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Հատոր Том Volume 125 | 2025 | № 2

MATHEMATICS

UDC 517.521
DOI: 10.54503/0321-1339-2024.125.2-7

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A Uniqueness Theorem for Stromberg Series

(Submitted on 13/V 2025)

Keywords: *Stromberg series, uniqueness problem, reconstruction theorem*

Introduction. Studies of the uniqueness of series in some classical orthogonal systems occupy an important place in the theory of series in these systems. In the theory of uniqueness of trigonometric series of fundamental importance is Cantor's theorem (see [1]), which says that if trigonometric series converges to zero everywhere, then all the coefficients of this series are equal to zero. Further, many authors were engaged in the study of the uniqueness of trigonometric series.

The investigation of the uniqueness of series in the Haar system started with the papers [6, 8, 9]. In these papers, a Cantor-type theorem is proved for Haar series.

In [7] it is proved counterparts of the de la Vallée-Poussin theorem for series in Haar and Walsh systems whose coefficients satisfy certain necessary conditions.

Investigations of uniqueness for series in the Franklin system began recently. The definition of the orthonormal Franklin system $\{f_n(x)\}_{n=0}^{\infty}$ will be given below. In [3] the following theorem is proved.

Theorem 1. *If the Franklin series converges to zero everywhere, then all the coefficients of this series are equal to zero.*

In [2] it is stated that any singleton is not a uniqueness set for Franklin series, i.e. for any $x_0 \in [0,1]$ there exists a nontrivial Franklin series which converges to zero everywhere except the point x_0. The coefficients of these series satisfy the relation $a_n = O(\sqrt{n})$.

Z. Wronicz constructed a non-trivial Franklin series for which

$$\lim_{k\to\infty}\sum_{n=0}^{2^k} a_n f_n(x) = 0, x \in [0,1]. \tag{1}$$

In [4], the following problem was posed: Will all coefficients a_n be equal to zero if both (1) is satisfied and $a_n = o(\sqrt{n})$? In [11] Z. Wronicz gave a positive answer to this question.

Theorem 2 (Z. Wronicz). *If $a_n = o(\sqrt{n})$ and (1) is satisfied, then all coefficients a_n are equal to zero.*

In [5], using methods different from those used by Z. Wronicz , the following theorem is proved.

Theorem 3. *Let $\sup_i \frac{n_{i+1}}{n_i} < \infty$ and partial sums $S^{(i)}(x) := \sum_{k=0}^{n_i} a_k f_k(x)$ of the series $\sum_{k=0}^{\infty} a_k f_k(x)$, with coefficients $a_n = o(\sqrt{n})$ converge in measure to a bounded function f. Then this series is the Fourier-Franklin series of the function f.*

Definition of the Stromberg system, its periodized version and statements of the main results

In [10] J.O. Stromberg introduced spline wavelet system on $\mathbb{R}^n$. Here we recall the definition only on $\mathbb{R}$.

Let $A_0 = \mathbb{Z}_+ \cup \{0\} \cup \frac{1}{2}\mathbb{Z}_-$ and $A_1 = A_0 \cup \{\frac{1}{2}\}$. A_0 splits $\mathbb{R}$ into intervals $\{I_\sigma\}_{\sigma\in A_0}$ (σ is the left endpoint of I_σ). Let $m \in \mathbb{Z}_+$ and S_0^m be the subspace of functions f in $L^2(\mathbb{R})$ such that $f \in C^m(\mathbb{R})$ and is real polynomial of degree $m+1$ on each I_σ, $\sigma \in A_0$. Let S_1^m be the corresponding subspace of $L^2(\mathbb{R})$ with set A_0 replaced by A_1. It is clear that S_0^m is a subspace of S_1^m and has codimension 1 in S_1^m. Therefore there is a function τ in $L^2(\mathbb{R})$ that is a uniquely defined up to sign by conditions $\tau \in S_1^m$,

$$\int_R \tau(x) f(x) dx = 0, \quad \forall f \in S_0^m \tag{2}$$

$\| \tau \|_{L^2} = 1.$

J.O. Stromberg proved that system $\{f_{j,k}(x)\}_{(j,k)\in\mathbb{Z}^2}$, defined by formula

$$f_{j,k}(x) = 2^{j/2}\tau(2^j x - k), \quad j,k \in \mathbb{Z}, \quad x \in \mathbb{R}. \tag{3}$$

is a complete orthonormal system in $L^2(\mathbb{R})$ and unconditional basis in $H^p(\mathbb{R})$, $p > \frac{1}{m+1}$. Recall that if the system defined by formulas (3) forms complete orthonormal system in $L^2(\mathbb{R})$ the function τ is called wavelet.

To prove the latter he obtained some inequalities for τ and derivatives of τ. We need the following inequality from [10]

$$|\tau(x)| \leq Cr^{|x|}, \qquad x \in R, \qquad r \in (0,1) \text{ and } C \text{ is a positive constant.} \quad (4)$$

From (2) and (4) it follows that τ is integrable on $\mathbb{R}$ and $\int_{-\infty}^{+\infty} \tau(t)dt = 0$. In (4) constants r and C depend on m. In this paper m is arbitrary fixed non-negative integer. Therefore we omit m and write C instead of C_m.

In [10] J.O. Stromberg introduced another orthonormal system $\{F_n(x)\}_{n=0}^{\infty}$ on $[0,1]$.

Denote

$$F_{j,k}(x) = \sum_{l \in \mathbb{Z}} f_{j,k}(x-l), \; j,k \in \mathbb{Z}.$$

It is clear that

$$F_{j,k}(x) = F_{j,k}(x+1) \text{ and } F_{j,k+2^j}(x) = F_{j,k}(x).$$

For fixed $j \geq 0$ there exist only 2^j distinct functions $F_{j,k}(x)$, $k = 0,1,\ldots,2^j - 1$.

Denote

$$F_n(x) := F_{j,k}(x), t_n := \frac{2k-1}{2^{j+1}},$$
$$\text{if } n = 2^j + k, \; j \geq 0, k = 0,\ldots,2^j - 1$$

and $F_0(x) = 1$.

It follows from (4) that

$$|F_n(x)| \leq C_1 2^{\frac{j}{2}} r^{2^j\left|x-\frac{k}{2^j}\right|} \leq C_2 \sqrt{n} q^{n|x-t_n|}, \text{ where } q = \sqrt{r}.$$

The main results being announced in this article are the following theorems:

Theorem 4. *Let n_i be an increasing subsequence of natural numbers so that*

$$\sup_i (n_{i+1} - n_i) =: M < \infty. \quad (5)$$

If the coefficients of the Stromberg series $S(x) := \sum_{n=0}^{\infty} a_n F_n(x)$ *satisfy the condition*

$$a_n = o(\sqrt{n}), \tag{6}$$

the partial sums $S^{(i)}(x) := \sum_{n=0}^{2^{n_i}-1} a_n F_n(x)$ *converge in measure to 0 and*

$$\sup_i S^{(i)}(x) < \infty \text{ for } x \notin B, \text{where } B \text{ is a countable set,} \tag{7}$$

then $a_n = 0$ for $n \geq 0.$

The following theorem is a generalization of the previous one.

Theorem 5. If an increasing sequence n_i satisfies (5), the coefficients of the Stromberg series $\sum_{n=0}^{\infty} a_n F_n(x)$ satisfy the condition (6), the partial sums $S^{(i)}(x)$ converge in measure to a bounded function f and (7) holds, then this series is the Fourier-Stromberg series of the function f.

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A uniqueness theorem for Stromberg series

Some uniqueness theorems for series in the Stromberg's periodized system are proved in this work. In particular, if the partial sums $S^{(i)}(x) := \sum_{n=0}^{2^{n_i}-1} a_n F_n(x)$ of the Stromberg series $\sum_{n=0}^{\infty} a_n F_n(x)$ converge in measure to a bounded integrable function f and $\sup_i |S^{(i)}(x)| < \infty$ when $x \notin B$, where B is some countable set, with $a_n = o(\sqrt{n})$ and $\sup_i (n_{i+1} - n_i) < \infty$, then this series is the Fourier-Stromberg series of the function f.

Ակադեմիկոս Գ.Գ. Գևորգյան, Կ.Ա. Քեռյան, Մ.Պ. Պողոսյան

Միակության թեորեմ Ստրոմբերգի շարքերի համար

Այս աշխատանքում ապացուցված են Ստրոմբերգի պարբերակա-նացված համակարգի շարքերի համար միակության որոշ թեորեմներ: Մասնավորապես, եթե Ստրոմբերգի շարքի $S^{(i)}(x) := \sum_{n=0}^{2^{n_i}-1} a_n F_n(x)$ մասնակի գումարները $\sum_{n=0}^{\infty} a_n F_n(x)$, ըստ չափի, զուգամիտում են f սահմանափակ ինտեգրելի ֆունկցիային և $\sup_i |S^{(i)}(x)| < \infty$, երբ

$x \notin B$, որտեղ B-ն որևէ հաշվելի բազմություն է, $a_n = o(\sqrt{n})$ և $\sup_i (n_{i+1} - n_i) < \infty$, ապա այս շարքը f ֆունկցիայի Ֆուրիե-Ստրոմ-բերգի շարքն է:

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Теорема единственности для рядов Стромберга

В работе доказаны некоторые теоремы единственности для рядов по периодической системе Стромберга. В частности, если частичные суммы $S^{(i)}(x) := \sum_{n=0}^{2^{n_i}-1} a_n F_n(x)$ ряда Стромберга $\sum_{n=0}^{\infty} a_n F_n(x)$ сходятся по мере к интегрируемой функции f и $\sup_i |S^{(i)}(x)| < \infty$ при x∉B, где B – некоторое счетное множество, причем $a_n = o(\sqrt{n})$ и $\sup_i (n_{i+1} - n_i) < \infty$, то этот ряд является рядом Фурье-Стромберга функции f.

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