbb{R}_+)$ are non-negative functions, then the following inequalities hold and are equivalent:*

$$\begin{aligned} &\frac{1}{\Gamma^2(1+\alpha)} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(1+xv(y))^{\alpha s}} (dx)^\alpha (dy)^\alpha \\ &\leq B_\alpha\left(\frac{s}{2}, \frac{s}{2}\right) \left[\frac{1}{\Gamma(1+\alpha)} \int_0^\infty x^{\alpha p(1-\frac{s}{2})-\alpha} f^p(x)(dx)^\alpha\right]^{\frac{1}{p}} \\ &\quad \times \left[\frac{1}{\Gamma(1+\alpha)} \int_0^\infty [v(y)]^{\alpha q(1-\frac{s}{2})-\alpha} [v'(y)]^{\alpha(1-q)} g^q(y)(dy)^\alpha\right]^{\frac{1}{q}} \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty [v(y)]^{\frac{\alpha p s}{2}-\alpha} [v'(y)]^\alpha \left[\frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{f(x)}{(1+xv(y))^{\alpha s}} (dx)^\alpha\right]^p\right)^{\frac{1}{p}} \\ &\leq B_\alpha\left(\frac{s}{2}, \frac{s}{2}\right) \left[\frac{1}{\Gamma(1+\alpha)} \int_0^\infty x^{\alpha p(1-\frac{s}{2})-\alpha} f^p(x)(dx)^\alpha\right]^{\frac{1}{p}}, \end{aligned}$$

where constant $B_\alpha\left(\frac{s}{2}, \frac{s}{2}\right)$ is the best possible.

The constant that appears in our second example is expressed in terms of local fractional hypergeometric function defined by

$${}_2F_1^\alpha(a, b; c; z) = \frac{1}{B_\alpha(b, c-b)} \frac{1}{\Gamma(1+\alpha)} \int_0^1 t^{(b-1)\alpha}(1-t)^{(c-b-1)\alpha}(1-zt)^{-a\alpha}(dt)^\alpha,$$

where $c > b > 0$, $|z| \leq 1$.

Let $h_2(t) = (1 + t + \max\{1, t\})^{-\alpha s}$, $0 < s < 2$, and let A_1 and A_2 be defined by (21). Using following formula (see also [16])

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_0^1 t^{\beta\alpha} \left(1 + \frac{1}{b} t\right)^{-\alpha s} (dt)^\alpha \\ &\qquad = B_\alpha(\beta+1, 1)\, {}_2F_1^\alpha\left(s, \beta+1, \beta+2; -\frac{a}{b}\right). \end{aligned}$$

for constant L^* from Theorem 4, we can write

$$\begin{aligned} L^* &= k(pA_2) = k\left(1-\frac{s}{2}\right) \\ &= \frac{1}{\Gamma(1+\alpha)}\int_0^\infty \frac{t^{\frac{\alpha s}{2}-\alpha}}{(1+t+\max\{1,t\})^{\alpha s}}(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{t^{\frac{\alpha s}{2}-\alpha}}{(2+t)^{\alpha s}}(dt)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_0^\infty \frac{t^{\frac{\alpha s}{2}-\alpha}}{(1+2t)^{\alpha s}}(dt)^\alpha \\ &= 2^{1-\alpha s}B_\alpha\left(\frac{s}{2},1\right)\ {}_2F_1^\alpha\left(s,\frac{s}{2};\frac{s}{2}+1;-\frac{1}{2}\right). \end{aligned} \tag{22}$$

Setting $v(y) = y$ and $h_2(t)$ in Theorem 4, we obtain the following result with non-homogeneous kernel.

Corollary 2 *Let $1/p + 1/q = 1$, $p > 1$, and let $0 < s < 2$. If $f, g \in C_\alpha(\mathbb{R}_+)$ are non-negative functions, then the following inequalities hold and are equivalent:*

$$\begin{aligned} &\frac{1}{\Gamma^2(1+\alpha)}\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(1+xy+\max\{1,xy\})^{\alpha s}}(dx)^\alpha(dy)^\alpha \\ &\le M\left[\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha p(1-\frac{s}{2})-\alpha} f^p(x)(dx)^\alpha\right]^{\frac{1}{p}} \\ &\quad\times\left[\frac{1}{\Gamma(1+\alpha)}\int_0^\infty y^{\alpha q(1-\frac{s}{2})-\alpha} g^q(y)(dy)^\alpha\right]^{\frac{1}{q}} \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty y^{\frac{\alpha p s}{2}-\alpha}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \frac{f(x)}{(1+xy+\max\{1,xy\})^{\alpha s}}(dx)^\alpha\right]^p\right)^{\frac{1}{p}} \\ &\quad\le M\left[\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha p(1-\frac{s}{2})-\alpha} f^p(x)(dx)^\alpha\right]^{\frac{1}{p}}, \end{aligned}$$

where the best possible constant M is defined by (22).

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References

[1] D. Baleanu, M. Krnić and P. Vuković, A class of fractal Hilbert-typeinequalities obtained via Cantor-type spherical coordinates. Math. Meth. Appl. Sci., **44** (2021), no. 7, pp. 6195–6208. https://doi.org/10.1002/mma.7180

[2] Y. Basci and D. Baleanu, Hardy-type inequalities within fractional derivatives without singular kernel. J. Inequal. Appl., **2018** (2018), Article number 304. https://doi.org/10.1186/s13660-018-1893-6

[3] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities.* Second edition. Cambridge University Press, Cambridge, 1967.

[4] G. Jumarie, Fractional Euler's integral of first and second kinds. Application to fractional Hermite's polynomials and to probability density of fractional order. J. Appl. Math. & Informatics, **28** (2010), no. 1-2, pp. 257–273.

[5] M. Krnić, J. Pečarić, I. Perić and P. Vuković, *Recent advances in Hilbert-type inequalities*, Element, Zagreb, 2012.

[6] M. Krnić and P. Vuković, Multidimensional Hilbert-type inequalities obtained via local fractional calculus. Acta Appl. Math., **169** (2020), pp. 667–680. https://doi.org/10.1007/s10440-020-00317-x

[7] Q. Liu, A Hilbert-type fractional integral inequality with the kernel of Mittag-Leffler function and its applications. Math. Inequal. Appl., **21** (2018), no. 3, pp. 729–737. http://dx.doi.org/10.7153/mia-2018-21-52

[8] T. Liu, R.A. Rahim and B. Yang, A new Hilbert-type integral inequality with the general nonhomogeneous kernel and applications. J. Math. Inequal., **17** (2023), no. 4, pp. 1581–1596. http://dx.doi.org/10.7153/jmi-2023-17-104

[9] M.Th. Rassias and B. Yang, On an equivalent property of a peverse Hilbert-type integral inequality related to the extended Hurwitz-zeta function. J. Math. Inequal., **13** (2019), no. 2, pp. 315–334. http://dx.doi.org/10.7153/jmi-2019-13-23

[10] M.Th. Rassias and B. Yang, On a Hilbert-type integral inequality in the whole plane related to the extended Riemann zeta function. Complex Anal. Oper. Theory, **13** (2019), pp. 1765–1782. https://doi.org/10.1007/s11785-018-0830-5

[11] M.Th. Rassias, B. Yang and G.C. Meletiou, A more accurate half-discrete Hilbert-type inequality in the whole plane and the reverses. Ann. Funct. Anal., **12** (2021), Article number 50. https://doi.org/10.1007/s43034-021-00133-w

[12] M.Th. Rassias, B. Yang and A. Raigorodskii, On a more accurate reverse Hilbert-type inequality in the whole plane. J. Math. Inequal., **14** (2020), no. 4, pp. 1359–1374. http://dx.doi.org/10.7153/jmi-2020-14-88

[13] M.Th. Rassias, B. Yang and A. Raigorodskii, A Hilbert-type integral inequality in the whole plane related to the arc tangent function. Symmetry, **13** (2021), no. 2, Article number 351. https://doi.org/10.3390/sym13020351

[14] M.Th. Rassias, B. Yang and A. Raigorodskii, Equivalent conditions of a multiple Hilbert-type integral inequality with the nonhohogeneous kernel. Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., **116** (2022), Article number 107. https://doi.org/10.1007/s13398-022-01238-0

[15] M.Th. Rassias, B. Yang and A. Raigorodskii, An equivalent form related to a Hilbert-type integral inequality. Axioms, **12** (2023), no.7, Article number 677. https://doi.org/10.3390/axioms12070677

[16] P. Vuković, Some local fractional Hilbert-type inequalities. Fractal Fract., **7** (2023), no. 2, Article number 205. https://doi.org/10.3390/fractalfract7020205

[17] X.J. Yang, *Local Fractional Functional Analysis and its Applications*, Asian Academic publisher Limited, Hong Kong, 2011.

[18] X.J. Yang, *Advanced Local Fractional Calculus and its Applications*, Word Science Publisher, New York, 2012.

[19] B. Yang, D. Andrica, D. Bagdasar and M.Th. Rassias, An equivalent property of a Hilbert-type integral inequality and its applications. Appl. Anal. Discrete Math., **16** (2022), no. 2, pp. 548–563. https://doi.org/10.2298/AADM220514025Y

[20] B. Yang, D. Andrica, O. Bagdasar and M.Th. Rassias, On a Hilbert-type integral inequality in the whole plane with the equivalent forms. Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., **117** (2023), Article number 57. https://doi 10.1007/s13398-023-01388-9

[21] B. Yang and M.Th. Rassias, *On Hilbert-Type and Hardy-Type Integral Inequalities and Applications*, Springer Cham, 2019. https://doi.org/10.1007/978-3-030-29268-3

Predrag Vuković
Faculty of Teacher Education,
University of Zagreb
Savska cesta 77, 10000 Zagreb, Croatia.
predrag.vukovic@ufzg.hr

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