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ROBUSTNESS OF UNIFORM CONTROL SYSTEMS WITH MULTIPLICATIVE UNCERTAINTIES

This paper aims to develop simple graphical tests for analyzing stability robustness of uniform control systems with respect to multiplicative uncertainties. The uniform systems are multi-input multi-output (MIMO) feedback control systems having identical transfer functions of separate channels and rigid cross-connections described by a square numerical matrix. The exposition is based on the method of characteristic transfer functions, which allows reducing the stability analysis of an interconnected MIMO system with N and N outputs to the analysis of N fictitious independent systems with one input and one output. The proposed robustness tests are in the form of N "forbidden" circles on the complex plane of characteristic gain loci of the open-loop uniform system. A numerical example illustrating application of the tests to the analysis of stability robustness of a three-dimensional uniform system is given.

Keywords: multivariable control system, uniform system, multiplicative uncertainty, stability robustness, characteristic transfer functions.

Introduction. The issue of robustness of multivariable or multi-input and multi-output (MIMO) control systems with respect to external disturbances and uncertainties has always been one of the central in modern feedback control [1-3]. The paper presents simple graphical tests for analyzing stability robustness to multiplicative perturbations of a special class of multivariable feedback control systems called uniform systems.

The uniform systems are MIMO systems with identical transfer functions of separate channels and rigid cross-connections described by a square numerical matrix. These specific structural features of uniform systems allow transforming the well-known sufficient conditions of stability robustness of MIMO systems to a very simple and visual graphical form, which is quite close to the sufficient conditions for single-input, single-output (SISO) control systems. The exposition is based on the method of characteristic transfer functions (CTFs) [4], which allows reducing the stability analysis of an interconnected MIMO control system with N and N outputs to the stability analysis of N fictitious, but independent SISO systems.

The proposed graphical tests of stability robustness of uniform systems to multiplicative perturbations or uncertainties are very similar to the corresponding analysis of SISO control systems by the Nyquist criterion, in which the critical point $-1, j0$ is replaced by some "forbidden" circles or areas on the complex plane of the characteristic gain loci of the open-loop uniform system.

Canonical representations and stability analysis of uniform MIMO systems. The matrix block diagram of a linear uniform system with N inputs and N outputs is shown in Fig. 1, where $w(s)$ is a scalar (SISO) transfer function of identical separat-e channels and R is an $N \times N$ numerical matrix of rigid cross-connections.

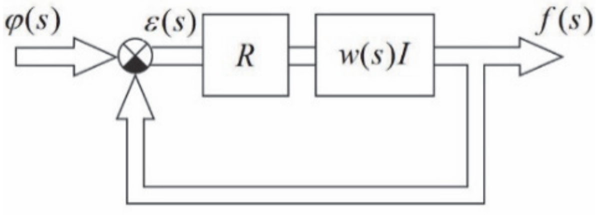


Fig. 1. The block diagram of a uniform MIMO system

The transfer matrix $W(s)$ of the open-loop uniform system in Fig.1:

$$W(s) = w(s)R \tag{1}$$

coincides, up to the complex scalar multiplier $w(s)$, with the numerical matrix of cross-connections R. The corresponding transfer matrix $T(s)$ of the closed-loop uniform system (complementary sensitivity transfer matrix) has the following standard forms [4]:

$$T(s) = [I + W(s)]^{-1} W(s). \tag{2}$$

Denoting by λ_i the eigenvalues of R, which for simplicity are supposed distinct, and by C the modal matrix composed of linearly independent eigenvectors c_i of R, the canonical representation of the open-loop uniform system via similarity transformation will have the following form [4]:

$$W(s) = C\, diag\{\lambda_i w(s)\} C^{-1}. \tag{3}$$

As can be seen from (1) and (3), the canonical basis of the linear uniform system is completely defined by the numerical matrix of cross-connections R and does not depend on the transfer function $w(s)$ of separate channels. Besides, all the CTFs

$$q_i(s) = \lambda_i w(s) \ \ (i = 1, 2, \ldots, N) \tag{4}$$

coincide, up to the constant "gains" λ_i, with the transfer function $w(s)$. Considering (3) and (4), the canonical representation of the transfer matrix $T(s)$ (2) is:

$$T(s) = C \, diag \left\{ \frac{\lambda_i w(s)}{1 + \lambda_i w(s)} \right\} C^{-1}. \tag{5}$$

The stability of the linear closed-loop uniform system is determined by the roots of the characteristic equation:

$$\det\left[1 + w(s)R\right] = \prod_{i=1}^{N} [1 + \lambda_i w(s)] = 0, \tag{6}$$

which is equivalent to a set of N equations:

$$1 + \lambda_i w(s) = 0 \quad (i = 1, 2, \ldots, N). \tag{7}$$

Basic perturbation models of uniform systems with multiplicative uncertainties. At present, there are various paradigms for modeling dynamic system uncertainties, e.g., structured, unstructured, highly structured (or parametric), etc. [3].

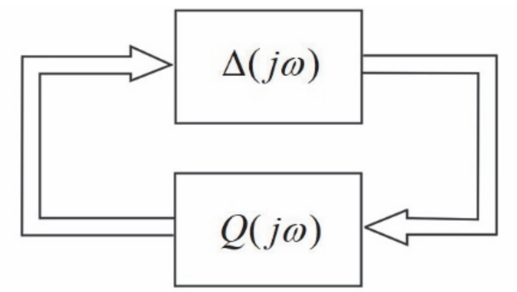


Fig. 2. Basic perturbation model of a MIMO control system

The most common approach to analyzing the influence of uncertainties on the stability of feedback control system assumes that uncertainties may be presented in the form of a Basic Perturbation Model (BPM) shown in Fig. 2 [3]. Here, $Q(j\omega)$ is the transfer matrix of the ideal (nominal) system, which is assumed to be stable, and the block $\Delta(j\omega)$ represents all uncertainties in the dynamics of the system.

One of the key results in the robust theory is based on the Small Gain Theorem [1,2], and is formulated for the systems in Fig. 2 as follows.

Let $Q(j\omega)$ and $\Delta(j\omega)$ be stable. Then, for stability of the MIMO system with uncertainty $\Delta(j\omega)$, it is sufficient that for all frequencies ω, the following condition holds:

$$\left\| Q(j\omega) \right\| < \frac{1}{\left\| \Delta(j\omega) \right\|} \quad \forall\ \omega \in [-\infty, \infty] \tag{8}$$

with $\|\cdot\|$ denoting the spectral norm of the corresponding matrix, or (another sufficient condition):

$$\left\| Q(j\omega) \right\|_{\infty} < \frac{1}{\left\| \Delta(j\omega) \right\|_{\infty}}, \tag{9}$$

where $\|\cdot\|_{\infty}$ stands for the Hardy norm [3] determined for any transfer matrix $\Phi(j\omega)$ as:

$$\| \Phi(j\omega) \|_{\infty} = \sup_{\omega} \left\| \Phi(j\omega) \right\|. \tag{10}$$

Two main types of uncertainties (perturbation) used in the BPM are called *additive* and *multiplicative* [3]. Uniform control systems with additive uncertainties are thoroughly considered in [5]. Below, we shall discuss the robustness of uniform systems with multiplicative uncertainties.

As can be seen from the matrix block diagram in Fig. 1, there are two essentially different structural blocks in the uniform system, namely, the numerical matrix of rigid cross-connections R and the scalar (diagonal) transfer matrix $w(s)I$ of identical separate channels. When analyzing the stability robustness of uniform systems, it is appropriate to consider the influence of uncertainties (perturbations) in these two blocks separately. Besides, it is also important to analyze the case of joint perturbations in matrices R and $w(s)I$.

Structurally, all these three cases are illustrated in Fig. 3-Fig. 5.

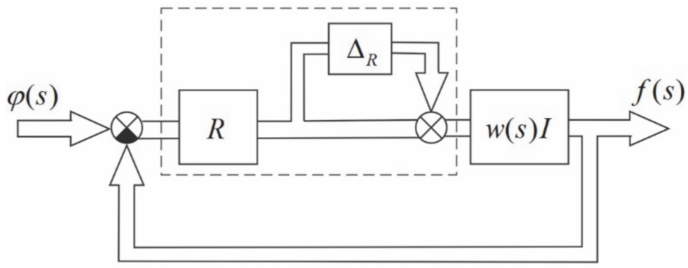


Fig. 3. Multiplicative perturbation of the numerical matrix R

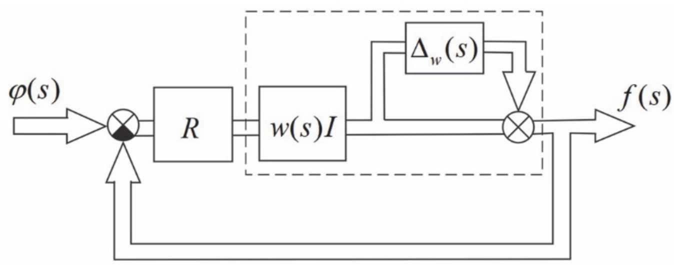


Fig. 4. Multiplicative perturbation of the transfer matrix $w(s)I$

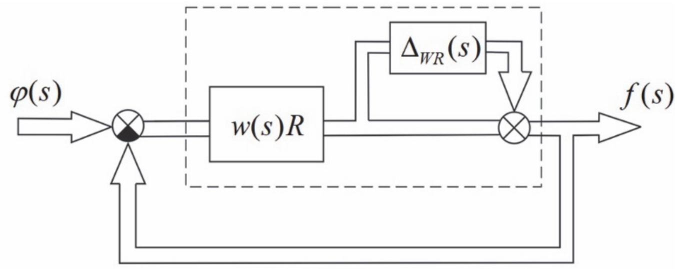


Fig. 5. Joint multiplicative perturbation of the open-loop transfer matrix $W(s)$

The matrix block diagrams in Fig. 3 and Fig. 4 represent the matrix block diagrams of the uniform system with multiplicative uncertainties in matrix R and the transfer matrix $w(s)I$. Note that the perturbation Δ_R in Fig. 3 is assumed to be numerical. As for the perturbations $\Delta_W(s)$ of $w(s)I$ in Fig. 4, generally they may be frequency-dependent and nondiagonal. The same concerns the case of joint perturbation $\Delta_{WR}(s)$ of the open-loop transfer matrix $W(s)$ (1) (Fig. 5).

Robustness analysis of uniform systems with multiplicative uncertainties (perturbations). Remember that the developed in [5] graphical tests for analyzing stability robustness of uniform systems have different forms for cases of additive uncertainties in matrices R and $w(s)I$. Besides, they are not applicable to the case of joint additive perturbations of the open-loop transfer matrix $W(s)$ (1).

In this respect, the situation with multiplicative perturbations shown in Fig. 3-Fig. 5 is drastically different. It can be shown that the matrix $Q(j\omega)$ in BPM in Fig. 2 for <u>all</u> perturbed uniform systems in Fig. 3 - Fig. 5 is the same and coincides with the transfer matrix of the closed-loop system $T(j\omega)$ (2) taken with the minus sign:

$$Q(j\omega) = -T(j\omega) = -C diag\left\{\frac{\lambda_i w(j\omega)}{1+\lambda_i w(j\omega)}\right\}C^{-1}. \tag{11}$$

Respectively, the robustness stability conditions (8) and (9) can be rewritten in general form, i.e. for all models in Fig. 3 - Fig. 5, as

$$\|T(j\omega)\| = \left\| C diag\left\{ \frac{\lambda_i w(j\omega)}{1+\lambda_i w(j\omega)} \right\} C^{-1} \right\| < \frac{1}{\|\Delta(j\omega)\|} \quad \forall\, \omega \in [-\infty, \infty] \tag{12}$$

and

$$\|T(j\omega)\|_\infty = \left\| C diag\left\{ \frac{\lambda_i w(j\omega)}{1+\lambda_i w(j\omega)} \right\} C^{-1} \right\|_\infty < \frac{1}{\|\Delta(j\omega)\|_\infty} \ , \tag{13}$$

where the specific form of $\Delta(j\omega)$ on the right-hand side depends on the analyzed perturbation model.

Using conventional rules of matrix multiplication and norms, we get the following estimate for the upper bound of the norm $\|T(j\omega)\|$:

$$\|T(j\omega)\| = \left\| C diag\left\{ \frac{\lambda_i w(j\omega)}{1+\lambda_i w(j\omega)} \right\} C^{-1} \right\| \le \nu(C) \max_i \left| \frac{\lambda_i w(j\omega)}{1+\lambda_i w(j\omega)} \right|, \tag{14}$$

where

$$\nu(C) = \|C\| \cdot \|C^{-1}\| \ge 1 \tag{15}$$

is the condition number of the modal matrix C in (3) and (5).

Based on (14), one can state that if the following condition:

$$\max_i \left| \frac{\lambda_i w(j\omega)}{1+\lambda_i w(j\omega)} \right| < \frac{1}{\nu(C)\|\Delta(j\omega)\|} \tag{16}$$

holds true for all frequencies ω, then the sufficient condition (12) of stability robustness of the uniform system with any type of multiplicative uncertainties also holds true.

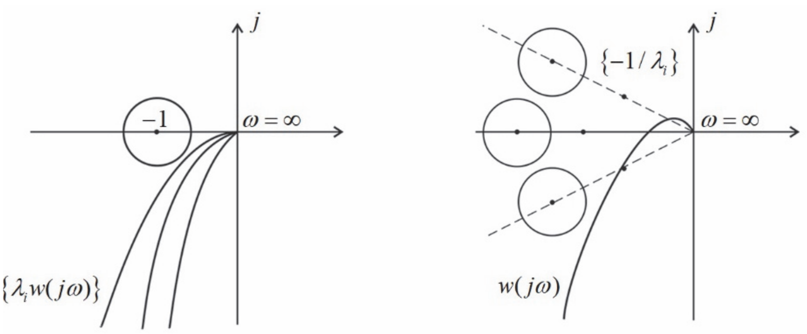
Expression (16) allows imparting two simple geometrical interpretations to the robust condition (12) assuming that the uncertainty $\Delta(j\omega)$ does not depend on the frequency ω or the norm $\|\Delta(j\omega)\|$ is replaced by the supreme value $\|\Delta(j\omega)\|_\infty$. In what follows, we shall just write in both cases $\|\Delta\|$. If we replace in (16) the sign < by the equality sign, then after some algebraic manipulations, that condition can be rewritten in the following form:

$$\left[\mathrm{Re}\{\lambda_i w(j\omega)\} - c\,\right]^2 + \left[\mathrm{Im}\{\lambda_i w(j\omega)\}\right]^2 = r^2, \tag{17}$$

where

$$c = \frac{1}{\left[\nu(C)\|\Delta\|\right]^2 - 1}, \qquad r = \frac{\nu(C)\|\Delta\|}{\left|1 - \left[\nu(C)\|\Delta\|\right]^2\right|}. \tag{18}$$

Geometrically, this expression determines the complex plane of N characteristic gain loci $\lambda_i w(j\omega)$ ($i = 1, 2, ..., N$) a circle with the center at the real point c with the radius r (Fig. 6a). The sufficient condition (12) is satisfied if the circle (17) does not intersect the graphs of $\lambda_i w(j\omega)$. Note that for $\|\Delta\| = 0$, the circle (17) reduces to the critical point $-1, j0$.



a) b)

Fig. 6. Analysis of stability robustness

Condition (16) can be rewritten in an equivalent form:

$$\max_i \left| \frac{w(j\omega)}{1/\lambda_i + w(j\omega)} \right| < \frac{1}{\nu(C)\|\Delta(j\omega)\|}. \tag{19}$$

Then, proceeding as before, we come to the following equation:

$$\left[\mathrm{Re}\{w(j\omega)\} - \mathrm{Re}\{c_i\}\right]^2 + \left[\mathrm{Im}\{w(j\omega)\} - \mathrm{Im}\{c_i\}\right]^2 = r_i^2, \tag{20}$$

where

$$c_i = \frac{1}{\lambda_i} \frac{1}{\left[\left(\nu(C)\|\Delta\|\right)^2 - 1\right]}, \quad r_i = \frac{1}{|\lambda_i|} \frac{\nu(C)\|\Delta\|}{\left|\left(\nu(C)\|\Delta\|\right)^2 - 1\right|}. \tag{21}$$

Geometrically, it determines on the complex plane of the hodograph $w(j\omega)$ of identical separate channels N circles with centers at the points c_i and the radi r_i, where the centers c_i lie on the half-lines starting on the origin of the coordinate axes and passing through the critical points $-1/\lambda_i$. This is illustrated for $N=3$ in Fig. 5. Again, the condition (19) is satisfied if none of the circles (20) intersects the graph of $w(j\omega)$.

It is important to note that, as can be seen from (17)-(21), the radii of the "forbidden" circles are proportional to the condition number $\nu(C)$ (15) of the modal matrix C. This means that the uniform systems with normal matrices R, that is the systems with orthogonal canonical bases, for which $\nu(C)=1$, are more robust as compared with uniform systems with all other types of the matrix R.

It should also be noted that the presented graphical tests of robustness belong to the so-called "very sufficient" criteria since an additional inequality is used in (14). On the other hand, the tests are very easy to use and, what is also important, they are based on the CTFs of the *open-loop* uniform systems.

Numerical example. Consider three-dimensional ($N=3$) uniform control system with the following transfer function $w(s)$ of separate channels and matrix R of cross-connections:

$$w(s)=\frac{600000000(s+3)}{s(s+0.33)(s+400)^2(s+500)} \tag{22}$$

$$R=\begin{bmatrix} 0.9 & 0.03 & -0.01 \\ -0.05 & 0.866 & 0.5 \\ 0.02 & -0.5 & 0.866 \end{bmatrix}. \tag{23}$$

The eigenvalues of the matrix R (23) are equal to:

$$\lambda_1=0.9,\ \lambda_2=0.866+j0.507,\ \ \lambda_3=0.866-j0.507, \tag{24}$$

and the condition number is $\nu(C)=1.112$.

The analysis of stability robustness of the system with respect to multiplicative perturbations is presented in Fig. 7 and Fig. 8. It shows that the "forbidden" circles touch the characteristic gain loci of $\lambda_i w(j\omega)$ ($i=1,2,3$) in Fig. 7 and $w(j\omega)$ in Fig. 8 for $\|\Delta\|=0.5134$. That value of the perturbation norm applies to all models of perturbed systems in Fig. 3-Fig. 5.

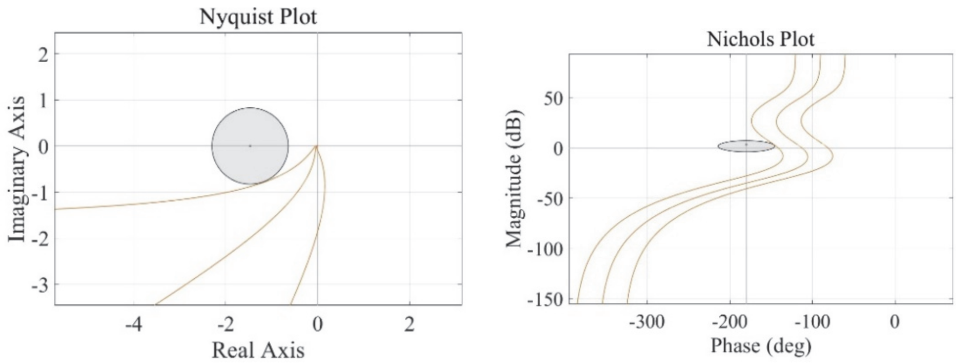


Fig. 7. Stability robustness analysis based on the condition (16)

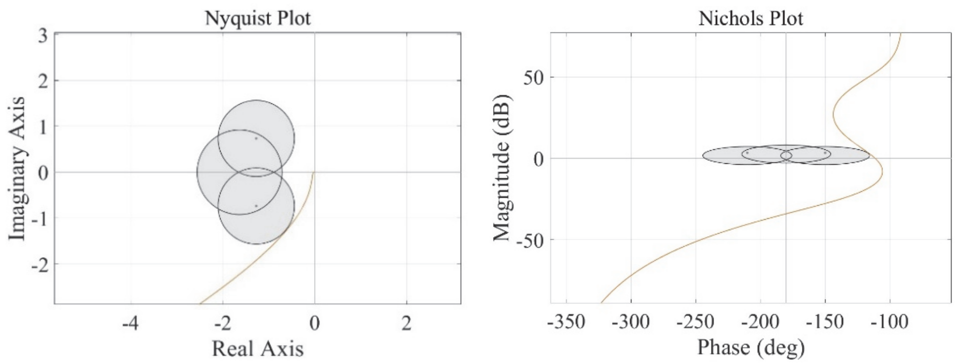


Fig. 8. Stability robustness analysis based on the condition (19)

In other words, the stability robustness of any of the perturbed models in Fig. 3-Fig.5 is guaranteed if the Hardy norm of uncertainties Δ_R . $\Delta_W(s)$, or $\Delta_{WR}(s)$ does not exceed 0.5134.

Conclusion. Simple graphical tests of the stability robustness of uniform systems to multiplicative perturbations or uncertainties are proposed in the paper. The analysis of the stability robustness of uniform systems is based on the method of characteristic transfer functions. It is very similar to the stability analysis of SISO control systems by the conventional Nyquist criterion, in which the critical point $-1, j0$ is replaced by some "forbidden" circles or areas on the complex plane of characteristic gain loci of the open-loop uniform system.

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REFERENCES

1. **Zhou K. and Doyle J.C.** Essentials of Robust Control. - Prentice Hall, Englewood Cliffs, 1996.- 425 p.
2. **Green M. and Limebeer D.J.N.** Linear Robust Control. - Prentice Hall, Englewood Cliffs, 1995.-558 p.
3. **Skogestad S. and Postlethwaite L**. Multivariable Feedback Control: Analysis and Design. - John Wiley & Sons, Ltd, Chichester, Sussex, UK, 2005.-585 p.
4. **Gasparyan O.N., Ispiryan V.H., Buniatyan L.M., Simonyan T.A.** Robustness Analysis of Multivariable Uniform Control Systems // International Conference on Computer Science and Information Technologies (CSIT-2023). -2023.- 1.- P. 182-185.
5. **Gasparyan O.N.** Linear and Nonlinear Multivariable Feedback Control: A Classical Approach. - John Wiley & Sons Ltd, UK, 2008.-347 p.

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ՄՈՒԼՏԻՊԼԻԿԱՏԻՎ ԱՆՈՐՈՇՈՒԹՅՈՒՆՆԵՐՈՎ ՄԻԱՏԻՊ ԿԱՌԱՎԱՐՄԱՆ ՀԱՄԱԿԱՐԳԵՐԻ ՌՈԲԱՍՏՈՒԹՅՈՒՆԸ

Մշակվել են միատիպ կառավարման համակարգերի մուլտիպլիկատիվ անորոշությունների նկատմամբ ռոբաստ կայունության վերլուծության պարզ գրաֆիկական չափանիշներ: Միատիպ կոչվում են հետադարձ կապով, մի քանի մուտքերով և ելքերով կառավարման համակարգերը, որոնց առանձին կապուղիների փոխանցման ֆունկցիաները միևնույնն են, իսկ կոշտ փոխադարձ կապերը նկարագրվում են քառակուսային թվային մատրիցով: Դիտարկումը հիմնվում է բնութագրիչ փոխանցման ֆունկցիաների մեթոդի վրա, որը հնարավորություն է տալիս N մուտքերով և N ելքերով փոխկապակցված բազմաչափ կառավարման համակարգերի կայունության վերլուծությունը հանգեցնել N հատ ֆիկտիվ, մեկ մուտքով և մեկ ելքով իրարից անկախ համակարգերի վերլուծությանը: Ռոբաստության վերլուծության առաջարկվող չափանիշները ներկայացնում են N «արգելված» շրջանակներ բաց միատիպ համակարգերի բնութագրիչ հոդոգրաֆների կոմպլեքս հարթության վրա: Բերված է թվային օրինակ, որտեղ ցույց է տրված վերը նշված չափանիշների կիրառությունը եռաչափ միատիպ համակարգի ռոբաստ կայունության վերլուծության դեպքում:

Առանցքային բառեր. բազմաչափ կառավարման համակարգեր, միատիպ համակարգ, մուլտիպլիկատիվ անորոշություն, ռոբաստ կայունություն, բնութագրիչ փոխանցման ֆունկցիաներ:

О.Н. ГАСПАРЯН, В.Г. ИСПИРЯН, Л.М. БУНИАТЯН, Г.А. МЕЛКОНЯН

РОБАСТНОСТЬ ОДНОТИПНЫХ СИСТЕМ УПРАВЛЕНИЯ С МУЛЬТИПЛИКАТИВНЫМИ НЕОПРЕДЕЛЕННОСТЯМИ

Целью статьи является разработка простых графических критериев анализа робастной устойчивости однотипных систем управления по отношению к мультипликативным неопределенностям. Однотипными называются системы управления с обратной связью с несколькими входами и выходами, имеющими одинаковыми передаточные функции отдельных каналов и жесткими взаимными связями, описываемыми квадратной числовой матрицей. Рассмотрение основано на методе характеристических передаточных функций, который позволяет свести анализ устойчивости взаимосвязанной многомерной системы управления с N входами и N выходами к анализу N фиктивных независимых систем с одним входом и одним выходом. Предлагаемые критерии анализа робастности имеют форму N "запретных" кругов на комплексной плоскости характеристических годографов разомкнутой однотипной системы. Приведен числовой пример, иллюстрирующий применение указанных критериев к анализу робастной устойчивости трехмерной однотипной системы.

Ключевые слова: многомерная система управления, однотипная система, мультипликативная неопределенность, робастная устойчивость, характеристические передаточные функции.