ISSN 00002-3043

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ИЗВЕСТИЯ
НАН АРМЕНИИ

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МАТЕМАТИКА

ТОМ 59 № 2 2024

Известия НАН Армении, Математика, том 59, н. 2, 2024, стр. 3 – 15.

ON AN EFFICIENT SOLUTION OF THE DIRICHLET PROBLEM FOR PROPERLY ELLIPTIC EQUATION IN THE ELLIPTIC DOMAIN

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Abstract. The fourth-order properly elliptic equation with multiple root is considered in the elliptic domain. The conditions, necessary and sufficient for the unique solvability of the Dirichlet problem for this equation are found, and if these conditions fail the defect numbers of this problem are determined. The solution of the problem is found in explicit form.

MSC2020 numbers: 35J40; 35E99; 35C11.

Keywords: properly elliptic equation; Dirichlet problem; non-trivial solutions of homogeneous Dirichlet problem; defect numbers.

1. Introduction. Formulation of the problem

Let Γ be an ellipse in the complex plane and $D = int\Gamma$. We consider the elliptic differential equation

$$\sum_{k=0}^{4} A_k \frac{\partial^4 U}{\partial x^k \partial y^{4-k}} = 0, \quad (x,y) \in D \tag{1.1}$$

where A_k are complex constants, such that the roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of the characteristic equation

$$A_0\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0 \tag{1.2}$$

satisfy conditions

$$\lambda_1 = \lambda_2 = \lambda, \ \Im m\lambda > 0; \ \lambda_3 \neq \lambda_4, \ \Im m\lambda_3 < 0, \ \Im m\lambda_4 < 0. \tag{1.3}$$

We a looking for a solution $U \in C^4(D) \bigcap C^{(1,\sigma)}(D \bigcup \Gamma)$ (the class $C^{(m,\sigma)}(X)$ is a class of functions, which with all derivatives of order up to m satisfy Hölder condition in the set X, $0 < \sigma \leq 1$), which on Γ satisfies to Dirichlet boundary conditions.

$$\left.\frac{\partial^k U}{\partial N^k}\right|_{\Gamma} = f_k(x,y), \ (x,y) \in \Gamma, \ k = 0, 1 \tag{1.4}$$

where $f_j \in C^{(1-j,\sigma)}(\Gamma)$ are the given functions.Here $\frac{\partial}{\partial N}$ and $\frac{\partial}{\partial s}$ are derivatives with respect to inner normal N and arc length respectively.

The conditions (1.3) imply that equation (1.1) is properly elliptic, therefore problem (1.1), (1.4) is Fredholmian, (see [1], [2]). In [3] it was proven that in the unit disk when $\lambda = i$, the problem (1.1), (1.4) is uniquely solvable. The same problem in the unit disc for the higher order properly elliptic equation was considered in [4]. In this paper, we will show, that for arbitrary ellipse problem (1.1), (1.4) is not uniquely solvable, find the condition to coefficients of the equation (1.1), and the parameters of the ellipse for which this problem is uniquely solvable, and determine the defect numbers of the problem (1.1), (1.4) in the general case.

For the exact formulation of the results, we rewrite equation (1.1) in the complex form, using equalities

$$x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}, \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Equation (1.1) transforms to

$$\left(\frac{\partial}{\partial \bar{z}} - \mu\frac{\partial}{\partial z}\right)^2 \left(\frac{\partial}{\partial z} - \nu_1\frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z} - \nu_2\frac{\partial}{\partial \bar{z}}\right) U = 0, \tag{1.5}$$

where

$$\mu = \frac{i-\lambda}{i+\lambda}, \quad \nu_j = \frac{i+\lambda_{2+j}}{i-\lambda_{2+j}}, \quad j = 1, 2.$$

Observe, that (1.3) implies

$$|\mu| < 1, \quad |\nu_j| < 1, \quad \nu_1 \neq \nu_2, \quad j = 1, 2. \tag{1.6}$$

Further, taking into account, that (1.1) and (1.4) are invariant under shift $U(x, y) \to U(x + a, y + b)$ and dilation $(U(x, y) \to U(rx, ry))$, without loss of generality, we suppose, that Γ is the ellipse with the center in origin and with semiaxises $1 + \rho$ and $1 - \rho$ where $0 < \rho < 1$ (if a and b, $a > b$ are semiaxises of start ellipse D, then using dilation $(x, y) \to (rx, ry)$, $r = \frac{2}{a+b}$, we get $\rho = \frac{a-b}{a+b}$). The equation of this ellipse in the complex form is

$$(z - \mu_0\bar{z})(\bar{z} - \bar{\mu}_0 z) = \left(1 - |\mu_0|^2\right)^2, \tag{1.7}$$

$$|\mu_0| = \left|\rho e^{i\delta}\right| < 1, \tag{1.8}$$

where $\delta/2$ is the angle, formed by the axis OX and the greater semiaxis of the ellipse. Further, we suppose, that we have an equation (1.5) with boundary conditions (1.4) and this problem is considered in the ellipse Γ given by equality (1.7). Using the preceding notation, we can formulate, the following statement.

Theorem 1.1. *Problem (1.5), (1.4) is uniquely solvable if and only if one of the following conditions holds:*

(1) $\mu = -\mu_0$,

(2) $\mu \neq -\mu_0$ *and*

$$\Delta_k = k\frac{\alpha^k - \beta^k}{\alpha - \beta} - \frac{\left(1-\alpha^k\right)\left(1-\beta^k\right)}{(1-\alpha)(1-\beta)} \neq 0, \quad k = 3, 4, \ldots. \tag{1.9}$$

where

$$\alpha = \frac{\mu + \mu_0}{1 + \mu\bar{\mu}_0} \cdot \frac{\nu_1 + \bar{\mu}_0}{1 + \mu_0\bar{\nu}_1}, \quad \beta = \frac{\mu + \mu_0}{1 + \mu_0\bar{\mu}} \cdot \frac{\nu_2 + \bar{\mu}_0}{1 + \mu_0\bar{\nu}_2}. \tag{1.10}$$

Remark 1.1. The conditions (1.6) and (1.8) imply $|\alpha| < 1, |\beta| < 1$ and $\alpha \neq \beta$. Therefore, for $k \to \infty$, we have $\Delta_k \to -(1-\alpha)^{-1}(1-\beta)^{-1}$.Hence, for every α and β the conditions (1.9) are fulfilled for sufficiently large k.

The conditions (1.9) show that the unique solvability of the problem (1.5), (1.4) depend both on the coefficients of the equation (1.5) and the shape of the Γ. Therefore, may be considered the following questions:

(1) What conditions on μ, ν_1 and ν_2 provide the unique solvability of the problem (1.5), (1.4) in an arbitrary ellipse Γ?
(2) For the given Γ (presented μ_0) describe the class of equations (1.5) (sufficient conditions on μ, ν_1, ν_2) for which the problem (1.5), (1.4) is uniquely solvable?
(3) Calculate the defect numbers, of the problem (1.5), (1.4), that is the number of linearly independent solutions of the homogeneous(when $f_k = 0$) problem (1.5), (1.4) and the number of the linearly independent conditions for the boundary functions f_j necessary and sufficient for the solvability of inhomogeneous problem (1.5), (1.4)?

Partly, the answers to these questions will be given in the final part of this paper.

2. Proof of the Theorem 1.1

We reduce the problem (1.5), (1.4) to the analogous problem in the unit disk. The transformation

$$\xi + i\eta = \zeta = \frac{z - \mu_0\bar{z}}{1 - |\mu_0|^2}, \quad z = x + iy \tag{2.1}$$

maps the point $(x, y) \in D$ to the corresponding point $(\xi, \eta) \in B = \{\zeta : |\zeta| < 1\}$. The inverse transformation is determined by the formula

$$z = \zeta + \mu_0\bar{\zeta}.$$

Using this change of variables, we represent the unknown function $U(z, \bar{z})$ in the form

$$U(z, \bar{z}) = U(\zeta + \mu_0\bar{\zeta}, \bar{\zeta} + \bar{\mu}_0\zeta) \equiv V(\zeta, \bar{\zeta}).$$

This denotion implies

$$V_\zeta = U_z + \bar{\mu}_0 U_{\bar{z}}, \quad V_{\bar{\zeta}} = \mu_0 U_z + U_{\bar{z}}$$

and, therefore

$$U_z = \frac{1}{1-|\mu_0|^2}\left(V_\zeta - \bar{\mu}_0 V_{\bar{\zeta}}\right), \quad U_{\bar{z}} = \frac{1}{1-|\mu_0|^2}\left(V_{\bar{\zeta}} - \mu_0 V_\zeta\right).$$

Substituting these equalities in (1.5), we get the final form of this equation:

$$\left(\frac{\partial}{\partial\bar{\zeta}} - \gamma\frac{\partial}{\partial\zeta}\right)^2 \left(\frac{\partial}{\partial\zeta} - \omega_1\frac{\partial}{\partial\bar{\zeta}}\right)\left(\frac{\partial}{\partial\zeta} - \omega_2\frac{\partial}{\partial\bar{\zeta}}\right) V = 0. \tag{2.2}$$

Here

$$\gamma = \frac{\mu+\mu_0}{1+\mu\bar{\mu}_0}, \quad \omega_j = \frac{\nu_j+\bar{\mu}_0}{1+\nu_j\mu_0}, \quad j=1,2. \tag{2.3}$$

Observe that (1.6) and (1.8) imply $|\gamma|<1$, $|\omega_j|<1$, $j=1,2$, $\omega_1\neq\omega_2$.

Let's reduce the boundary conditions (1.4) to the equivalent boundary conditions on the $\Gamma_1 = \partial B$. First, we must mention, that the domain D (interior of the ellipse(1.7)) in the polar coordinates is defined by the formulas

$$D = \left\{z = re^{i\theta}; r^2\left(e^{i\theta} - \mu_0 e^{-i\theta}\right)\left(e^{-i\theta} - \bar{\mu}_0 e^{i\theta}\right) \le \left(1-|\mu_0|^2\right)^2\right\},$$

where $0\le r\le 1$, $\theta\in[-\pi,\pi]$, or, as a set of (x,y)

$$\begin{cases} x = r(1+\rho)\cos\left(\theta - \frac{\delta}{2}\right) \\ y = r(1-\rho)\sin\left(\theta - \frac{\delta}{2}\right) \end{cases}.$$

Therefore, the conditions (1.4) are equivalent to conditions:

$$\left.\frac{\partial U}{\partial\theta}\right|_\Gamma = \frac{df_0}{d\theta}, \quad \left.\frac{\partial U}{\partial r}\right|_\Gamma = \widetilde{f}_1(\theta), \quad U\left(1,\frac{\delta}{2}\right) = f_0\left(1,\frac{\delta}{2}\right). \tag{2.4}$$

We don't change the denotation of the unknown function U and in different places may use $(x,y),(r,\theta),(z,\bar{z})$ independent variables (for the same domain D). In (2.4) differentiation by r is not coincide with $\frac{\partial}{\partial N}$, but the angle between $\bar{r}$ and normal to Γ is acute, hence the function $\widetilde{f}_1\in C^{(\sigma)}(\Gamma)$ is the real-valued function, uniquely determined by f_0 and f_1. The conditions (2.4) using operators of complex differentiation, may be written in the form

$$\left.\frac{\partial U}{\partial z}\right|_\Gamma = \frac{\bar{z}}{2}\left(\widetilde{f}_1(\theta) - i\frac{df_0}{d\theta}\right) \equiv G_1(\theta),$$

$$\left.\frac{\partial U}{\partial\bar{z}}\right|_\Gamma = \frac{z}{2}\left(\widetilde{f}_1(\theta) + i\frac{df_0}{d\theta}\right) \equiv \overline{G_1(\theta)}, z = e^{i\theta}. \tag{2.5}$$

Using the representation of complex derivatives in polar coordinates

$$zU_z = \frac{1}{2}\left(rV_r - iV_\theta\right), \quad \bar{z}U_{\bar{z}} = \frac{1}{2}\left(rV_r + iV_\theta\right)$$

and representation of derivatives in ζ variables

$$V_\zeta = U_z + \bar{\mu}_0 U_{\bar{z}}, \quad V_{\bar{\zeta}} = U_{\bar{z}} + \mu_0 U_z, \tag{2.6}$$

we get the boundary conditions, equivalent to (1.4), which we will use:

$$V_\zeta\big|_\Gamma = F_1(\theta), \quad V_{\bar\zeta}\big|_\Gamma = F_2(\theta), \quad V(1,0) = f_0\left(1, \frac{\delta}{2}\right). \tag{2.7}$$

Here

$$F_1(\theta) = \frac{e^{-i\theta} + \bar\mu_0 e^{i\theta}}{2}\widetilde{f}_1(\theta) + \frac{i\left(\bar\mu_0 e^{i\theta} - e^{-i\theta}\right)}{2}\frac{df_0}{d\theta}, \tag{2.8}$$

$$F_2(\theta) = \frac{e^{i\theta} + \mu_0 e^{-i\theta}}{2}\widetilde{f}_1(\theta) + \frac{i\left(e^{i\theta} - \mu_0 e^{-i\theta}\right)}{2}\frac{df_0}{d\theta}. \tag{2.9}$$

We must mention, that the functions F_j are satysfy the Hölder condition, $F_j \in C^{(\sigma)}(\Gamma)$, on the Γ. Thus, we have to solve the boundary value problem (2.2), (2.7).

The general solution of the equation (2.2) can be represented in the form ([4]):

$$V(\xi,\eta) = \Phi_0\left(\zeta + \gamma\bar\zeta\right) + \frac{\partial}{\partial\theta}\Phi_1\left(\zeta + \gamma\bar\zeta\right) + \Psi_1\left(\bar\zeta + \omega_1\zeta\right) + \Psi_2\left(\bar\zeta + \omega_2\zeta\right) \tag{2.10}$$

where functions Φ_0, Φ_1 are analytic in the domain $D(\gamma) = \left\{\zeta + \gamma\bar\zeta : |\zeta| < 1\right\}$ and Ψ_j $(j = 1,2)$ are analytic in $D_1(\omega_j) = \left\{\bar\zeta + \omega_j\zeta : |\zeta| < 1\right\}$. For the determination of unknown functions, we substitute the function (2.10) in the boundary conditions (2.7) and use the operator identities:

$$\frac{\partial}{\partial\zeta}\frac{\partial}{\partial\theta} = \left(\frac{\partial}{\partial\theta} + iI\right)\frac{\partial}{\partial\zeta}, \quad \frac{\partial}{\partial\bar\zeta}\frac{\partial}{\partial\theta} = \left(\frac{\partial}{\partial\theta} - iI\right)\frac{\partial}{\partial\zeta}.$$

We get

$$\begin{gathered}\Phi_0'\left(\zeta + \gamma\bar\zeta\right) + \left(\frac{\partial}{\partial\theta} + iI\right)\Phi_1'\left(\zeta + \gamma\bar\zeta\right) + \omega_1\Psi_1'\left(\bar\zeta + \omega_1\zeta\right) + \\ +\omega_2\Psi_2'\left(\bar\zeta + \omega_2\zeta\right) = F_1(\theta),\end{gathered} \tag{2.11}$$

$$\begin{gathered}\gamma\Phi_0'\left(\zeta + \gamma\bar\zeta\right) + \gamma\left(\frac{\partial}{\partial\theta} - iI\right)\Phi_1'\left(\zeta + \gamma\bar\zeta\right) + \Psi_1'\left(\bar\zeta + \omega_1\zeta\right) + \\ +\Psi_2'\left(\bar\zeta + \omega_2\zeta\right) = F_2(\theta).\end{gathered} \tag{2.12}$$

These equalities hold for $|\zeta| = 1$, and, as it was shown in [1], the functions Φ_j' for $j = 0,1$ and Ψ_j' for $j = 1,2$ may be represented in the form:

$$\Phi_j'\left(\zeta + \gamma\bar\zeta\right) = \phi_j(\zeta) + \phi_j(\gamma\bar\zeta) = \sum_{k=0}^{\infty} A_{kj}\zeta^k + \sum_{k=0}^{\infty} A_{kj}\gamma^k\zeta^{-k}, \quad |\zeta| = 1, \tag{2.13}$$

$$\Psi_j'\left(\bar\zeta + \omega_j\zeta\right) = \psi_j(\bar\zeta) + \psi_j(\omega_j\zeta) = \sum_{k=0}^{\infty} B_{kj}\zeta^{-k} + \sum_{k=0}^{\infty} B_{kj}\omega_j^k\zeta^k, \quad |\zeta| = 1. \tag{2.14}$$

The functions ϕ_j, ψ_j are analytic in the unit disk, therefore, may be represented by Taylor series. We substitute the representations (2.13), (2.14) and representation of the boundary functions F_j by the Fourier series

$$F_j(\theta) = \sum_{k=-\infty}^{\infty} d_{kj}\zeta^k, \quad |\zeta| = |e^{i\theta}| = 1, \quad j = 1,2 \tag{2.15}$$

in the boundary equations (2.11), (2.12). We get

$$\sum_{k=0}^{\infty} A_{k0}\zeta^k + \sum_{k=0}^{\infty} A_{k0}\gamma^k\zeta^{-k} + \sum_{k=0}^{\infty} A_{k1}\left(ik+i\right)\zeta^k + \sum_{k=0}^{\infty} A_{k1}\left(-ik+i\right)\gamma^k\zeta^{-k} +$$

$$+\sum_{k=0}^{\infty} B_{k1}\omega_1\zeta^{-k} + \sum_{k=0}^{\infty} B_{k1}\omega_1^{k+1}\zeta^k + \sum_{k=0}^{\infty} B_{k2}\omega_2\zeta^{-k} +$$

$$+\sum_{k=0}^{\infty} B_{k2}\omega_2^{k+1}\zeta^k = \sum_{k=-\infty}^{\infty} d_{k1}\zeta^k, \quad |\zeta| = 1, \tag{2.16}$$

$$\sum_{k=0}^{\infty} A_{k0}\gamma\zeta^k + \sum_{k=0}^{\infty} A_{k0}\gamma^{k+1}\zeta^{-k} + \sum_{k=0}^{\infty} A_{k1}\gamma\left(ik-i\right)\zeta^k +$$

$$+\sum_{k=0}^{\infty} A_{k1}\left(-ik-i\right)\gamma^{k+1}\zeta^{-k} + \sum_{k=0}^{\infty} B_{k1}\zeta^{-k} + \sum_{k=0}^{\infty} B_{k1}\omega_1^k\zeta^k +$$

$$+\sum_{k=0}^{\infty} B_{k2}\zeta^{-k} + \sum_{k=0}^{\infty} B_{k2}\omega_2^k\zeta^k = \sum_{k=-\infty}^{\infty} d_{k2}\zeta^k, \quad |\zeta| = 1. \tag{2.17}$$

The Fourier series expansion is unique, and, therefore, equating the coefficients by ζ^k and $\bar{\zeta}^k$, we get the system of linear equations for determination of the coefficients A_{kj}, B_{kj}.

For $k = 0$ we have:

$$\begin{cases} 2A_{00} + 2iA_{01} + 2\omega_1 B_{01} + 2B_{02}\omega_2 = d_{01}, \\ 2\gamma A_{00} + 2{-}iA_{01} + 2B_{01} + 2B_{02} = d_{02}. \end{cases} \tag{2.18}$$

For $k \geqslant 1$ we get four equations with four unknown

$$\begin{cases} A_{k0} + i(k+1)A_{k1} + \omega_1^{k+1}B_{k1} + \omega_2^{k+1}B_{k2} = d_{k1}, \\ \gamma A_{k0} + i(k-1)\gamma A_{k1} + \omega_1^k B_{k1} + \omega_2^k B_{k2} = d_{k2}, \\ \gamma^k A_{k0} + i(-k+1)\gamma^k A_{k1} + \omega_1 B_{k1} + \omega_2 B_{k2} = d_{-k1}, \\ \gamma^{k+1} A_{k0} + i(-k-1)\gamma^{k+1} A_{k1} + B_{k1} + B_{k2} = d_{-k2}. \end{cases} \tag{2.19}$$

Thus, if $\gamma \neq 0$, we reduce the problem (1.5), (1.4) to the solution of the linear systems (2.18), (2.19).

Let's consider the determinant of the main matrix Ω_k of the system (2.19):

$$\det\Omega_k = \begin{vmatrix} 1 & i(k+1) & \omega_1^{k+1} & \omega_2^{k+1} \\ \gamma & i(k-1)\gamma & \omega_1^k & \omega_2^k \\ \gamma^k & i(-k+1)\gamma^k & \omega_1 & \omega_2 \\ \gamma^{k+1} & i(-k-1)\gamma^{k+1} & 1 & 1 \end{vmatrix}. \tag{2.20}$$

Denoting

$$\alpha = \gamma\omega_1, \quad \beta = \gamma\omega_2 \tag{2.21}$$

we get

$$\det \Omega_k = i \begin{vmatrix} 1 & k+1 & \alpha^{k+1} & \beta^{k+1} \\ 1 & k-1 & \alpha^k & \beta^k \\ 1 & -k+1 & \alpha & \beta \\ 1 & -k-1 & 1 & 1 \end{vmatrix} = 2i(1-\alpha)(\beta-1) \begin{vmatrix} 1 & \alpha^k & \beta^k \\ k & \frac{\alpha^k-1}{\alpha-1} & \frac{\beta^k-1}{\beta-1} \\ 1 & 1 & 1 \end{vmatrix} =$$

(2.22)

$$= -2i(\alpha-1)(\beta-1)\left(\frac{\left(\alpha^k-1\right)\left(\beta^k-1\right)(\alpha-\beta)}{(\alpha-1)(\beta-1)} - k\left(\alpha^k-\beta^k\right)\right).$$

Using the denotion (1.9), we have

$$\det \Omega_k = 2i(\alpha-1)(\beta-1)(\alpha-\beta)\Delta_k. \tag{2.23}$$

The condition (1.10) imply that $|\alpha| < 1, |\beta| < 1$ and $\alpha \neq \beta$, therefore $\det \Omega_k = 0$ if and only if $\Delta_k = 0$.

Let's suppose, that condition 2 of Theorem 1 holds. Taking into account, that the problem (1.5), (1.4) is Fredholmian, we may consider only a homogeneous problem. Conditions (1.9) imply, that for $k > 2$ the homogeneous system (2.19) (if $d_{kj} \equiv 0$) has only zero solution, $\Delta_2 = (\alpha-1)(\beta-1) \neq 0$ also, therefore $A_{kj} = B_{kj} = 0$ for all $k \geqslant 2$. It means, that the functions ϕ_j, ψ_j may be only of the first-degree polynomials.

Thus, the solution of the homogeneous problem (1.5), (1.4) is at most second degree polynomial of ζ and $\bar{\zeta}$. On the other side in [5] (T.5.1, page 84), it was shown, that arbitrary polynomial satisfying the homogeneous conditions (1.4) admits the representation $\left(1-\zeta\bar{\zeta}\right)^2 Q\left(\zeta,\bar{\zeta}\right)$ that is a polynomial of degree not less than four. Therefore, if the conditions (1.9) hold, the homogeneous problem (1.5), (1.4) has only trivial solution. As this problem is Fredholmian, the latter implies the unique solvability of the primary problem. Vice verse, assume that $\Delta_j = 0$ for $j_1, \ldots, j_m$. Then direct computation shows that the homogeneous problem (1.5), (1.4) has m linearly independent solutions $V_{j1}, \ldots, V_{jm}$ each of which is a polynomial of degree $j_p + 1, p = \overline{1,m}$. For example, if $\Delta_3 = 0$ then $1 + 2(\alpha+\beta) + \alpha\beta = 0$, and the function $P_4\left(\zeta,\bar{\zeta}\right) = \left(\left(\zeta-\mu_0\bar{\zeta}\right)\left(\bar{\zeta}-\bar{\mu}_0\zeta\right) - \left(1-|\mu|^2\right)^2\right)^2$ is a non-trivial solution of the homogeneous problem (1.5), (1.4). This completes the proof of the second part of Theorem 1 (for $\gamma \neq 0$).

Now, we pass to the proof of the first point of Theorem 1. This part was proved in [3], but we get the proof for the completeness of considerations.

Let's suppose, that $\mu = -\mu_0$ or $\gamma = 0$ in the equation (2.2). In this case, the general solution of the (2.2) may be represented in the form:

$$V(\xi,\eta) = \Phi_0(\zeta) + \bar{\zeta}\Phi_1(\zeta) + \Psi_1(\bar{\zeta} + \omega_1\zeta) + \Psi_2(\bar{\zeta} + \omega_2\zeta), \tag{2.24}$$

where Φ_0 and Φ_1 are analytic functions in the unit disk B_1. Substituting this function in the boundary conditions (2.7), we get

$$\Phi_0'(\zeta) + \bar{\zeta}\Phi_1'(\zeta) + \omega_1\Psi_1'(\bar{\zeta} + \omega_1\zeta) + \omega_2\Psi_2'(\bar{\zeta} + \omega_2\zeta) = F_1(\theta), \tag{2.25}$$

$$\Phi_1(\zeta) + \Psi_1'(\bar{\zeta} + \omega_1\zeta) + \Psi_2'(\bar{\zeta} + \omega_2\zeta) = F_2(\theta). \tag{2.26}$$

Analogously the case 2, we can consider only homogeneous problem ($F_1 \equiv F_2 \equiv 0$). We substitute the representation (24) in homogeneous equalities (2.25), (2.26). We get for $|\zeta| = 1$

$$\Phi_0'(\zeta) + \bar{\zeta}\Phi_1'(\zeta) + \omega_1\psi_1(\bar{\zeta}) + \omega_1\psi_1(\omega_1\zeta) + \omega_2\psi_2(\bar{\zeta}) + \omega_2\psi_2(\omega_2\zeta) = 0, \tag{2.27}$$

$$\Phi_1(\zeta) + \psi_1(\bar{\zeta}) + \psi_1(\omega_1\zeta) + \psi_2(\bar{\zeta}) + \psi_2(\omega_2\zeta) = 0. \tag{2.28}$$

Taking into account inequality $\omega_1 \neq \omega_2$, we can determine $\psi_1(\bar{\zeta})$ and $\psi_2(\bar{\zeta})$ from (2.27), (2.28), and as we know, that $\Phi_0, \Phi_1, \psi_1(\omega_1\zeta), \psi_2(\omega_2,\zeta)$ are analytic in B_1, we have:

$$\psi_1\left(\bar{\zeta}\right) = H_1 + S_1\bar{\zeta}, \psi_2\left(\bar{\zeta}\right) = H_2 + S_2\bar{\zeta}.$$

It means, that the solution V may be a polynomial of order not more than two. But from homogeneous conditions (2.7) the polynomial V, if it is not identically zero, must be represented in the form

$$V = \left(1 - \zeta\bar{\zeta}\right)^2 Q\left(\zeta, \bar{\zeta}\right)$$

(see [5]), that is has a degree not less than four. Hence, the homogeneous problem (1.5), (1.4) has only a trivial solution, therefore this problem is uniquely solvable. Theorem 1 is proved.

Remark 2.1. We have from the conditions (1.6) and (1.8) that $|\alpha| < 1, |\beta| < 1$ and $\alpha \neq \beta$. Therefore, we may transform the determinant (2.22) and the condition (1.9), to omit the roots $\alpha = 1$, $\beta = 1$ and, $\alpha = \beta$.

We have

$$\det\Omega_k = -2i(\alpha-1)(\beta-1)\begin{vmatrix} 1 & \alpha^k & \beta^k \\ k & \sum_{j=0}^{k-1}\alpha^j & \sum_{j=0}^{k-1}\beta^j \\ 1 & 1 & 1 \end{vmatrix} =$$

$$= -2i(\alpha-1)(\beta-1)\begin{vmatrix} 1 & \alpha^k - 1 & \beta^k - 1 \\ k & \sum_{j=0}^{k-1}\left(\alpha^j - 1\right) & \sum_{j=0}^{k-1}\left(\beta^j - 1\right) \\ 1 & 0 & 0 \end{vmatrix} =$$

$$(2.29) \qquad = 2i(\alpha-1)^2(\beta-1)^2 \begin{vmatrix} \sum\limits_{m=0}^{k-1} \alpha^m & \sum\limits_{m=0}^{k-1} \beta^m \\ \sum\limits_{m=0}^{k-1} m\alpha^m & \sum\limits_{m=0}^{k-1} m\beta^m \end{vmatrix} \equiv 2i(\alpha-1)^2(\beta-1)^2 T_k.$$

We see, that the conditions (1.9) hold if and only if $T_k \neq 0$, for $k = 3, 4, \ldots$. Introducing the notion

$$(2.30) \qquad P_{k-1}(z) = \sum_{m=0}^{k-1} z^m,$$

the conditions (1.9) may be represented in equivalent form:

$$(2.31) \qquad \frac{T_k}{\beta-\alpha} = \frac{P_{k-1}(\alpha)\beta P'_{k-1}(\beta) - P_{k-1}(\beta)\alpha P'_{k-1}(\alpha)}{\beta-\alpha} \neq 0, \quad k = 3, 4, \ldots.$$

It was taken into account, that $\alpha \neq \beta$.

This condition may be represented in the form:

$$(2.32) \qquad S_{k-2}(\alpha,\beta) = - \begin{vmatrix} \sum\limits_{m=1}^{k-1}\sum\limits_{l=0}^{m-1} \alpha^l \beta^{m-1-l} & \sum\limits_{m=0}^{k-1} \beta^m \\ \sum\limits_{m=1}^{k-1} m \sum\limits_{l=0}^{m-1} \alpha^l \beta^{m-1-l} & \sum\limits_{m=0}^{k-1} m\beta^m \end{vmatrix} \neq 0, \quad k = 3, 4, \ldots.$$

Finally, expanding the determinant, we get, that $\Delta_k \neq 0$ if and only if

$$(2.33) \qquad S_{k-2}(\alpha,\beta) = \sum_{l=0}^{k-2} c_l(\beta)\alpha^l \neq 0, \quad k = 3, 4, \ldots.$$

where

$$(2.34) \qquad c_l(\beta) = \sum_{s=0}^{k-2-l} \beta^s (s+1)(l+1) + \sum_{s=k-1-l}^{k-2} \beta^s (k-1-s)(k-1-l).$$

Introducing the denotation

$$(2.35) \qquad Q_{k-2}(\alpha,\beta) = \sum_{l=0}^{k-2} (l+1)\alpha^l \sum_{s=0}^{k-2-s} (s+1)\beta^s, \quad k = 3, 4, \ldots,$$

the condition (2.33) may be represented in the form;

$$(2.36) \qquad S_{k-2}(\alpha,\beta) = Q_{k-2}(\alpha,\beta) + \alpha^{k-2}\beta^{k-2} Q_{k-3}\left(\frac{1}{\alpha}, \frac{1}{\beta}\right) \neq 0, \quad k = 3, 4, \ldots.$$

3. Some corollaries and final remarks

In this section we try to get some corollaries of the proved theorem.

Let's calculate the defect numbers of the problem (1.5), (1.4), that is, K - the number of the linearly independent solutions of the homogeneous problem (when boundary functions identically zero) and K_1 - the number of the linearly independent conditions to the boundary functions, provided solvability of the inhomogeneous problem.

Corollary 3.1. *As we see in the previous consideration the defect numbers of the problem are equal*

$$K = K_1 = \sum_{k=3}^{\infty} (4 - rank\Omega_k), \tag{3.1}$$

where Ω_k is a main matrix of the system (2.19). Taking into account, that if $k \to \infty$ $det\Omega_k \to 2i(\beta - \alpha) \neq 0$; we see that in the sum only a finite number of summands differ from zero.

We can calculate $rank\Omega_k$ more exactly. Using representation of $\det \Omega_k$ (2.22) we can calculate third order minor of the matrix Ω_k:

$$M_3 = \begin{vmatrix} 1 & k-1 & \alpha^k \\ 1 & -k+1 & \alpha \\ 1 & -k-1 & 1 \end{vmatrix} = 2(\alpha - 1)\left(k - \sum_{j=0}^{k-1} \alpha^j\right) \neq 0,$$

for all $k \geqslant 3$ (because of $|\alpha| < 1$). Hence, we can formulate the following statement.

Corollary 3.2. *We see, that $rank\Omega_k \geqslant 3$, therefore the difference, in the formula (3.1) may be equal zero or one only. That is, the defect numbers of the problem (1.5), (1.4) are equal to the number of the Δ_k ($k \geqslant 3$), equal to zero. Δ_k determined in (1.9).*

Now, let's find some sufficient conditions for the unique solvability of the problem (1.5), (1.4).

Corollary 3.3. *We consider the problem (1.5), (1.4) in the ellipse (1.7). If we have*

$$|\alpha| < \frac{20}{77}, \quad |\beta| < \frac{20}{77},$$

α and β determined in (1.10), then that problem is uniquely solvable.

Proof. Let's suppose, that $|\alpha| < 20/77$ and $|\beta| < 20/77$, and estimate the $|\Delta_k|$ for $k \geqslant 3$. We have:

$$\left|\frac{(1-\alpha^k)(1-\beta^k)}{(1-\alpha)(1-\beta)}\right| \geqslant \frac{(1-(20/77)^3)^2}{(1+20/77)^2} > 0,60825. \tag{3.2}$$

From the other side

$$\left|k\frac{\alpha^k - \beta^k}{\alpha - \beta}\right| = \left|k\sum_{j=0}^{k-1} \alpha^j \beta^{k-1-j}\right| \leqslant k^2 (20/77)^{k-1} \equiv \chi(k). \tag{3.3}$$

Let's calculate the maximal value of the function χ. We have

$$\chi'(k) = k(20/77)^{k-1}(2 - k\ln 3.85),$$

therefore, $k_{max} = 2/\ln 3.85 \approx 1.484$. It means, that for $k \geqslant 3$ we have

$$\chi(k) \leqslant \chi(3) = 9(20/77)^2 \approx 0.6071.$$

From this inequality, (3.2) and (3.3), we get

$$|\Delta_k| \geqslant \left|\frac{(1-\alpha^k)(1-\beta^k)}{(1-\alpha)(1-\beta)}\right| - \left|k\frac{\alpha^k-\beta^k}{\alpha-\beta}\right| > 0$$

for $k \geqslant 3$. Corollary is proved. □

Now, we use the formulas (2.33), (2.34) for the more exact determination of the of the defect numbers of the problem (1.5), (1.4). First, let's consider the case, when the parameter β is zero. In this case, from (2.34) we get

$$c_l(0) = l + 1, \tag{3.4}$$

therefore, by (2.33), we have, that the conditions (1.9) are equivalent to conditions

$$P_{k-2}(\alpha) = \sum_{m=0}^{k-2} (m+1)\alpha^m \neq 0, \quad k = 3, 4, \ldots. \tag{3.5}$$

Hence, we have to find the roots of the polynomial (3.5). We will use the Eneström-Kakeya theorem ([7], p. 12):

Theorem 3.1. *(Eneström-Kakeya) If all coefficients of the polynomial*

$$G_n(x) = \sum_{k=0}^{n-1} a_{n-1-k}x^k$$

are positive, then all roots ξ of this polynomial are in the ring

$$\min_{1\leqslant i\leqslant n-1}\left(\frac{a_i}{a_{i-1}}\right) \leqslant |\xi| \leqslant \max_{1\leqslant i\leqslant n-1}\left(\frac{a_i}{a_{i-1}}\right). \tag{3.6}$$

Let's apply this theorem to the polynomial P_{k-2}. If α is the root of this polynomial, we have

$$\frac{1}{2} = \min_{1\leqslant i\leqslant k-2}\left(\frac{i}{i+1}\right) \leqslant |\alpha| \leqslant \max_{1\leqslant i\leqslant k-2}\left(\frac{i}{i+1}\right) = \frac{k-2}{k-1}. \tag{3.7}$$

This estimation shows, that for arbitrary k_0 we can find $|\alpha| < 1$, for which $P_{k_0-2}(\alpha) = 0$, and, therefore, the problem (1.5), (1.4) is not uniquely solvable. If $|\alpha| < 0.5$, then the conditions (3.5) hold, therefore, the problem (1.5), (1.4) is uniquely solvable.

Now, let's suppose, that for some $n > 2$ we have $P_{n-2}(\alpha) = 0$, and for $m > 0$ $P_{n+m-2}(\alpha) = 0$ also. That is

$$\sum_{j=0}^{n-2}(j+1)\alpha^j = 0, \qquad \sum_{j=0}^{n+m-2}(j+1)\alpha^j = 0.$$

Subtracting first equality from the second, we get

$$\sum_{j=n-1}^{n+m-2}(j+1)\alpha^j = 0,$$

or taking into account that $\alpha \neq 0$

$$\sum_{l=0}^{m-1}(n+l)\alpha^l = 0. \tag{3.8}$$

Applying Eneström-Kakeya theorem, we see, that

$$\frac{n}{n+1} = \min_{1\leqslant i\leqslant m-1}\left(\frac{n+i-1}{n+i}\right) \leqslant |\alpha| \leqslant \max_{1\leqslant i\leqslant m-1}\left(\frac{n+i-1}{n+i}\right) = \frac{n+m-2}{n+m-1}. \tag{3.9}$$

But from equality $P_{n-2}(\alpha) = 0$ we have

$$\frac{1}{2} = \min_{1\leqslant i\leqslant n-2}\left(\frac{i}{i+1}\right) \leqslant |\alpha| \leqslant \max_{1\leqslant i\leqslant n-2}\left(\frac{i}{i+1}\right) = \frac{n-2}{n-1}.$$

This inequality contradicts to (3.9), because $\frac{n-2}{n-1} < \frac{n}{n+1}$, therefore, our assumption, that two polynomials P_{n-2} and P_{n+m-2} have the same root, was wrong.

Thus, we prove the following result.

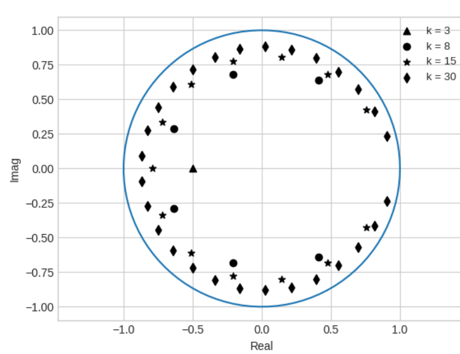
Theorem 3.2. *We consider the Dirichlet problem (1.5), (1.4) in the ellipse (1.7). If $\bar{\mu}_0 = -\nu_2$ and α determined in (1.10), then:*

1. The problem (1.5), (1.4) is uniquely solvable if and only if the conditions (3.5) hold.

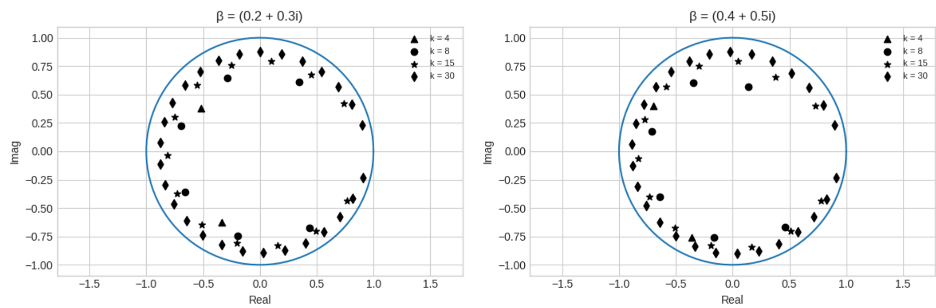
2. If $|\alpha| < 0.5$ then the problem (1.5), (1.4) is uniquely solvable.

3. The conditions (3.5) may fail for one k only, therefore the defect numbers of the problem (1.5), (1.4) may be equal one or zero only.

We see, that for $\beta = 0$ different polynomials P_{k-2} has different roots, or for different values k the roots of $S_{k-2}(\alpha, 0)$ are different (on the picture we show the roots of polynomials P_{k-2} for $k = 3, 8, 15, 30$).



We did the following numerical experiment. In the formula (2.33) we fix β_0 in the unit disc and find the roots of the polynomial $S_{k-2}(\alpha, \beta_0)$. For all considered values of β_0 we obtain the same result, that is we get that for different values k the roots of $S_{k-2}(\alpha, \beta_0)$ are different. As an illustration of the obtained result, we show the roots of the polynomials $S_{k-2}(\alpha, \beta_0)$ for values $\beta_0 = 0.2 + 0.3i$, $\beta_0 = 0.4 + 0.5i$ and for $k = 4, 8, 15, 30$):



Therefore, we see that if for some (α, β) the condition (2.33) failed, that is $S_{k_0}(\alpha, \beta) = 0$ $(k_0 > 2)$, then we have $S_k(\alpha, \beta) \neq 0$ for $k \neq k_0$, $k > 2$.

Thus, we may suppose, that the third point of the theorem 3.2 is true in general case, that is the defect numbers of the problem (1.5), (1.4) may be equal one or zero only for arbitrary α and β, but it should be proved.

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Поступила 29 марта 2023

После доработки 29 марта 2023

Принята к публикации 13 июля 2023

Известия НАН Армении, Математика, том 59, н. 2, 2024, стр. 16 – 34.

A NEW REGULARLY VARYING DISCRETE DISTRIBUTION GENERATED BY WARING-TYPE PROBABILITY

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Abstract. In this paper, based on the discretization method, we construct a new 2-parameter regularly varying discrete distribution generated by Waring-type probability (2-RDWP). Some useful plots are displayed for the model. From the mathematical point of view, to suggest 2-RDWP as a new discrete probability distribution in bioinformatics, some statistical facts such as unimodality, skewness to the right, upward/downward convexity, regular variation at infinity and asymptotically constant slowly varying component are established for the model. We provide the conditions of coincidence of solution for the system of likelihood equations with the maximum likelihood estimators for the unknown parameters. Simulation studies are performed using the Monte Carlo method and Nelder-Mead optimization algorithm to obtain maximum likelihood estimations of the unknown parameters. Asymptotic expansion of the probability function with two terms is considered, and then the moment's existence of integer orders is investigated. Finally, a real count data set is used to show the applicability of the new model compared to other models in bioinformatics.

MSC2020 numbers: 60E05, 62E10, 62F10, 62P10.

Keywords: asymptotic expansion; discretization; maximum likelihood; Monte Carlo method; statistical facts; Waring-type probability.

1. Introduction

Probability distributions are commonly applied to describe phenomena in biomolecular systems, bioinformatics, etc. Due to the usefulness of probability distributions in bioinformatics, their mathematical theory is widely studied, and new discrete distributions (frequency distributions) are developed. According to the variety, diversity and complexity of real data sets in bioinformatics and biomolecular systems, it is impossible to figure out and suggest a universal model suitable for all situations. Hence, the interest in developing discrete distributions in bioinformatics and biomolecular systems remain strong in probability and statistics.

Many discrete probability distributions have been introduced based on different methods for the needs of bioinformatics systems. For a review of different methods, see, for example, [3]. Let us point out two of the known producers as follows.

Using the method of birth-death process, we refer the readers to, for example, [2, 5, 13, 14, 15]. Besides the method of birth-death process, there are other methods, in particular by *discretization method* which we refer to, for example, [6, 7, 8, 10].

The advantage of constructing new probability distribution is proposed to parametric ones because by changing the parameters, one hopes to find the best approximation for the unknown model. Because of the wide variety of phenomena in bioinformatics, we shall attempt to introduce new parametric distribution (based on discretization method).

A continuous analog of the 2-parameter regularly varying Waring probability was given by dediscretization method [1, 2, 9]. Its probability density function is stated as

$$f_x(\alpha) = \frac{1}{c(\alpha)} \times \frac{(r+x-1)^{(r+x-1)}}{(q+x)^{(q+x)}}, \quad x \in (0, \infty) \tag{1.1}$$

where $\alpha = (r, q)$ is the unkown parameter such that $r > 0, \quad q > 0$. r is called numerator parameter and q denominator parameter and $q - r > 0$. Also, $c(\alpha)$ is the normalization factor and $c(\alpha) = \int_0^\infty \frac{(r+t-1)^{(r+t-1)}}{(q+t)^{(q+t)}} dy$.

We note that the continuous analog of the 2-parameter regularly varying Waring probability (1.1) is a continuous probability distribution. Here, let us call the model (1.1) as *Waring-type probability.*

The novelty and the motivation to write this paper is to construct a new skewed discrete probability model (frequency distribution) for the needs of biosystems using (1.1). We use discretization method, and then study mathematical properties, statistical inferences and applications.

2. The 2-RDWP distribution

The desired discrete probability distribution is possible to obtain using the discretization method. We use a type of discretization of densities used by, for example, Farbod [6, 8] and Farbod and Gasparian [10]. Let us consider the numerator of (1.1) as follows:

$$p_x(\alpha) = \frac{(r+x-1)^{r+x-1}}{(q+x)^{q+x}}, \qquad x > 0. \tag{2.1}$$

To have $p_x(\alpha)$ (2.1) as a probability mass function (pmf), we use discretization method [6, 7, 8, 10] to get a new *discrete probability distribution*, denoted by $g_x(\alpha)$, with the following pmf:

$$g_x(\alpha) = \big(d(\alpha)\big)^{-1} \times \frac{(r+x-1)^{r+x-1}}{(q+x)^{q+x}}, \tag{2.2}$$

where $x = 1, 2, ...,$ and $d(\alpha)$ is the *normalization factor (normalization constant)* given by

$$d(\alpha) = \sum_{y=1}^{\infty} \frac{(r+y-1)^{r+y-1}}{(q+y)^{q+y}} \tag{2.3}$$

and $\alpha = (r, q)$ is the unknown parameter such that $r > 0$ and $q > r$.

Remark 2.1. *It is obvious that* $g_x(\alpha) \geq 0$ *and also* $\sum_{x=1}^{\infty} g_x(\alpha) = 1$*. Thus, function (2.2) is a probability function and can be considered as a new pmf on the set of positive integers* $x \in \mathbb{N}_+ = \{1, 2, 3, ...\}$.

A probability measure (distribution function of random variable X) is given by

$$F_x(\alpha) = P(X \leq x) = \big(d(\alpha)\big)^{-1} \sum_{m=1}^{x} g_m(\alpha) = \big(d(\alpha)\big)^{-1} \sum_{m=1}^{x} \frac{(r+m-1)^{r+m-1}}{(q+m)^{q+m}}. \tag{2.4}$$

We call model (2.4) a "*2-parameter regularly varying discrete distribution generated by Waring-type probability*"(in short, 2-RDWP). The pmf of 2-RDWP is given by Eq.(2.2). This paper investigates some mathematical properties, statistical inferences and applications for the model (2.2).

The remaining sections of the paper can be summarized as follows. Section 3 presents some plots of pmf and log-log plot of the 2-RDWP model for different values of parameters. *Statistical facts*, for our model, are verified for the mathematical needs of bioinformatics in Section 4. In Section 5, we propose maximum likelihood (ML) estimators of the 2-RDWP's parameters, which are coincided with some moment estimators. Section 6 uses the Monte Carlo method and Nelder-Mead optimization algorithm to simulate for obtaining the ML estimations of parameters. Section 7 gives an asymptotic expansion with two terms for the pmf, tail behavior of distribution function, and also the moment's existence of integer orders is investigated. Section 8 presents application of the proposed model and compares it with other rival models. The study is concluded in Section 9. Section 10 considers an Appendix containing the pmfs of some rival models arising in bioinformatics.

3. Figures

This section presents two types of figures for the 2-RDWP model (2.2). To depict figures, we need to consider the model's pmf as truncated. First, some pmfs for different possible values of parameters r and q are plotted in Figures 1(A-J). Second, some log-log plots ($\ln g_x(\alpha)$ versus $\ln x$) are displayed in Figures 2(A-J). Figures 1(A-J) show skewness to the right and also unimodality of the pmf and Figures 2(A-J) show the deviations of $\ln g_x(\alpha)$ versus $\ln x$ from the straight line, which is discussed in Section 4.

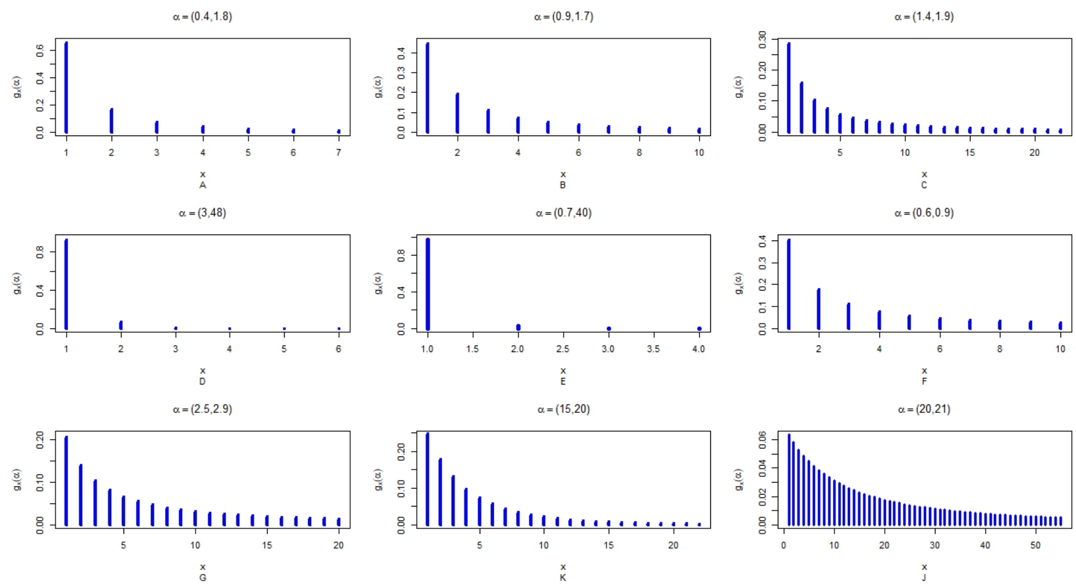

Рис. 1. Illustrations of the pmf of 2-RDWP model (2.2) for possible values of two parameters r and q.

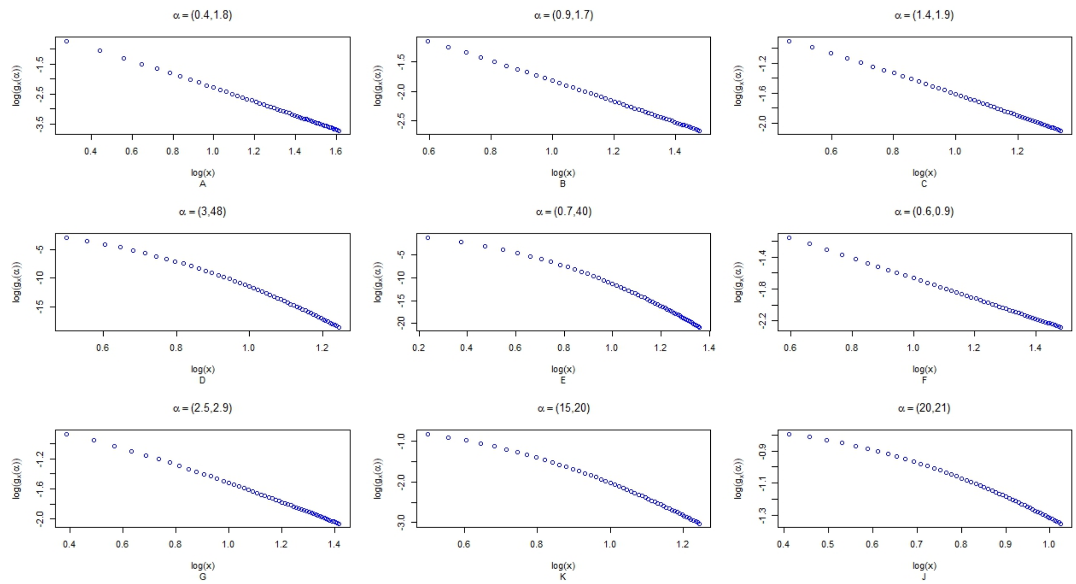

Рис. 2. Illustrations of the log-log plot of 2-RDWP model (2.2) for possible values of two parameters r and q.

4. Statistical facts

From the mathematical point of view, to suggest a discrete probability distribution as a new model in bioinformatics, we need to verify some common *statistical facts (empirical facts)* such as unimodality, skewness to the right, upward/downward convexity, regularly varying at infinity, and slowly varying at infinity. In other words, it was established that if a pmf (probability law) holds these *statistical facts*, then the corresponding pmf could be a mathematical framework for bioinformatics applications [2, 3, 5, 15]. So, to apply the 2-RDWP model (2.2) as a new probability model in bioinformatics, we need to check out the validity of such known *statistical facts*, mathematically, numerically and intuitively.

We notice that *statistical facts* (*empirical facts*) are common *mathematical properties* of the empirical frequency distributions (with complex forms and long right-side tails) observed in bioinformatics data sets and are systematically reproducible in biomolecular systems [2, 3, 5, 15].

4.1. **Log-log plot.** Biologists prefer to deal with log-log plot of distribution instead of its shape [2]. One of the statistical facts is that log-log plot of discrete distributions arising in bioinformatics systematically deviated from the straight line and shows upward/downward convexity [2, 3, 15]. It means that the deviations of log-log plot of $g_x(\alpha)$ from the straight line must be *not too large*.

Let us investigate the log-log plot of our model. Namely, we deal with $\ln g_x(\alpha)$ versus $\ln x$. We write the log-log plot of the model ($\ln g_x(\alpha)$ versus $\ln x$) as follows:

$$\frac{\ln g_x(\alpha)}{\ln x} = \frac{(x+r-1)\ln(x+r-1) - (x+q)\ln(x+q) - \ln(d(\alpha))}{\ln x} \tag{4.1}$$

It is obvious that, for sufficiently large x (sufficiently large x means $x \longrightarrow \infty$), $\ln(x+a) \approx \ln x$ (a is some real constant). Therefore from Eq.(4.1), for sufficiently large x, we have

$$\frac{\ln g_x(\alpha)}{\ln x} \approx r - q - 1 - \frac{\ln d(\alpha)}{\ln x}. \tag{4.2}$$

Due to Eq.(4.2) we conclude that the deviations of $\ln g_x(\alpha)$ versus $\ln x$ from the straight line *constant* $= (r-q-1)$ are small, at least for large values of x which it turns out upward/downward convexity.

Remark 4.1. *We note that there are not any specific definitions for upward/downward convexity concept in bioinformatics and it is issue of the mathematical disciplines. In other words, some of the peculiarities of the shapes of empirical frequency distributions in bioinformatics are: upward/downward convexity, the only point where the frequency distribution achieves it's maximal value, etc. For more details about mathematical and applied concepts of upward/downward convexity, we refer the readers to Astola and Danielian* [2, Sec.1.4, Sec.2.5].

Additionally, Figures 2(A-J) show the log-log plot of 2-RDWP with different values of parameters. Figures 2(A-J) provide that the deviations of $\ln g_x(\alpha)$ versus $\ln x$ from the straight line may be small, at least for some large values of x. From Figures 2(A-J), we observed a significant shift and variation of the power law-like right-side tail of the pmfs.

4.2. **Regular variation.** This subsection shows that the model $g_x(\alpha)$ varies regularly at infinity and also we present an asymptotically constant slowly varying component for it. Compared to Astola and Danielian [2], let us state two definitions for our model (2.2).

Definition 4.1. *The frequency distribution $g_x(\alpha)$ varies regularly at infinity with exponent $(-\rho)$ if it may be presented in the form*

$$g_x(\alpha) = x^{-\rho} \cdot R(x)(1+o(1)), \quad x \longrightarrow \infty, \quad \rho \in (-\infty, \infty),$$

where $R(x) > 0$ for $x = 1, 2, ...,$ and for $\kappa = 2, 3, ...,$ $\lim_{x\longrightarrow\infty} \frac{R(\kappa x)}{R(x)} = \kappa^{-\rho}$.

Definition 4.2. *Let, for $\kappa = 2, 3, ...,$ the limit exists*

$$\lim_{x \longrightarrow \infty} \frac{R(\kappa x)}{R(x)} = 1$$

then $g_x(\alpha)$ exhibits the asymptotically constant slowly varying component L if we have

$$\lim_{x \longrightarrow \infty} R(x) = L \in (0, \infty).$$

Remark 4.2. *It is clear that Definition 4.2 is a particular case of Definition 4.1. Thus, a function varying regularly at infinity with exponent $\rho = 0$ varies slowly at infinity* [2].

Let us establish the function $g_x(\alpha)$ varies regularly at infinity with exponent $(-\rho)$ having $-\rho = -(q+1-r)$. We propose theorem, remark and numerical example as follows.

Theorem 4.1. *The model $g_x(\alpha)$ (2.2) varies regularly at infinity with exponent $(-\rho)$ and*

$$-\rho = -(q+1-r) < -1. \tag{4.3}$$

Proof. From (2.2) and (2.3), for sufficiently large x, we get

$$g_x(\alpha) \approx (d(\alpha))^{-1} \cdot e^{-(q+1-r)} x^{-(q+1-r)} \approx x^{-(q+1-r)}. \tag{4.4}$$

It follows from (4.4) that $g_x(\alpha)$ (2.2) varies regularly at infinity if $\rho = q+1-r > 1$. Theorem 4.1 is proved. □

Remark 4.3. *From (4.4) and based on Remark 3, we observe that $L = (d(\alpha))^{-1}$ is an asymptotically constant slowly varying component for the model $g_x(\alpha)$ (2.2). In other words, $g_x(\alpha)$ exhibits the asymptotically constant slowly varying component given by $L = (d(\alpha))^{-1}$.*

Let us give a numerical example as follows.

Example 4.1. *Let us compute the value of ρ corresponding to selected two parameters r and q (used in Figures 1 and 2) by:*

$$\begin{array}{ll}
\alpha = (0.4, 1.8), & -\rho = -2.4 < -1 \\
\alpha = (0.9, 1.7), & -\rho = -1.8 < -1 \\
\alpha = (1.4, 1.9), & -\rho = -1.5 < -1 \\
\alpha = (3, 48), & -\rho = -46 < -1 \\
\alpha = (0.7, 40), & -\rho = -40.3 < -1 \\
\alpha = (0.6, 0.9), & -\rho = -1.3 < -1 \\
\alpha = (2.5, 2.9), & -\rho = -1.4 < -1 \\
\alpha = (15, 20), & -\rho = -6 < -1 \\
\alpha = (20, 21), & -\rho = -2 < -1
\end{array}$$

We see that our numerical values are agreed with the variation of the value of regular variation exponent $(-\rho)$ and are met in the condition (4.3).

4.3. **Unimodality.** Unimodality is an essential feature for discrete distributions arising in bioinformatics. For details about this, we refer the readers to, for example, [2, 13, 14, 15]. In this subsection, we study such feature for the 2-RDWP model. Compared to Bhati and Bakouch [4], let us give a proposition as follows.

Proposition 4.1. *The pmf (2.2) is unimodal with mode value at $x = 1$.*

Proof. Let us consider pmf (2.2) for the positive integer value of x. Then for $x \geq 1$, we get

(4.5)
$$\frac{dg_x(\alpha)}{dx} = \frac{d}{dx}\left(\frac{1}{d(\alpha)} \cdot \frac{(r+x-1)^{r+x-1}}{(q+x)^{q+x}}\right)$$

$$= \frac{1}{d(\alpha)} \cdot \frac{1}{\left((q+x)^{q+x}\right)^2} \cdot (r+x-1)^{r+x-1} \cdot (q+x)^{q+x} \cdot \left[\ln(r+x-1) - \ln(q+x)\right].$$

It is obvious that for $0 < r < q$

$$\ln(r+x-1) - \ln(q+x) < 0.$$

So, we conclude that $\frac{dg_x(\alpha)}{dx}$ given by (4.5) is always negative. It implies that $g_x(\alpha)$ decreases and takes its mode at $x = 1$. The proof is completed. $\square$

In addition to Proposition 4.1, let us investigate unimodality as numerical. We have a recursive formula given by

$$\frac{g_{x+1}(\alpha)}{g_x(\alpha)} = \frac{(r+x)^{r+x}(q+x)^{q+x}}{(r+x-1)^{r+x-1}(q+x+1)^{q+x+1}}, \qquad x = 1, 2, ... \tag{4.6}$$

Numerically, it can be shown that $\frac{g_{x+1}(\alpha)}{g_x(\alpha)} < 1$. Let us have the following example.

Example 4.2. *Let us consider some values of parameters $(r = 0.7, q = 1)$ and $(r = 1.5, q = 2.5)$. From (4.6), we calculate $\frac{g_{x+1}(\alpha)}{g_x(\alpha)}$, for $x = 1, 2, 3, 4, 5, 6,$ in Table 1 as follows:*

Таблица 1. The behavior of $\frac{g_{x+1}(\alpha)}{g_x(\alpha)}$ (4.6) for different values of parameters r and q

$\alpha = (r, q)$	$\frac{g_2(\alpha)}{g_1(\alpha)}$	$\frac{g_3(\alpha)}{g_2(\alpha)}$	$\frac{g_4(\alpha)}{g_3(\alpha)}$	$\frac{g_5(\alpha)}{g_4(\alpha)}$	$\frac{g_6(\alpha)}{g_5(\alpha)}$	$\frac{g_7(\alpha)}{g_6(\alpha)}$
$\alpha = (0.7, 1)$	*0.46869*	*0.62523*	*0.70967*	*0.76288*	*0.79955*	*0.82638*
$\alpha = (1.5, 2.5)$	*0.49602*	*0.59820*	*0.66569*	*0.71370*	*0.74962*	*0.77752*

From Table 1, we see that the expression, as in (4.6), increases when x increases and also for $x = 1, 2, 3, 4, 5, 6,$ the values $\frac{g_{x+1}(\alpha)}{g_x(\alpha)} < 1$. Numerically, it seems that $g_x(\alpha)$ defined by (2.2) decreases and is downward convex. Automatically, the unimodality of $g_x(\alpha)$ is received.

Moreover, in Section 3, we plotted the pmf of 2-RDWP (2.2) for different values of parameters. In other words, intuitively and from the graphical approach in Figures 1(A-J), it is readily seen that the pmf of 2-RDWP is unimodal. The modes are observed for all plots in Figures 1(A-J) at $x = 1$.

4.4. **Skewness to the right.** One of the essential properties of discrete distributions (frequency distributions) arising in biomolecular systems is the skewness to the right of the pmf. This property has been discovered by experimental methods based on the observation of various data sets of such systems. The conception of skewness for biologists is based on intuition and the shapes of graphs of discrete distributions [2, 3]. Section 3 displayed the plots of the pmf of 2-RDWP (2.2) for different possible parameter values. Intuitively and from the graphical approach in Figures 1(A-J), it can be observed that the pmf of 2-RDWP (2.2) is skewed to the right. Here, let us have a numerical example.

Example 4.3. *Let us have some real data that includes the number of proteins assigned to Panther families or subfamilies as follows* [18]*:*
1, 17, 11, 22, 16, 10, 61, 10, 12, 15, 22, 10, 5, 1, 33, 6, 11, 1, 5, 3, 2, 9, 22, 10, 3, 86, 1, 1, 15, 5, 8, 26, 2, 14, 2, 9, 62, 7, 114, 113, 20, 22, 14, 12, 13, 6, 24, 26, 22, 51, 56, 106, 59, 55, 29, 1, 141, 168, 607, 395, 616, 1, 7, 19, 3, 29, 59, 4, 4, 1, 3, 18, 60, 46, 11, 56, 269, 812.

The value of skewness for these data is 3.960*.*

The following mathematical result is received from Subsections 4.1 − 4.4.

Corollary 4.1. *The common statistical facts (unimodality, skewness to the right, upward/downward convexity, regular variation at infinity, asymptotically constant slowly varying component) hold for the model $g_x(\alpha)$ (2.2). Therefore, from the mathematical point of view, the 2-RDWP model (2.2) may be considered as a new regularly varying frequency distribution for the needs of large-scale biomolecular systems, bioinformatics, etc. For details about this, see* [2, 3, 5, 15].

5. On the ML estimators

This section gives ML estimators for the model (2.2). We get the conditions of coincidence of solution for the system of likelihood equations with the ML estimators for the unknown parameters. Let us define the functions $h(x;\alpha)$ and $t(x;\alpha)$ by

$$h(x;\alpha) = \ln(r + x - 1) + 1, \quad t(x;\alpha) = -\big(\ln(q + x) + 1\big),$$

and $\overline{h_n(\alpha)} = \frac{1}{n}\sum_{i=1}^{n} h(x_i;\alpha)$, $\overline{t_n(\alpha)} = \frac{1}{n}\sum_{i=1}^{n} t(x_i;\alpha)$. We state a lemma for the model (2.2).

Lemma 5.1. *For model (2.2), we have the following*

$$E\big[h(\xi;\alpha)\big] < \infty, \qquad E\big[t(\xi;\alpha)\big] < \infty,$$

where $E[\cdot]$ is the mathematical expectation.

Proof. Based on the definition of mathematical expectation, the proof is satisfied, obviously.

From Lemma 5.1, and compared to Farbod and Gasparian [11], let us present a theorem.

Theorem 5.1. *The likelihood equations for obtaining the ML estimators of parameter α with the model (2.2) have the following moments equations*

$$\begin{cases} E\big[h(\xi;\alpha)\big] = \overline{h_n(\alpha)} \\ E\big[t(\xi;\alpha)\big] = \overline{t_n(\alpha)} \end{cases} \tag{5.1}$$

Proof. We consider the likelihood function $L(X^n;\alpha) = \prod_{i=1}^{n} g_{x_i}(\alpha)$. The logarithm of the likelihood function is given by

$$l(X^n;\alpha) = \ln L(X^n;\alpha) = \sum_{i=1}^{n} \ln \frac{(r + x_i - 1)^{r + x_i - 1}}{(q + x_i)^{q + x_i}} - n \ln d(\alpha) \tag{5.2}$$

If the following conditions hold

$$\frac{\partial l(X^n;\alpha)}{\partial r} = 0, \quad \frac{\partial l(X^n;\alpha)}{\partial q} = 0,$$

then the ML estimators of the parameters $\alpha = (r, q)$ exist.

Let us obtain derivatives by parameters r and q. We have

$$\frac{\partial l(X^n;\alpha)}{\partial r} = \sum_{i=1}^{n} \Big(\ln(r + x_i - 1) + 1\Big) - n\frac{1}{d(\alpha)}\frac{\partial d(\alpha)}{\partial r}$$

where $\frac{1}{d(\alpha)}\frac{\partial d(\alpha)}{\partial r} = E\big[h(\xi;\alpha)\big]$. From $\frac{\partial l(X^n;\alpha)}{\partial r} = 0$, we get $E\big[h(\xi;\alpha)\big] = \overline{h_n(\alpha)}$.

Meanwhile, we have

$$\frac{\partial l(X^n;\alpha)}{\partial q} = \sum_{i=1}^{n} -\Big(\ln(q + x_i) + 1\Big) - n\frac{1}{d(\alpha)}\frac{\partial d(\alpha)}{\partial q}$$

where $\frac{1}{d(\alpha)}\frac{\partial d(\alpha)}{\partial q} = E\big[t(\xi;\alpha)\big]$. From $\frac{\partial l(X^n;\alpha)}{\partial q} = 0$, we obtain $E\big[t(\xi;\alpha)\big] = \overline{t_n(\alpha)}$.

The Theorem 5.1 is proved. □

We aim to show that the solution $\hat{\alpha}$ of the system (5.1) is the ML estimator of the parameter α. It is sufficient to establish that the matrix $\hat{M}_n = (\hat{M}^n_{ij})^2_{i,j=1}$ with $\hat{M}^n_{ij} = \hat{M}^n_{ij}(\hat{\alpha}),\ \hat{M}^n_{ij}(\hat{\alpha}) = \frac{\partial^2 l(X^n;\alpha)}{\partial r \partial q}|_{\alpha=\hat{\alpha}}$ is negative definite. Let us state two lemmas.

Lemma 5.2. *Consider the model (2.2). Assuming the solution $\widehat{\alpha}$ of the system (5.1) (if it exists) holds in the following conditions*

$$\begin{cases} E\big[\psi(\xi;\alpha)\big] = \overline{\psi_n(\alpha)} \\ \\ E\big[\eta(\xi;\alpha)\big] = \overline{\eta_n(\alpha)} \end{cases} \tag{5.3}$$

where

$$\psi(\xi;\alpha) = \frac{1}{r+x-1},\ \ \overline{\psi_n(\alpha)} = \frac{1}{n}\sum_{i=1}^{n}\psi(x_i;\alpha);\quad \eta(\xi;\alpha) = -\frac{1}{q+x},\ \ \overline{\eta_n(\alpha)} = \frac{1}{n}\sum_{i=1}^{n}\eta(x_i;\alpha).$$

Then, the elements of the matrix $\hat{M}_n$ are as follows ($Var(\cdot)$ is the variance and $Cov(\cdot,\cdot)$ is the covariance):

$$\hat{M}_{11} = -n\ Var(h(\xi;\alpha)),$$

$$\hat{M}_{12} = \hat{M}_{21} = -n\ Cov\big(h(\xi;\alpha), t(\xi;\alpha)\big),$$

$$\hat{M}_{22} = -n\ Var(t(\xi;\alpha)).$$

Proof. We obtain second derivatives of the logarithm of likelihood functions by

$$\frac{\partial^2 l(X_n;\alpha)}{\partial r^2} = -n\Big(\frac{1}{d(\alpha)}\frac{\partial^2 d(\alpha)}{\partial r^2} - \big(\frac{1}{d(\alpha)}\frac{\partial d(\alpha)}{\partial r}\big)^2\Big) + n\overline{\psi_n(\alpha)}$$

$$\frac{\partial^2 l(X_n;\alpha)}{\partial r \partial q} = \frac{\partial^2 l(X_n;\alpha)}{\partial q \partial r} = -n\Big[\frac{1}{d(\alpha)}\frac{\partial^2 d(\alpha)}{\partial r \partial q} - \Big(\frac{1}{d(\alpha)}\frac{\partial d(\alpha)}{\partial r}\Big)\Big(\frac{1}{d(\alpha)}\frac{\partial d(\alpha)}{\partial q}\Big)\Big]$$

$$\frac{\partial^2 l(X_n;\alpha)}{\partial q^2} = -n\Big(\frac{1}{d(\alpha)}\frac{\partial^2 d(\alpha)}{\partial q^2} - \big(\frac{1}{d(\alpha)}\frac{\partial d(\alpha)}{\partial q}\big)^2\Big) + n\overline{\eta_n(\alpha)}$$

After some simplification, we have

$$M_{11} = -n\ Var\big(h(\xi;\alpha)\big) - n\big(E\big[\psi(\xi;\alpha)\big] - \overline{\psi_n(\alpha)}\big)$$

$$M_{12} = M_{21} = -n\ Cov\big(h(\xi;\alpha), t(\xi;\alpha)\big)$$

$$M_{22} = -n\ Var\big(t(\xi;\alpha)\big) - n\big(E\big[\eta(\xi;\alpha)\big] - \overline{\eta_n(\alpha)}\big)$$

With the help of (5.3) the proof of Lemma 5.2 is finished. □

Lemma 5.3. *Consider the model (2.2). Under the conditions (5.3), the matrix $\hat{M}_n$ is negative definite.*

Proof. It suffices to show $\hat{M}_{11}^{n} < 0$ and $\det(\hat{M}^{n}) > 0$. From Lemma 5.2, it can be concluded that $\hat{M}_{11}^{n} < 0$. To show that $\det(\hat{M}^{n}) > 0$ we give

$$\det(\hat{M}^{n}) = \hat{M}_{11}^{n}\hat{M}_{22}^{n} - (\hat{M}_{12}^{n})^{2}$$

In accord with the value of $\hat{M}_{11}^{n}$, $\hat{M}_{22}^{n}$, $\hat{M}_{12}^{n}$ and based on Cauchy-Bunyakovski-Schwartz inequality the proof is completed. □

From Lemmas 5.2 and 5.3, the following result is given.

Corollary 5.1. *Suppose that the solution of the system (5.1) satisfies the conditions (5.3), then it coincides with the ML estimators of parameters.*

6. ML estimaton and simulation

Based on systems (5.1) and (5.3), it is not easy to derive closed forms for the solutions, analytically. So, we need to use a numerical method for the ML estimations of unknown parameters. Compared to Farbod [8], Nelder-Mead optimization algorithm (or *simplex search algorithm*) is suggested. Let us notice that the Nelder-Mead optimization algorithm is a free-derivative optimization method to nonlinear optimization problems and is suggested to apply for models with more than one parameter. This algorithm was introduced by Nelder and Mead [16]. See also [17].

For sampling, a simple stochastic sampling with replacement with the probability of variables is considered in which the probability of variables are probability functions. Simulation studies are proposed using the Monte Carlo method [17] with 1000 iterations to calculate ML estimations, biases and mean square errors (MSEs).

Remark 6.1. *First, we performed our simulation for the model (2.2). Based on (2.2), our simulation works well when $x = 1, 2, ..., xmax$ ($xmax = 100$). But we have some computational problems for large values x, such as $xmax = 150$ and bigger than 150. Let us notice that a type of function x^x exists in our pmf's form (2.2), and hence it raises problems for simulations and numerical calculations when x is large. For example, if $x = 500$, then using R statistical software (Version 4.2.2) $x^x = 500^{500} = \infty$. To solve this computational problem, without loss of generality and after some mathematical simplification, our pmf (2.2) can be written as follows:*

$$g_x^*(\alpha) = \left(\sum_{y=1}^{\infty} \frac{\left(1+\frac{r-1}{y}\right)^{y}\left(y+r-1\right)^{r-1}}{\left(1+\frac{q}{y}\right)^{y}(y+q)^{q}}\right)^{-1} \cdot \frac{\left(1+\frac{r-1}{x}\right)^{x}\left(x+r-1\right)^{r-1}}{\left(1+\frac{q}{x}\right)^{x}\left(x+q\right)^{q}}. \tag{6.1}$$

From Remark 6.1 and (6.1), we have the following corollary.

Corollary 6.1. *It is readily seen that the pmf (2.2) equals the pmf (6.1). So, for simulation studies, the pmf (6.1) is considered. In the formula (6.1), we need to have*

x as truncated. For simulation aims, let us set $x = 1$ to $xmax$ ($xmax = 10000$). Namely, we have $x = 1, 2, ..., 10000$ and $y = 1, 2, ..., 10000$.

Let us consider the logarithm of the likelihood function (5.2). Based on (6.1) and $x = 1, 2, ..., 10000$, the ML estimations, biases, and MSEs are calculated. To simulation studies, we consider the values $(r = 0.4, q = 0.6), (r = 1, q = 2), (r = 2.5, q = 3.2)$ as true values, different sample sizes $n = 50, 100, 200, 500, 1000, 5000$, and using 1000 iterations.

Using R statistical software, the simulation results are given in Table 2. Our simulation studies work well and have satisfactory results for the model. The differences between real and estimated values of the parameters are small, in particular for large sample sizes.

Table 2 shows that when the sample size n increases, bias and MSE decrease. Moreover, from Table 2, we observe that when the true values of r and q are smaller (also a small value of $q-r$), the results are better, i.e. biases and MSEs are smaller. Let us notice that for the ML estimations, the conditions $\hat{q} - \hat{r} > 0$ and $\hat{\rho} > 1$ are satisfied.

7. Asymptotic expansion

Considering that our proposed model has no closed form for the pmf, obtaining some useful asymptotic expansion with two terms for the pmf is interesting. From (2.2) and (6.1), we get

$$g_x(\alpha) = (d(\alpha))^{-1} \cdot x^{r-q-1} \cdot \frac{\left(1+\frac{r-1}{x}\right)^x \left(1+\frac{r-1}{x}\right)^{r-1}}{\left(1+\frac{q}{x}\right)^x \left(1+\frac{q}{x}\right)^q} \tag{7.1}$$

We use two known asymptotic expansions as follows. For $x \longrightarrow \infty$, we have [5, 12]:

$$(1 + \tfrac{c}{x})^x = e^c \cdot \left(1 - \tfrac{c^2}{2x} + O(\tfrac{1}{x^2})\right). \tag{7.2}$$

Also, for $x \longrightarrow 0$ we have

$$(1 + x)^\alpha = 1 + \alpha x + O(x^2). \tag{7.3}$$

From (7.2) and (7.3), the formula (7.1) may be given by

(7.4)

$$\begin{aligned} g_x(\alpha) &= (d(\alpha))^{-1} \cdot x^{r-q-1} \cdot e^{r-q-1} \frac{\left(1+\frac{r-1}{x}\right)^x \left(1+\frac{r-1}{x}\right)^{r-1}}{\left(1+\frac{q}{x}\right)^x \left(1+\frac{q}{x}\right)^q} \\ &= (d(\alpha))^{-1} \cdot x^{r-q-1} \cdot e^{r-q-1} \cdot \frac{\left(1-\frac{(r-1)^2}{2x}+O(\frac{1}{x^2})\right)}{\left(1-\frac{q^2}{2x}+O(\frac{1}{x^2})\right)} \cdot \frac{\left(1+\frac{(r-1)^2}{x}+O(\frac{1}{x^2})\right)}{\left(1+\frac{q^2}{x}+O(\frac{1}{x^2})\right)} \end{aligned}$$

Таблица 2. Simulation results: The values of ML estimations $(\hat{r}, \hat{q})$, biases, and
MSEs for the 2-RDWP model (6.1)

	$(r = 0.4, q = 0.6); \quad x = 1, 2, ..., 10000$		
n	$(\hat{r}, \hat{q})$	Bias	MSE
50	(0.6156,0.8434)	(0.2156,0.2434)	(0.3234,0.4047)
100	(0.4844,0.6955)	(0.0844,0.0955)	(0.1013,0.1307)
200	(0.4318,0.6346)	(0.0318,0.0346)	(0.0420,0.0560)
500	(0.4108,0.6115)	(0.0108,0.0115)	(0.0150,0.0204)
1000	(0.4049,0.6051)	(0.0049,0.0051)	(0.0072,0.0098)
5000	(0.4002,0.6001)	(0.0002,0.0001)	(0.0014,0.0020)
	$(r = 1, q = 2); \quad x = 1, 2, ..., 10000$		
n	$(\hat{r}, \hat{q})$	Bias	MSE
50	(1.4683,2.6768)	(0.4683,0.6768)	(1.9440,3.8043)
100	(1.1716,2.2475)	(0.1716,0.2475)	(0.4026,0.7901)
200	(1.0673,2.0963)	(0.0673,0.0963)	(0.1254,0.2417)
500	(1.0236,2.0330)	(0.0236,0.0330)	(0.0461,0.0872)
1000	(1.0158,2.0229)	(0.0158,0.0229)	(0.0227,0.0430)
5000	(1.0039,2.0051)	(0.0039,0.0051)	(0.0041,0.0079)
	$(r = 2.5, q = 3.2); \quad x = 1, 2, ..., 10000$		
n	$(\hat{r}, \hat{q})$	Bias	MSE
50	(3.1808,3.9675)	(0.6808,0.7675)	(5.4404,6.8484)
100	(2.748,3.4778)	(0.2480,0.2778)	(1.3875,1.7214)
200	(2.6214,3.3357)	(0.1214,0.1357)	(0.5669,0.6986)
500	(2.5541,3.2609)	(0.0541,0.0609)	(0.2028,0.2494)
1000	(2.5367,3.2415)	(0.0367,0.0415)	(0.0992,0.1226)
5000	(2.5093,3.2102)	(0.0093,0.0102)	(0.0191,0.0238)

Let $\rho = q + 1 - r$. From (7.4), we get

(7.5)
$$g_x(\alpha) \approx \big(d(\alpha)\big)^{-1} \cdot x^{-\rho} \cdot e^{-\rho} \times \Big(1 + \tfrac{1}{2x}\big((r-1)^2 - q^2\big) + \big((r-1)^2 - q^2\big)O(\tfrac{1}{x^2})\Big)$$
$$\approx \big(d(\alpha)\big)^{-1} \cdot x^{-\rho} \cdot e^{-\rho} \cdot \Big(1 + \tfrac{1}{2x}\big((r-1)^2 - q^2\big) + O(\tfrac{1}{x^2})\Big).$$

7.1. **Tail behavior.** Using asymptotic expansions (7.2), (7.3) and based on (7.4), let us propose tail behavior of distribution function $F_x(\alpha)$ (2.4) when $x \longrightarrow \infty$.

From (2.2), we get

$$1 - F_x(\alpha) = P(X > x) = \sum_{m=x+1}^{\infty} g_x(\alpha) \tag{7.6}$$

By substituting (7.4) and (7.5) into (7.6), when $x \longrightarrow \infty$, we have

$$1 - F_x(\alpha) \approx \big(d(\alpha)\big)^{-1} e^{-(q+1-r)} \sum_{m=x+1}^{\infty} m^{-(q+1-r)}. \tag{7.7}$$

The following corollary is given.

Corollary 7.1. *It follows from (7.7) that the condition (4.3) must be met.*

7.2. **Moments.** It is known that some moments are undefined for every power law-like distribution. We investigate the moment's existence of the model (2.2). To do that, using asymptotic expansion (7.5), we shall propose the moment's existence of integer orders of the 2-RDWP model (2.2). Let $\rho = q + 1 - r$.

From (7.5), it is readily seen that the first-order moment of X is finite if $q-r > 1$ (or equivalently $\rho > 2$). In other words, for model (2.2):

$$E(X) < \infty, \quad if\ \rho > 2.$$

For the second-order moment, it is easy to see that

$$E(X^2) < \infty, \quad if \ \ \rho > 3.$$

Hence, the variance for the model (2.2) is also finite if $\rho = q + 1 - r > 3$. In other words, we have

$$Var(X) = E(X^2) - E^2(X) < \infty, \quad if\ \rho > 3.$$

In the general case, if $q - r > j$ then

$$E(X^j) < \infty, \quad j = 1, 2, ...; \quad if \ \ \rho > j + 1.$$

Corollary 7.2. *Assume that X is a regularly varying random variable with a distribution (2.2) and index ρ. Then the moment of order j is infinite if $\rho \leq j + 1$.*

Moreover, evaluating the mean and variance of the model (2.2) for practical needs is of interest. From the proposed asymptotic expansion (7.5), we can present an approximate form with two terms for the mean and variance. Let us obtain mean as a practical form for the truncated function with two terms as

$$E(X) = \sum_{x=1}^{\infty} g_x(\alpha) \approx (d(\alpha))^{-1} e^{-\rho} \Big[\sum_{x=1}^{\infty} x^{-\rho+1} + \tfrac{1}{2}((r-1)^2 - q^2) \sum_{x=1}^{\infty} x^{-\rho} \Big]. \tag{7.8}$$

Compared to Astola and Danielian [2, p.29], we have

$$\begin{cases} \sum_{x=1}^{\infty} x^{-\rho} = \frac{1}{\Gamma(\rho-1)} \lim_{\lambda \longrightarrow 1} \int_0^1 \ln(1-\lambda t)(\ln \frac{1}{t})^{\rho-2} \frac{dt}{t} \\ \sum_{x=1}^{\infty} x^{-\rho+1} = \frac{1}{\Gamma(\rho-2)} \lim_{\lambda \longrightarrow 1} \int_0^1 \ln(1-\lambda t)(\ln \frac{1}{t})^{\rho-3} \frac{dt}{t} \end{cases} \tag{7.9}$$

where $0 < \lambda < 1$ is some small constant and $\Gamma(\cdot)$ is the Gamma function. Substituting (7.9) into (7.8), an approximate form with two terms for the mean is given in the practical form and integral representation. Similarly, we can provide integral representations for the second order moment and also variance.

8. Application to data and comparison

As we pointed out in Section 2 and verified in Section 4, our new discrete distribution (2.2) may be considered in bioinformatics, biosystems, etc. Let us fit our model with a real count data set (Example 4.3) and then compare it with the other models in bioinformatics. Again, for simulation and fitting aims, we consider the pmf form (6.1).

The given real data set is the number of proteins in a biological system. In other words, we consider some real data that includes the number of proteins assigned to Panther families or subfamilies (see Subsection 4.4, Example 4.3). These data are collected from Venter et al. [18].

For these 78 data (data used in the Example 4.2), using (6.1) we obtain the ML estimations for two parameters r and q. ML estimations are given by $\hat{r} = 9.620101$, $\hat{q} = 10.378817$. It implies $-\hat{\rho} = -1.758716 < -1$. In addition, $\ln L = -357.063$ and p-value=0.713. Based on the Kolmogorov-Smirnov test and the 2-RDWP model, the p-value equals 0.713, which is a good fit for such real data. Additionally, based on some well-known statistical criteria such as:

Akaike information criterion (AIC) is given by $AIC = 2\ln L + 2k$ where k the number of parameters in the model; $-\ln L$ is the maximized value of the likelihood function for the estimated model; AIC with corrected (AICc) is given by $AICc = AIC + \frac{2k^2+2k}{n-k-1}$ where n is the sample size; and p-value, we compare the 2-RDWP with other discrete models arising in bioinformatics, such as the one-parameter skewed discrete Levy distribution (DLD) (10.1) [7], one-parameter skewed Power-Law (PL) model (10.2) [2], one-parameter truncated skewed discrete stable distribution (T-SDSD) (10.3) [8], one-parameter truncated skewed discrete stable distribution (T-DSD) (10.4) [8], and two-parameter truncated skewed discrete stable distribution (T-2SDSD) (10.5) [8], all having support on the set of positive integers, i.e. $x \in \mathbb{N}_+ = \{1, 2, 3, ...\}$.

Using R statistical software, our results are presented in Table 3. It can be observed from Table 3 that the 2-RDWP model has the smallest $-\ln L$, AIC, AICc, and the largest p-value. Accordingly, we can conclude that the 2-RDWP model provides the best fit among the compared models (DLD, PL, T-SDSD, T-DSD and T-2SDSD models). The pmfs of DLD, PL, T-SDSD, T-DSD and T-2SDSD are given in Section 10 (Appendix).

Таблица 3. Comparing results for 2-RDWP, DLD, PL, T-SDSD, T-DSD, and T-2SDSD models for data of Example 4.2

Model	$\ln L$	k	AIC	AICc	p-value
2-RDWP	-357.063	2	718.126	718.286	0.713
DLD	-360.51055	1	723.0211	723.07373	0.04948
PL	-366.2436	1	734.4872	734.53983	0.01773
T-SDSD	-362.7622	1	727.5244	727.57703	0.09018
T-DSD	-360.57675	1	723.1535	723.20613	0.1746
T-2SDSD	-360.55815	2	725.1163	725.2763	0.1723

9. Conclusions

In this paper, using the discretization methods, we formulated a new skewed regular varying discrete distribution, the so-called 2-RDWP, given by Eq.(2.2). Some plots for the pmf and log-log plots of the model have been illustrated for the different values of parameters satisfying the condition in Eq.(4.3). Figures 1(A-J) indicated the pmfs for the used parameters are skewed to the right and unimodal with mode value at $x = 1$. Significantly, Figures 1(A-J) showed that the length and shape of the right-side tails varied with parameter value changes. Figures 2(A-J) established the log-log plots of the 2-RDWP (2.2). The log-log plots of Figures 2(A-J) illustrated that the right-side tails could significantly deviate from the straight line, at least for large values of observed x.

The known common *statistical facts* *(empirical facts)*, including unimodality, skewness to the right, upward/downward convexity, stability by estimated parameters values, regular variation at infinity and asymptotically constant slowly varying component, have been proved for the 2-RDWP model. Hence, mathematically, we concluded that our model (2.2) could be used as a new probability distribution for the needs of bioinformatics and biomolecular systems.

ML estimators have been obtained based on some moment equations. The conditions of coincidence of solution for the system of likelihood equations with the ML estimators for the parameters have been proposed. Based on Monte Carlo method and Nelder-Mead optimization algorithm simulation studies have been given to get ML estimations, biases and MSEs. Simulation studies presented satisfactory results. We noted that for simulation aims, instead of Eq.(2.2), we considered the pmf in Eq.(6.1). The ML estimations $\hat{r}$ and $\hat{q}$ have met in the conditions, namely based on simulation studies $\hat{q} - \hat{r} > 0$ and $\hat{\rho} > 1$.

An asymptotic expansion with two terms for the pmf (2.2) has been given. Using asymptotic expansions, we proposed tail behavior of distribution function. Also, we investigated the moment's existence of integer orders. Then, based on asymptotic expansion, useful formulas for the mean and variance in the truncated forms have been provided.

Finally, we successfully applied 2-RDWP to a real data set. Based on well-known statistical criteria, we compared our results for the proposed model with other known models in biosystems. Our model gives better results than the other models for this real data set (Table 3).

The 2-RDWP model has a long right-side tail and power law-like behavior. It can be helpful in biomolecular systems, bioinformatics and other areas such as economics and physics.

10. Appendix

We present the pmfs of some rival models, used in Table 3. The pmf of the one-parameter DLD model is given by [7]

$$p_x(\gamma) = \frac{x^{-\frac{3}{2}} \exp(-\frac{\gamma}{2x})}{\sum_{y=1}^{\infty} y^{-\frac{3}{2}} \exp(-\frac{\gamma}{2y})}, \quad x = 1, 2, ...; \quad \gamma > 0. \tag{10.1}$$

The pmf of the one-parameter PL model is as [2]

$$p_x(\nu) = \frac{x^{-\nu}}{\sum_{y=1}^{\infty} y^{-\nu}}, \quad x = 1, 2, ...; \quad \nu > 1. \tag{10.2}$$

The pmf of T-SDSD when $0 < \theta < 1$, and $x = 1, 2, \ldots$, is given by [8]

$$p_x(\theta, 1) = \frac{\Gamma(\theta+1)x^{-\theta-1}\sin(\pi\theta) - \frac{1}{2}\Gamma(2\theta+1)x^{-2\theta-1}\sin(2\pi\theta)}{\sum_{y=1}^{\infty}\left(\Gamma(\theta+1)y^{-\theta-1}\sin(\pi\theta) - \frac{1}{2}\Gamma(2\theta+1)y^{-2\theta-1}\sin(2\pi\theta)\right)}. \tag{10.3}$$

The pmf of T-DSD when $0 < \theta < 2$, and $x = 1, 2, \ldots$, is given by [8]

$$p_x(\theta, 0) = \frac{\Gamma(\theta+1)x^{-\theta-1}\sin(\frac{\pi\theta}{2}) - \frac{1}{2}\Gamma(2\theta+1)x^{-2\theta-1}\sin(\pi\theta)}{\sum_{y=1}^{\infty}\left(\Gamma(\theta+1)y^{-\theta-1}\sin(\frac{\pi\theta}{2}) - \frac{1}{2}\Gamma(2\theta+1)y^{-2\theta-1}\sin(\pi\theta)\right)}. \tag{10.4}$$

The pmf of T-2SDSD when $0 < \theta < 2$, $0 < \beta < 1$, and $x = 1, 2, \ldots$, is given by [8]

$$p_x(\theta, \beta) = \frac{\Gamma(\theta+1)x^{-\theta-1}\sin(\frac{\pi\theta(1+\beta)}{2}) - \frac{1}{2}\Gamma(2\theta+1)x^{-2\theta-1}\sin(\pi\theta(1+\beta))}{\sum_{y=1}^{\infty}\left(\Gamma(\theta+1)y^{-\theta-1}\sin(\frac{\pi\theta(1+\beta)}{2}) - \frac{1}{2}\Gamma(2\theta+1)y^{-2\theta-1}\sin(\pi\theta(1+\beta))\right)}. \tag{10.5}$$

Code availability. All computational, fitting and simulation studies have been done using R statistical software. The R codes are available from the author upon request.

Acknowledgements. The author is grateful to the anonymous reviewers and the handling editor for their valuable comments and suggestions, which definitely improved the quality and presentation of the paper.

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Поступила 19 мая 2023

После доработки 02 июня 2023

Принята к публикации 04 октября 2023

Известия НАН Армении, Математика, том 59, н. 2, 2024, стр. 35 – 55.

UNIQUENESS OF MEROMORPHIC FUNCTIONS WITH RESPECT TO THEIR SHIFTS CONCERNING DERIVATIVES

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Abstract. An example in the article shows that the first derivative of $f(z)=\frac{2}{1-e^{-2z}}$ sharing 0 CM and $1,\infty$ IM with its shift πi cannot obtain they are equal. In this paper, we study the uniqueness of meromorphic function sharing small functions with their shifts concerning its $k-th$ derivatives. We use a different method from Qi and Yang [18] to improves entire function to meromorphic function, the first derivative to the $k-th$ derivatives, and also finite values to small functions. As for $k=0$, we obtain: Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f)<1$, let c be a nonzero finite value, and let $a(z)\not\equiv\infty, b(z)\not\equiv\infty\in\hat{S}(f)$ be two distinct small functions of $f(z)$ such that $a(z)$ is a periodic function with period c and $b(z)$ is any small function of $f(z)$. If $f(z)$ and $f(z+c)$ share $a(z),\infty$ CM, and share $b(z)$ IM, then either $f(z)\equiv f(z+c)$ or

$$e^{p(z)}\equiv\frac{f(z+c)-a(z+c)}{f(z)-a(z)}\equiv\frac{b(z+c)-a(z+c)}{b(z)-a(z)},$$

where $p(z)$ is a non-constant entire function of $\rho(p)<1$ such that $e^{p(z+c)}\equiv e^{p(z)}$.

MSC2020 numbers: 30D35; 39A46.

Keywords: meromorphic functions; shifts; derivatives; small functions.

1. Introduction and main results

Throughout this paper, we assume that the reader have a knowledge of the fundamental results and the standard notations of the Nevanlinna value distribution theory. See([6, 20, 21]). In the following, a meromorphic function f means meromorphic in the whole complex plane. Define

$$\rho(f)=\varlimsup_{r\to\infty}\frac{log^{+}T(r,f)}{logr},$$

$$\rho_2(f)=\varlimsup_{r\to\infty}\frac{log^{+}log^{+}T(r,f)}{logr}$$

by the order and the hyper-order of f, respectively. When $\rho(f)<\infty$, we say f is of finite order.

By $S(r,f)$, we denote any quantity satisfying $S(r,f)=o(T(r,f))$, as $r\to\infty$ outside of a possible exceptional set of finite logarithmic measure. A meromorphic function $a(z)$ satisfying $T(r,a)=S(r,f)$ is called a small function of f. We denote $S(f)$ as the family of all small meromorphic functions of f which includes the constants in $\mathbb{C}$. Moreover, we define $\hat{S}(f)=S(f)\cup\{\infty\}$. We say that two non-constant meromorphic functions f and g share small function a CM(IM) if $f-a$ and

$g-a$ have the same zeros counting multiplicities (ignoring multiplicities). Moreover, we introduce the following notation: $S_{(m,n)}(a) = \{z|z$ is a common zero of $f(z+c)-a(z)$ and $f(z)-a(z)$ with multiplicities m and n respectively$\}$. $\overline{N}_{(m,n)}(r,\frac{1}{f-a})$ denotes the counting function of f with respect to the set $S_{(m,n)}(a)$. $\overline{N}_{n)}(r,\frac{1}{f-a})$ denotes the counting function of all distinct zeros of $f-a$ with multiplicities at most n. $\overline{N}_{(n}(r,\frac{1}{f-a})$ denotes the counting function of all zeros of $f-a$ with multiplicities at least n.

We say that two non-constant meromorphic functions f and g share small function a CM(IM)almost if

$$N(r,\frac{1}{f-a})+N(r,\frac{1}{g-a})-2N(r,f=a=g)=S(r,f)+S(r,g),$$

or

$$\overline{N}(r,\frac{1}{f-a})+\overline{N}(r,\frac{1}{g-a})-2\overline{N}(r,f=a=g)=S(r,f)+S(r,g),$$

respectively.

For a meromorphic function $f(z)$, we denote its shift by $f_c(z)=f(z+c)$.

Rubel and Yang [19] studied the uniqueness of an entire function concerning its first order derivative, and proved the following result.

Theorem A. Let $f(z)$ be a non-constant entire function, and let a,b be two finite distinct complex values. If $f(z)$ and $f'(z)$ share a,b CM, then $f(z)\equiv f'(z)$.

Zheng and Wang [23] improved Theorem A and proved

Theorem B. Let $f(z)$ be a non-constant entire function, and let $a(z)\not\equiv\infty, b(z)\not\equiv\infty$ be two distinct small functions of $f(z)$. If $f(z)$ and $f^{(k)}(z)$ share $a(z),b(z)$ CM, then $f(z)\equiv f^{(k)}(z)$.

Li and Yang [15] improved Theorem B and proved

Theorem C. Let $f(z)$ be a non-constant entire function, and let $a(z)\not\equiv\infty, b(z)\not\equiv\infty$ be two distinct small functions of $f(z)$. If $f(z)$ and $f^{(k)}(z)$ share $a(z)$ CM, and share $b(z)$ IM. Then $f(z)\equiv f^{(k)}(z)$.

Recently, the value distribution of meromorphic functions concerning difference analogue has become a popular research, see [1, 2, 4 – 9, 12 – 14, 16 – 18]. Heittokangas et al [7] obtained a similar result analogue of Theorem A concerning shifts.

Theorem D. Let $f(z)$ be a non-constant entire function of finite order, let c be a nonzero finite complex value, and let a,b be two finite distinct complex values. If $f(z)$ and $f(z+c)$ share a,b CM, then $f(z)\equiv f(z+c)$.

In [17], Qi-Li-Yang investigated the value sharing problem with respect to $f'(z)$ and $f(z+c)$. They proved

Theorem E. Let $f(z)$ be a non-constant entire function of finite order, and let a, c be two nonzero finite complex values. If $f'(z)$ and $f(z+c)$ share $0, a$ CM, then $f'(z) \equiv f(z+c)$.

Recently, Qi and Yang [18] improved Theorem E and proved

Theorem F. Let $f(z)$ be a non-constant entire function of finite order, and let a, c be two nonzero finite complex value. If $f'(z)$ and $f(z+c)$ share 0 CM and a IM, then $f'(z) \equiv f(z+c)$.

Of above theorem, it's naturally to ask whether the condition $0, a$ can be replaced by two distinct small functions, and f' can be replaced by $f^{(k)}$?

In this article, we give a positive answer. In fact, we prove the following more general result.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$, let c be a nonzero finite value, k be a positive integer, and let $a(z) \not\equiv \infty, b(z) \not\equiv \infty \in \hat{S}(f)$ be two distinct small functions. If $f^{(k)}(z)$ and $f(z+c)$ share $a(z), \infty$ CM, and share $b(z)$ IM, then $f^{(k)}(z) \equiv f(z+c)$.*

Example 1.1. [9] *Let $f(z) = \frac{2}{1-e^{-2z}}$, and let $c = \pi i$. Then $f'(z)$ and $f(z+c)$ share 0 CM and share $1, \infty$ IM, but $f'(z) \not\equiv f(z+c)$.*

This example shows that for meromorphic functions, the conclusion of Theorem 1 doesn't hold even when sharing ∞ CM is replaced by sharing ∞ IM when $k = 1$. We believe there are examples for any k, but we can not construct them.

As for $k = 0$, Li and Yi [13] obtained

Theorem G. Let $f(z)$ be a transcendental entire function of $\rho_2(f) < 1$, let c be a nonzero finite value, and let $a(z) \not\equiv \infty, b(z) \not\equiv \infty \in \hat{S}(f)$ be two distinct small functions. If $f(z)$ and $f(z+c)$ share $a(z)$ CM, and share $b(z)$ IM, then $f(z) \equiv f(z+c)$.

Remark 1.1. *Theorem G holds when $f(z)$ is a non-constant meromorphic function of $\rho_2(f) < 1$ such that $N(r, f) = S(r, f)$.*

Theorem H. [8] Let $f(z)$ be a non-constant meromorphic function of finite order, let c be a nonzero finite value, and let $a(z) \not\equiv \infty$, $b(z) \not\equiv \infty$ and $d(z) \not\equiv \infty \in \hat{S}(f)$ be three distinct small functions such that $a(z)$, $b(z)$ and $d(z)$ are periodic functions with period c. If $f(z)$ and $f(z+c)$ share $a(z), b(z)$ CM, and $d(z)$ IM, then $f(z) \equiv f(z+c)$.

We can ask a question that whether the small periodic function $d(z)$ of $f(z)$ can be replaced by any small function of $f(z)$?

In this paper, we obtain our second result.

Theorem 1.2. *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$, let c be a nonzero finite value, and let $a(z) \not\equiv \infty, b(z) \not\equiv \infty \in \hat{S}(f)$ be two distinct small functions of $f(z)$ such that $a(z)$ is a periodic function with period c and $b(z)$ is a small function of $f(z)$. If $f(z)$ and $f(z+c)$ share $a(z), \infty$ CM, and share $b(z)$ IM, then either $f(z) \equiv f(z+c)$ or*

$$e^{p(z)} \equiv \frac{f(z+c) - a(z+c)}{f(z) - a(z)} \equiv \frac{b(z+c) - a(z+c)}{b(z) - a(z)},$$

where $p(z)$ is a non-constant entire function of $\rho(p) < 1$ such that $e^{p(z+c)} \equiv e^{p(z)}$.

We can obtain the following corollary from the proof of Theorem 1.2.

Corollary 1.1. *Under the same condition as in Theorem 2, then $f(z) \equiv f(z+c)$ holds if one of conditions satisfies*
(i) $b(z)$ is a periodic function with period nc ;
(ii) $\rho(b(z)) < \rho(e^{p(z)})$;
(iii) $\rho(b(z)) < 1$.

Example 1.2. *Let $f(z) = \frac{e^z}{1-e^{-2z}}$, and let $c = \pi i$. Then $f(z+c) = \frac{-e^z}{1-e^{-2z}}$, and $f(z)$ and $f(z+c)$ share $0, \infty$ CM, but $f(z) \not\equiv f(z+c)$.*

Example 1.3. *Let $f(z) = e^z$, and let $c = \pi i$. Then $f(z+c) = -e^z$, and $f(z)$ and $f(z+c)$ share $0, \infty$ CM, $f(z)$ and $f(z+c)$ attain different values everywhere in the complex plane, but $f(z) \not\equiv f(z+c)$.*

Above two examples of show that "2CM+1IM"is necessary.

Example 1.4. *Let $f(z) = e^{e^z}$, then $f(z+\pi i) = \frac{1}{e^{e^z}}$. It is easy to verify that $f(z)$ and $f(z+\pi i)$ share $0, 1, \infty$ CM, but $f(z) = \frac{1}{f(z+\pi i)}$. On the other hand, we obtain $f(z) = f(z+2\pi i)$.*

Example 1.4 tells us that if we drop the assumption $\rho_2(f) < 1$, we can get another relation.

By Theorem 1.1 and Theorem 1.2, we still believe the latter situation of Theorem 2 can be removed, that is to say, only the case $f(z) \equiv f(z+c)$ occurs. So we raise a conjecture here.

Conjecture. Under the same condition as in Theorem 1.2, is $f(z) \equiv f(z+c)$?

2. Some lemmas

Lemma 2.1. [6] *Let f be a non-constant meromorphic function of $\rho_2(f) < 1$, and let c be a non-zero complex number. Then*

$$m(r, \frac{f(z+c)}{f(z)}) = S(r, f),$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Lemma 2.2. [10, 20, 21] *Let f_1 and f_2 be two non-constant meromorphic functions in $|z|<\infty$, then*

$$N(r,f_1f_2)-N(r,\frac{1}{f_1f_2})=N(r,f_1)+N(r,f_2)-N(r,\frac{1}{f_1})-N(r,\frac{1}{f_2}),$$

where $0<r<\infty$.

Lemma 2.3. [6] *Let f be a non-constant meromorphic function of $\rho_2(f)<1$, and let c be a non-zero complex number. Then*

$$T(r,f(z))=T(r,f(z+c))+S(r,f),$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Lemma 2.4. *Let f be a transcendental meromorphic function of $\rho_2(f)<1$ such that $\overline{N}(r,f)=S(r,f)$, let c be a nonzero constant, k be a positive integer, and let $a(z)$ be a small function of $f(z+c)$ and $f^{(k)}(z)$. If $f(z+c)$ and $f^{(k)}(z)$ share $a(z),\infty$ CM, and $N(r,\frac{1}{f^{(k)}(z+c)-a^{(k)}(z)})=S(r,f)$, then $T(r,e^p)=S(r,f)$, where p is an entire function of order less than 1.*

Proof. Since f is a transcendental meromorphic function of $\rho_2(f)<1$, $\overline{N}(r,f)=S(r,f)$, and f_c and $f^{(k)}$ share a and ∞ CM, then there is an entire function p of order less than 1 such that

$$f_c-a=e^p(f^{(k)}-a_{-c}^{(k)})+e^p(a_{-c}^{(k)}-a). \tag{2.1}$$

Suppose on the contrary that $T(r,e^p)\not\equiv S(r,f)$.
Set $g=f_c^{(k)}-a^{(k)}$. Differentiating (2.1) k times we have

$$g=(e^p)^{(k)}g_{-c}+k(e^p)^{(k-1)}g'_{-c}+\cdots+k(e^p)'g_{-c}^{(k-1)}+e^pg_{-c}^{(k)}+B^{(k)}, \tag{2.2}$$

where $B=e^p(a_{-c}^{(k)}-a)$.
It is easy to see that $g\not\equiv 0$. Then we rewrite (2.2) as

$$1-\frac{B^{(k)}}{g}=De^p, \tag{2.3}$$

where

$$\begin{aligned}D=e^{-p}[(e^p)^{(k)}\frac{g_{-c}}{g}+k(e^p)^{(k-1)}\frac{g'_{-c}}{g}+\cdots\\ +k(e^p)'\frac{g_{-c}^{(k-1)}}{g}+(e^p)\frac{g_{-c}^{(k)}}{g}].\end{aligned} \tag{2.4}$$

Since f is a transcendental meromorphic function with $\rho_2(f)<1$ and $f^{(k)}$ and f_c share ∞ CM, we can see from $\overline{N}(r,f)=S(r,f)$, Lemma 2.1 and Lemma 2.3 that

$$(1+o(1))N(r,f)+S(r,f)=N(r,f_c)=N(r,f^{(k)}),$$

and on the other hand

$$k\overline{N}(r,f_c)+N(r,f_c)=N(r,f_c^{(k)}),\overline{N}(r,f_c)=\overline{N}(r,f^{(k)})=\overline{N}(r,f),$$

which follows from above equalities that $N(r,f^{(k)}) = N(r,f_c^{(k)}) + S(r,f)$, and thus we can know that g and g_{-c} share ∞ CM almost. It is easy to see from the assumption f_c and $f^{(k)}$ share ∞ CM that there exists no simple pole point of f_c. Now we estimate $N(r,\frac{g_{-c}^{(i)}}{g})$. Let z_0 be a pole of f with multiplicity n, than z_0 is a pole of g with multiplicity $n+2k$, and also z_0 is a pole of $g_{-c}^{(i)}$ with multiplicity $n+k+i$. Then we can see that z_0 is a zero point of $\frac{g_{-c}^{(i)}}{g}$ with $k-i$. Let z_1 be a pole of f_c with multiplicity m, then z_1 is a pole of g with multiplicity $m+k$, and also z_1 is a pole of $g_{-c}^{(i)}$ with multiplicity $m+i$. Then we can see that z_1 is a zero point of $\frac{g_{-c}^{(i)}}{g}$ with $k-i$. Note that $N(r,\frac{1}{f_c^{(k)}-a^{(k)}}) = N(r,\frac{1}{g}) = S(r,f)$, then $N(r,\frac{g_{-c}^{(i)}}{g}) = S(r,f)$, and hence

$$\begin{aligned}T(r,D) &\le \sum_{i=0}^{k}(T(r,\frac{(e^p)^{(i)}}{e^p})+T(r,\frac{C_k^i g_{-c}^{(k-i)}}{g}))+S(r,f)\\ &\le \sum_{i=0}^{k}(S(r,e^p)+m(r,\frac{g_{-c}^{(i)}}{g_{-c}})+N(r,\frac{g_{-c}^{(i)}}{g}))+S(r,f)\\ &= S(r,e^p)+S(r,f),\end{aligned}\tag{2.5}$$

where C_k^i is a combinatorial number. By (2.1) and Lemma 2.1, we get

$$T(r,e^p)\le T(r,f_c)+T(r,f^{(k)})+S(r,f)\le 2T(r,f)+S(r,f).\tag{2.6}$$

Then it follows from (2.5) that $T(r,D)=S(r,f)$. Next we discuss two cases.

Case 1. $e^{-p}-D\not\equiv 0$. Rewrite (2.3) as

$$ge^p(e^{-p}-D)=B^{(k)}.\tag{2.7}$$

We claim that $D\equiv 0$. Otherwise, using the Lemma 2.8 to e^{-p}, we get

$$\begin{aligned}&m(r,\frac{1}{e^{-p}-D})+N(r,\frac{1}{e^{-p}-D})=T(r,e^{-p})\\ &\le \overline{N}(r,e^{-p})+\overline{N}(r,\frac{1}{e^{-p}})+\overline{N}(r,\frac{1}{e^{-p}-D})\\ &+S(r,e^p)=\overline{N}(r,\frac{1}{e^{-p}-D})+S(r,f)\le T(r,e^{-p})+S(r,f),\end{aligned}$$

that is to say

$$T(r,e^p)=T(r,e^{-p})+O(1)=\overline{N}(r,\frac{1}{e^{-p}-D})+S(r,f)$$

and

$$N(r,\frac{1}{e^{-p}-D})=N_1(r,\frac{1}{e^{-p}-D})+S(r,f).$$

It follows form above two equalities that

$$T(r,e^p)=N_1(r,\frac{1}{e^{-p}-D})+S(r,f).$$

Because the numbers of zeros and poles of $B^{(k)}$ are $S(r,f)$, we can see from (2.7) and $\overline{N}(r,f)=S(r,f)$ that the multiplicities of poles of g are almost 1. And then

$$\begin{aligned}N(r,f)+k\overline{N}(r,f)&=N(r,g)+S(r,f)=N(r,\frac{1}{e^{-p}-D})+S(r,f)\\&=N_1(r,f)+S(r,f)\leq\overline{N}(r,f)+S(r,f)=S(r,f).\end{aligned}$$

it follows from above that $\overline{N}(r,\frac{1}{e^{-p}-D})=S(r,f)$. Then by Lemma 2.8 in the following we can obtain

$$\begin{aligned}T(r,e^p)&=T(r,e^{-p})+O(1)\\&\leq\overline{N}(r,e^{-p})+\overline{N}(r,\frac{1}{e^{-p}})+\overline{N}(r,\frac{1}{e^{-p}-D})\\&+S(r,e^p)=S(r,f),\end{aligned}\tag{2.8}$$

which contradicts with present assumption. Thus $D\equiv 0$. Then by (2.7) we get

$$g=B^{(k)}.\tag{2.9}$$

Integrating (2.9), we get

$$f_c=e^p(a_{-c}^{(k)}-a)+P+a,\tag{2.10}$$

where P is a polynomial of degree at most $k-1$. (2.10) implies

$$T(r,f_c)=T(r,e^p)+S(r,f).\tag{2.11}$$

Substituting (2.9) and (2.10) into (2.1) we can obtain

$$e^p(a_{-c}^{(k)}-a)+P=e^{p+p_{-c}}L_{-c},\tag{2.12}$$

where L_{-c} is the differential polynomial in

$$p'_{-c},\ldots,p_{-c}^{(k)},a_{-2c}-a_{-c},(a_{-2c}-a_{-c})',\ldots,(a_{-2c}-a_{-c})^{(k)},$$

and it is a small function of $f(z+c)$. On the one hand

$$2T(r,e^p)=T(r,e^{2p})=m(r,e^{2p})\leq m(r,e^{p+p_{-c}})+m(r,\frac{e^p}{e^{p_{-c}}})\leq T(r,e^{p+p_{-c}})+S(r,f).\tag{2.13}$$

On the other hand, we can prove similarly that

$$T(r,e^{p+p_{-c}})\leq 2T(r,e^p)+S(r,f).\tag{2.14}$$

So

$$T(r,e^{p+p_{-c}})=2T(r,e^p)+S(r,f).\tag{2.15}$$

By (2.11), (2.12) and (2.15) we can get $T(r,e^p)=2T(r,e^p)+S(r,f)$, which is $T(r,e^p)=S(r,f)$, a contradiction.

Case 2. $e^{-p} - D \equiv 0$. Immediately, we get $T(r, e^p) = S(r, f)$, but it's impossible.

Of above discussions, we conclude that $T(r, e^p) = S(r, f)$. □

Lemma 2.5. *Let f be a transcendental meromorphic function of $\rho_2(f) < 1$ such that $\overline{N}(r, f) = S(r, f)$, let k be a positive integer and $c \neq 0$ a complex value, and let $a \not\equiv \infty$ and $b \not\equiv \infty$ be two distinct small functions of f. Suppose*

$$L(f_c) = \begin{vmatrix} f_c - a & a - b \\ f_c' - a' & a' - b' \end{vmatrix}$$

and

$$L(f^{(k)}) = \begin{vmatrix} f^{(k)} - a & a - b \\ f^{(k+1)} - a' & a' - b' \end{vmatrix},$$

and f_c and $f^{(k)}$ share a, ∞ CM, and share b IM, then $L(f_c) \not\equiv 0$ and $L(f^{(k)}) \not\equiv 0$.

Proof. Suppose that $L(f_c) \equiv 0$, then we can get $\frac{f_c' - a'}{f_c - a} \equiv \frac{a' - b'}{a - b}$. Integrating both side of above we can obtain $f_c - a = C_1(a - b)$, where C_1 is a nonzero constant. So by Lemma 2.3, we have $T(r, f) = T(r, f_c) + S(r, f) = T(r, C(a - b) + a) = S(r, f)$, a contradiction. Hence $L(f_c) \not\equiv 0$.

Since $f^{(k)}$ and f_c share a CM and b IM, and f is a transcendental meromorphic function of $\rho_2(f) < 1$ such that $\overline{N}(r, f) = S(r, f)$, then by the Lemma 2.8, we get

$$\begin{aligned} T(r, f_c) &\leq \overline{N}(r, \frac{1}{f_c - a}) + \overline{N}(r, \frac{1}{f_c - b}) + \overline{N}(r, f_c) + S(r, f) \\ &= \overline{N}(r, \frac{1}{f^{(k)} - a}) + \overline{N}(r, \frac{1}{f^{(k)} - b}) + S(r, f) \\ &\leq 2T(r, f^{(k)}) + S(r, f). \end{aligned} \tag{2.16}$$

Hence a and b are small functions of $f^{(k)}$. If $L(f^{(k)}) \equiv 0$, then we can get $f^{(k)} - a = C_2(a - b)$, where C_2 is a nonzero constant. And we get $T(r, f^{(k)}) = S(r, f^{(k)})$. Combing (2.16) we obtain $T(r, f) = T(r, f_c) + S(r, f) = T(r, C(a - b) + a) = S(r, f)$, a contradiction. □

Lemma 2.6. *Let f be a transcendental meromorphic function, let $k_j (j = 1, 2, \ldots, q)$ be distinct constants, and let $a \not\equiv \infty$ and $b \not\equiv \infty$ be two distinct small functions of f . Again let $d_j = a - k_j(a - b)$ $(j = 1, 2, \ldots, q)$. Then*

$$m(r, \frac{L(f_c)}{f_c - a}) = S(r, f), \quad m(r, \frac{L(f_c)}{f_c - d_j}) = S(r, f).$$

for $1 \leq i \leq q$ and

$$m(r, \frac{L(f_c) f_c}{(f_c - d_1)(f_c - d_2) \cdots (f_c - d_m)}) = S(r, f),$$

where $L(f_c)$ is defined as in Lemma 2.5, and $2 \leq m \leq q$.

Proof. Obviously, we have

$$m(r, \frac{L(f_c)}{f_c - a}) \leq m(r, \frac{(a' - b')(f_c - a)}{f_c - a}) + m(r, \frac{(a - b)(f'_c - a')}{f_c - a}) = S(r, f),$$

and

$$\frac{L(f_c)f_c}{(f_c - d_1)(f_c - d_2)\cdots(f_c - d_q)} = \sum_{i=1}^{q} \frac{C_i L(f_c)}{f_c - d_i},$$

where $C_i = \frac{d_j}{\prod\limits_{j\neq i}(d_i - d_j)}$ are small functions of f. By Lemma 2.1 and above, we have

$$m(r, \frac{L(f_c)f_c}{(f_c - d_1)(f_c - d_2)\cdots(f_c - d_q)}) = m(r, \sum_{i=1}^{q} \frac{C_i L(f_c)}{f_c - d_i})$$

$$\leq \sum_{i=1}^{q} m(r, \frac{L(f_c)}{f_c - d_i}) + S(r, f) = S(r, f). \tag{2.17}$$

□

Lemma 2.7. *Let f and g be are two non-constant meromorphic functions such that $\overline{N}(r, f) = S(r, f)$, and let $a \not\equiv \infty$ and $b \not\equiv \infty$ be two distinct small functions of f and g. If*

$$H = \frac{L(f)}{(f - a)(f - b)} - \frac{L(g)}{(g - a)(g - b)} \equiv 0,$$

where

$$L(f) = (a' - b')(f - a) - (a - b)(f' - a')$$

and

$$L(g) = (a' - b')(g - a) - (a - b)(g' - a').$$

And if f and g share a, ∞ CM, and share b IM, then either $2T(r, f) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{f-b}) + S(r, f)$, or $f = g$.

Proof. Integrating H which leads to

$$\frac{g - b}{g - a} = C\frac{f - b}{f - a},$$

where C is a nonzero constant.

If $C = 1$, then $f = g$. If $C \neq 1$, then from above, we have

$$\frac{a - b}{g - a} \equiv \frac{(C - 1)f - Cb + a}{f - a},$$

and

$$T(r, f) = T(r, g) + S(r, f) + S(r, g).$$

It follows that $N(r, \frac{1}{f - \frac{Cb-a}{C-1}}) = N(r, \frac{1}{a-b}) = S(r, f)$. Then by Lemma 2.8 in the following,

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f - a}) + \overline{N}(r, \frac{1}{f - \frac{Cb-a}{C-1}}) + S(r, f)$$

$$\leq \overline{N}(r, \frac{1}{f - a}) + S(r, f) \leq T(r, f) + S(r, f),$$

and

$$T(r,f) \leq \overline{N}(r,f) + \overline{N}(r,\frac{1}{f-b}) + \overline{N}(r,\frac{1}{f-\frac{Cb-a}{C-1}}) + S(r,f)$$
$$\leq \overline{N}(r,\frac{1}{f-b}) + S(r,f) \leq T(r,f) + S(r,f),$$

that is $T(r,f) = \overline{N}(r,\frac{1}{f-a}) + S(r,f)$ and $T(r,f) = \overline{N}(r,\frac{1}{f-b}) + S(r,f)$, and hence $2T(r,f) = \overline{N}(r,\frac{1}{f-a}) + \overline{N}(r,\frac{1}{f-b}) + S(r,f)$. □

Lemma 2.8. [22] *Let $f(z)$ be a non-constant meromorphic function, and let $a_j \in \hat{S}(f)$ be q distinct small functions for all $j = 1, 2, \ldots, q$. Then*

$$(q-2-\epsilon)T(r,f) \leq \sum_{j=1}^{q} \overline{N}(r,\frac{1}{f-a_j}) + S(r,f), r \notin E,$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Remark 2.1. *Lemma 2.8 is true when $\infty, a_1, a_2, \cdots, a_q \in \hat{S}(f)$ with $S(r,f)$ in our notation, in other words, even if exceptional sets are of infinite linear measure. But they are not of infinite logarithmic measure.*

Lemma 2.9. [11] *Let f and g be two non-constant meromorphic functions. If f and g share $0, 1, \infty$ IM, and f is a bilinear transformation of g, then f and g assume one of the following six relations: (i) $fg = 1$; (ii) $(f-1)(g-1) = 1$; (iii) $f + g = 1$; (iv) $f = cg$; (v) $f - 1 = c(g-1)$; (vi) $[(c-1)f+1][(c-1)g-c] = -c$, where $c \neq 0, 1$ is a complex number.*

Lemma 2.10. [3] *Let f, F and g be three non-constant meromorphic functions, where $g = F(f)$. Then f and g share three values IM if and only if there exist an entire function h such that, by a suitable linear fractional transformation, one of the following cases holds:*

(i) $f \equiv g$;

(ii) $f = e^h$ and $g = a(1 + 4ae^{-h} - 4a^2e^{-2h})$ have three IM shared values $a \neq 0$, $b = 2a$ and ∞;

(iii) $f = e^h$ and $g = \frac{1}{2}(e^h + a^2e^{-h})$ have three IM shared values $a \neq 0$, $b = -a$ and ∞;

(iv) $f = e^h$ and $g = a + b - abe^{-h}$ have three IM shared values $ab \neq 0$ and ∞;

(v) $f = e^h$ and $g = \frac{1}{b}e^{2h} - 2e^h + 2b$ have three IM shared values $b \neq 0$, $a = 2b$ and ∞;

(vi) $f = e^h$ and $g = b^2e^{-h}$ have three IM shared values $a \neq 0$, 0 and ∞.

Lemma 2.11. [10, 20, 21] *Let f and g be two non-constant meromorphic functions, and let $\rho(f)$ and $\rho(g)$ be the order of f and g, respectively. Then $\rho(fg) \leq \max\{\rho(f), \rho(g)\}$.*

Remark 2.2. *We can see from the proof that Lemma 2.9* [11] *and Lemma 2,10* [20] *are still true when f and g share three value IM almost.*

3. The proof of Theorem 1.1

If $f_c \equiv f^{(k)}$, there is nothing to prove. Suppose $f_c \not\equiv f^{(k)}$. Since f is a non-constant meromorphic function of $\rho_2(f) < 1$, f_c and $f^{(k)}$ share a, ∞ CM, then we get

$$\frac{f^{(k)} - a}{f_c - a} = e^h, \tag{3.1}$$

where h is an entire function, and it is easy to know from (2.1) that $h = -p$.

Since f is a transcendental meromorphic function of $\rho_2(f) < 1$ and $f^{(k)}$ and f_c share ∞ CM, we can see from Lemma 2.1 and Lemma 2.3 that

$$(1 + o(1))N(r, f) + S(r, f) = N(r, f_c) = N(r, f^{(k)}),$$

which implies

$$\overline{N}(r, f) = S(r, f).$$

Furthermore, from the assumption that $f^{(k)}$ and f_c share a and ∞ CM and b IM, then by Lemma 2.1, Lemma 2.8 and above equality, we get

$$\begin{aligned}
T(r, f_c) &\leq \overline{N}(r, \frac{1}{f_c - a}) + \overline{N}(r, \frac{1}{f_c - b}) + \overline{N}(r, f_c) + S(r, f) \\
&= \overline{N}(r, \frac{1}{f^{(k)} - a}) + \overline{N}(r, \frac{1}{f^{(k)} - b}) + S(r, f) \\
&\leq N(r, \frac{1}{f_c - f^{(k)}}) + S(r, f) \leq T(r, f_c - f^{(k)}) + S(r, f) \\
&\leq m(r, f_c - f^{(k)}) + N(r, f_c - f^{(k)}) + S(r, f) \\
&\leq m(r, f_c) + m(r, 1 - \frac{f^{(k)}}{f_c}) + N(r, f_c) + S(r, f) \leq T(r, f_c) + S(r, f).
\end{aligned}$$

That is

$$T(r, f_c) = \overline{N}(r, \frac{1}{f_c - a}) + \overline{N}(r, \frac{1}{f_c - b}) + S(r, f). \tag{3.2}$$

By (3.1) and (3.2) we have

$$T(r, f_c) = T(r, f_c - f^{(k)}) + S(r, f) = N(r, \frac{1}{f_c - f^{(k)}}) + S(r, f). \tag{3.3}$$

and by Lemma 2.1,

$$\begin{aligned}
T(r, e^h) = m(r, e^h) &= m(r, \frac{f^{(k)} - a_{-c}^{(k)} + a_{-c}^{(k)} - a}{f_c - a}) \leq m(r, \frac{a_{-c}^{(k)} - a}{f_c - a}) \\
&+ m(r, \frac{f^{(k)} - a_{-c}^{(k)}}{f_c^{(k)} - a^{(k)}}) + m(r, \frac{f_c^{(k)} - a^{(k)}}{f_c - a}) \leq m(r, \frac{1}{f_c - a}) + S(r, f).
\end{aligned} \tag{3.4}$$

Then it follows from (3.1) and (3.3) that

(3.5)

$$m(r,\frac{1}{f_c-a})=m(r,\frac{e^h-1}{f^{(k)}-f_c})\leq m(r,\frac{1}{f^{(k)}-f_c})+m(r,e^h-1)\leq T(r,e^h)+S(r,f).$$

Then by (3.4) and (3.5)

$$T(r,e^h)=m(r,\frac{1}{f_c-a})+S(r,f). \tag{3.6}$$

On the other hand, (3.1) can be rewritten as

$$\frac{f^{(k)}-f_c}{f_c-a}=e^h-1, \tag{3.7}$$

which implies

$$\overline{N}(r,\frac{1}{f_c-b})\leq\overline{N}(r,\frac{1}{e^h-1})+S(r,f)=T(r,e^h)+S(r,f). \tag{3.8}$$

Thus, by (3.2), (3.6) and (3.8)

$$\begin{aligned} m(r,\frac{1}{f_c-a})+N(r,\frac{1}{f_c-a}) &= \overline{N}(r,\frac{1}{f_c-a})+\overline{N}(r,\frac{1}{f_c-b})+S(r,f)\\ &\leq \overline{N}(r,\frac{1}{f_c-a})+\overline{N}(r,\frac{1}{e^h-1})+S(r,f)\\ &\leq \overline{N}(r,\frac{1}{f_c-a})+m(r,\frac{1}{f_c-a})+S(r,f), \end{aligned}$$

which implies

$$N(r,\frac{1}{f_c-a})=\overline{N}(r,\frac{1}{f_c-a})+S(r,f). \tag{3.9}$$

And then

$$\overline{N}(r,\frac{1}{f_c-b})=T(r,e^h)+S(r,f). \tag{3.10}$$

Set

$$\varphi=\frac{L(f_c)(f_c-f^{(k)})}{(f_c-a)(f_c-b)}, \tag{3.11}$$

and

$$\psi=\frac{L(f^{(k)})(f_c-f^{(k)})}{(f^{(k)}-a)(f^{(k)}-b)}. \tag{3.12}$$

It is easy to know that $\varphi\not\equiv 0$ because of Lemma 2.5 and $f\not\equiv f^{(k)}$. We know that $N(r,\varphi)\leq\overline{N}(r,f)=S(r,f)$ by (3.11). By Lemma 2.1 and Lemma 2.6 we have

$$\begin{aligned} T(r,\varphi)=m(r,\varphi)+N(r,\varphi) &= m(r,\frac{L(f_c)(f_c-f^{(k)})}{(f_c-a)(f_c-b)})+S(r,f)\\ &\leq m(r,\frac{L(f_c)f_c}{(f_c-a)(f_c-b)})+m(r,1-\frac{f^{(k)}}{f_c})+S(r,f)=S(r,f), \end{aligned}$$

that is

$$T(r,\varphi)=S(r,f). \tag{3.13}$$

Let $d = a - j(a-b)(j \neq 0,1)$. Obviously, by Lemma 2.1 and Lemma 2.6, we obtain

$$\begin{aligned} m(r,\frac{1}{f_c}) &= m(r,\frac{1}{(b-a)\varphi}(\frac{L(f_c)}{f_c-a}-\frac{L(f_c)}{f_c-b})(1-\frac{f^{(k)}}{f_c})) \\ &\leq m(r,\frac{1}{\varphi})+m(r,\frac{L(f_c)}{f_c-a}-\frac{L(f_c)}{f_c-b}) \\ &+ m(r,1-\frac{f^{(k)}}{f_c})+S(r,f)=S(r,f). \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} m(r,\frac{1}{f_c-d}) &= m(r,\frac{L(f_c)(f_c-f^{(k)})}{\varphi(f_c-a)(f_c-b)(f_c-d)}) \\ &\leq m(r,1-\frac{f^{(k)}}{f_c})+m(r,\frac{L(f_c)f_c}{(f_c-a)(f_c-b)(f_c-d)}) \\ &+ S(r,f)=S(r,f). \end{aligned} \tag{3.15}$$

Set

$$\phi = \frac{L(f_c)}{(f_c-a)(f_c-b)} - \frac{L(f^{(k)})}{(f^{(k)}-a)(f^{(k)}-b)}. \tag{3.16}$$

We discuss two cases.

Case 1 $\phi \equiv 0$. Integrating the both sides of (3.16) which leads to

$$\frac{f_c-a}{f_c-b} = C\frac{f^{(k)}-a}{f^{(k)}-b}, \tag{3.17}$$

where C is a nonzero constant. Then by Lemma 2.7 we get

$$2T(r,f_c) = \overline{N}(r,\frac{1}{f_c-a})+\overline{N}(r,\frac{1}{f_c-b})+S(r,f), \tag{3.18}$$

which contradicts with (3.2).

Case 2 $\phi \not\equiv 0$. By (3.3), (3.13) and (3.16) we can obtain

$$\begin{aligned} T(r,f_c) &= T(r,f_c-f^{(k)})+S(r,f) = T(r,\frac{\phi(f_c-f^{(k)})}{\phi})+S(r,f) \\ &= T(r,\frac{\varphi-\psi}{\phi})+S(r,f) \leq T(r,\varphi-\psi)+T(r,\phi)+S(r,f) \\ &\leq T(r,\psi)+T(r,\phi)+S(r,f) \leq T(r,\psi)+\overline{N}(r,\frac{1}{f_c-b})+S(r,f). \end{aligned} \tag{3.19}$$

On the other hand,

$$\begin{aligned} T(r,\psi) &= T(r,\frac{L(f^{(k)})(f_c-f^{(k)})}{(f^{(k)}-a)(f^{(k)}-b)}) \\ &= m(r,\frac{L(f^{(k)})(f_c-f^{(k)})}{(f^{(k)}-a)(f^{(k)}-b)})+N(r,\psi) \\ &\leq m(r,\frac{L(f^{(k)})}{f^{(k)}-b})+m(r,\frac{f_c-f^{(k)}}{f^{(k)}-a})+\overline{N}(r,f)+S(r,f) \\ &\leq m(r,\frac{1}{f_c-a})+S(r,f)=\overline{N}(r,\frac{1}{f_c-b})+S(r,f). \end{aligned} \tag{3.20}$$

Hence combining (3.19) and (3.20), we obtain

$$T(r,f_c)\le 2\overline{N}(r,\frac{1}{f_c-b})+S(r,f). \tag{3.21}$$

If $a_{-c}^{(k)}\equiv a$, then by (3.1) and Lemma 2.1 we can get

$$\begin{aligned}T(r,e^h)&=m(r,e^h)=m(r,\frac{f^{(k)}-a_{-c}^{(k)}}{f_c-a})\\&\le m(r,\frac{f^{(k)}-a_{-c}^{(k)}}{f_c^{(k)}-a^{(k)}})+m(r,\frac{f_c^{(k)}-a^{(k)}}{f_c-a})=S(r,f).\end{aligned}\tag{3.22}$$

It follows from (3.10), (3.21), (3.22) and Lemma 2.3 that $T(r,f)=T(r,f_c)+S(r,f)=S(r,f)$. It's impossible.

If $a_{-c}^{(k)}\equiv b$, then by (3.10), (3.21) and and Lemma 2.1,

$$\begin{aligned}T(r,f_c)&\le m(r,\frac{1}{f_c-a})+\overline{N}(r,\frac{1}{f^{(k)}-b})+S(r,f)\\&\le m(r,\frac{f^{(k)}-a_{-c}^{(k)}}{f_c^{(k)}-a^{(k)}})+m(r,\frac{f_c^{(k)}-a^{(k)}}{f_c-a})+m(r,\frac{1}{f^{(k)}-b})\\&+\overline{N}(r,\frac{1}{f^{(k)}-b})+S(r,f)\le T(r,f^{(k)})+S(r,f),\end{aligned}$$

which implies

$$T(r,f_c)\le T(r,f^{(k)})+S(r,f). \tag{3.23}$$

Lemma 2.3 implies

$$T(r,f^{(k)})\le T(r,f)+k\overline{N}(r,f)+S(r,f)=T(r,f_c)+S(r,f), \tag{3.24}$$

and it follows from the fact f_c and $f^{(k)}$ share a CM and b IM, (3.2) and (3.23) that

$$\begin{aligned}T(r,f^{(k)})&=T(r,f_c)+S(r,f)\\&=\overline{N}(r,\frac{1}{f_c-a})+\overline{N}(r,\frac{1}{f_c-b})+S(r,f)\\&=\overline{N}(r,\frac{1}{f^{(k)}-a})+\overline{N}(r,\frac{1}{f^{(k)}-b})+S(r,f).\end{aligned}\tag{3.25}$$

By Lemma 2.1, Lemma 2.8, (3.2) and (3.25), we have

$$\begin{aligned}2T(r,f^{(k)})&\le\overline{N}(r,\frac{1}{f^{(k)}-a})+\overline{N}(r,\frac{1}{f^{(k)}-b})+\overline{N}(r,\frac{1}{f^{(k)}-d})+\overline{N}(r,f^{(k)})\\&+S(r,f)\le 2T(r,f^{(k)})-m(r,\frac{1}{f^{(k)}-d})+S(r,f)\end{aligned}$$

Immediately,

$$m(r,\frac{1}{f^{(k)}-d})=S(r,f). \tag{3.26}$$

By the First Fundamental Theorem, Lemma 2.1, Lemma 2.2, (3.14), (3.25), (3.26) and f is a transcendental meromorphic function of $\rho_2(f) < 1$, we obtain

$$\begin{aligned} m(r, \frac{f_c - d}{f^{(k)} - d}) &\le m(r, \frac{f_c}{f^{(k)} - d}) + m(r, \frac{d}{f^{(k)} - d}) + O(1) \\ &\le T(r, \frac{f_c}{f^{(k)} - d}) - N(r, \frac{f_c}{f^{(k)} - d}) + S(r,f) \\ &= m(r, \frac{f^{(k)} - d}{f_c}) + N(r, \frac{f^{(k)} - d}{f_c}) - N(r, \frac{f_c}{f^{(k)} - d}) + S(r,f) \\ &\le N(r, \frac{1}{f_c}) - N(r, \frac{1}{f^{(k)} - d}) + N(r, f^{(k)}) - N(r,f) + S(r,f) \\ &= T(r, \frac{1}{f_c}) - T(r, \frac{1}{f^{(k)} - d}) + S(r,f) \\ &= T(r, f_c) - T(r, f^{(k)}) + S(r,f) = S(r,f). \end{aligned}$$

Thus

$$m(r, \frac{f_c - d}{f^{(k)} - d}) = S(r,f). \tag{3.27}$$

It's easy to see that $N(r,\psi) = S(r,f)$ and (3.12) can be rewritten as

$$\psi = [\frac{a-d}{a-b}\frac{L(f^{(k)})}{f^{(k)} - a} - \frac{b-d}{a-b}\frac{L(f^{(k)})}{f^{(k)} - b}][\frac{f_c - d}{f^{(k)} - d} - 1]. \tag{3.28}$$

Then by Lemma 2.6, (3.27) and (3.28) we can get

$$T(r,\psi) = m(r,\psi) + N(r,\psi) = S(r,f). \tag{3.29}$$

By (3.2), (3.19) and (3.29) we get

$$\overline{N}(r, \frac{1}{f_c - a}) = S(r,f). \tag{3.30}$$

Moreover, by Lemma 2.1, (3.2), (3.25) and (3.30), we have

$$m(r, \frac{1}{(f_c - a)^{(k)}}) = m(r, \frac{1}{f_c^{(k)} - b_c}) = m(r, \frac{1}{f^{(k)} - b}) + S(r,f) = S(r,f), \tag{3.31}$$

and it follows from above, (3.6) and (3.10) that

$$\begin{aligned} \overline{N}(r, \frac{1}{f_c - b}) &= m(r, \frac{1}{f_c - a}) + S(r,f) \\ &\le m(r, \frac{1}{(f_c - a)^{(k)}}) + m(r, \frac{(f_c - a)^{(k)}}{f_c - a}) + S(r,f) = S(r,f). \end{aligned} \tag{3.32}$$

Then by (3.2), (3.30), (3.32) and Lemma 2.3, we obtain

$$\begin{aligned} T(r,f) = T(r,f_c) + S(r,f) &= \overline{N}(r, \frac{1}{f_c - a}) \\ &+ \overline{N}(r, \frac{1}{f_c - b}) + S(r,f) = S(r,f), \end{aligned} \tag{3.33}$$

which implies $T(r,f) = S(r,f)$, a contradiction.

So by (3.6), (3.10), (3.21), the First Fundamental Theorem, Lemma 2.8 and Remark 2.1 we can get

$$\begin{aligned} T(r,f_c) &\leq 2m(r,\frac{1}{f_c-a})+S(r,f)\leq 2m(r,\frac{1}{f^{(k)}-a_{-c}^{(k)}}) \\ &+S(r,f)=2T(r,f^{(k)})-2N(r,\frac{1}{f^{(k)}-a_{-c}^{(k)}})+S(r,f) \\ &\leq \overline{N}(r,\frac{1}{f^{(k)}-a})+\overline{N}(r,\frac{1}{f^{(k)}-b})+\overline{N}(r,\frac{1}{f^{(k)}-a_{-c}^{(k)}}) \\ &+\overline{N}(r,f^{(k)})-2N(r,\frac{1}{f^{(k)}-a_{-c}^{(k)}})+S(r,f) \\ &\leq T(r,f_c)-N(r,\frac{1}{f^{(k)}-a_{-c}^{(k)}})+S(r,f), \end{aligned}$$

which implies that

$$N(r,\frac{1}{f^{(k)}-a_{-c}^{(k)}})=S(r,f). \tag{3.34}$$

Consequently, Lemma 2.1 and Lemma 2.3 can deduce

$$N(r,\frac{1}{f^{(k)}-a_{-c}^{(k)}})=N(r,\frac{1}{f_c^{(k)}-a^{(k)}})=S(r,f).$$

Then applying Lemma 2.4, we have $T(r,e^h)=T(r,e^p)+O(1)=S(r,f)$, and it follows from (3.10) and (3.21) we can get $T(r,f)=T(r,f_c)+S(r,f)=S(r,f)$, a contradiction. This completes the proof of Theorem 1.

4. The Proof of Theorem 1.2

If $f(z)\equiv f(z+c)$, there is nothing to do. Assume that $f(z)\not\equiv f(z+c)$. Since $f(z)$ is a transcendental meromorphic function of $\rho_2(f)<1$, f and $f(z+c)$ share $a(z),\infty$ CM, then there is a nonzero entire function $p(z)$ of order less than 1 such that

$$\frac{f(z+c)-a(z)}{f(z)-a(z)}=e^{p(z)}, \tag{4.1}$$

then by Lemma 2.1 and $a(z)$ is a periodic function with period c,

$$T(r,e^p)=m(r,e^p)=m(r,\frac{f(z+c)-a(z+c)}{f(z)-a(z)})=S(r,f). \tag{4.2}$$

On the other hand, (4.1) can be rewritten as

$$\frac{f(z+c)-f(z)}{f(z)-a(z)}=e^{p(z)}-1, \tag{4.3}$$

and then we get

$$\overline{N}(r,\frac{1}{f(z)-b(z)})\leq N(r,\frac{1}{e^{p(z)}-1})=S(r,f). \tag{4.4}$$

Denote $N_{(m,n)}(r, \frac{1}{f(z)-b(z)})$ by the zeros of $f(z)-b(z)$ with multiplicities m and the zeros of $f_c(z)-b(z)$ with multiplicities n, where m, n are two positive integers. Thus, we can obtain

$$N(r, \frac{1}{f(z)-b(z)}) = \sum_{k=2}^{n} N_{(1,k)}(r, \frac{1}{f(z)-b(z)}) + \sum_{l=2}^{m} N_{(l,1)}(r, \frac{1}{f(z)-b(z)})$$
$$+ \sum_{l=2}^{m}\sum_{k=2}^{n} N_{(l,k)}(r, \frac{1}{f(z)-b(z)}) \le \overline{N}(r, \frac{1}{f(z)-b(z)}) + m\overline{N}(r, \frac{1}{f(z+c)-b(z)})$$
$$+ N(r, \frac{1}{e^{p(z)}-1}) \le (m+1)\overline{N}(r, \frac{1}{f(z)-b(z)}) + S(r,f) = S(r,f), \tag{4.5}$$

that is

$$N(r, \frac{1}{f(z+c)-b(z+c)}) = N(r, \frac{1}{f(z)-b(z)}) = S(r,f). \tag{4.6}$$

Similarly, we also have

$$N(r, \frac{1}{f(z+c)-b(z)}) = S(r,f). \tag{4.7}$$

Set

$$\psi(z) = \frac{f(z+c)-b(z+c)}{f(z)-b(z)}. \tag{4.8}$$

It is easy to see that

$$N(r, \frac{1}{\psi(z)}) \le N(r, \frac{1}{f(z+c)-b(z+c)}) + N(r, b(z)) = S(r,f), \tag{4.9}$$

$$N(r, \psi(z)) \le N(r, \frac{1}{f(z)-b(z)}) + N(r, b(z)) = S(r,f). \tag{4.10}$$

Hence by Lemma 2.1 and above,

$$T(r, \psi(z)) = m(r, \psi(z)) + N(r, \psi(z)) = S(r,f) \tag{4.11}$$

According to (4.1) and (4.8),we have

$$(e^{p(z)} - \psi(z))f(z) + \psi(z)b(z) + a(z) - b(z+c) - a(z)e^{p(z)} \equiv 0. \tag{4.12}$$

We discuss following two cases.

Case 1 $e^{p(z)} \not\equiv \psi(z)$. Then by (4.2), (4.11) and (4.12) we obtain $T(r,f) = S(r,f)$, a contradiction.

Case 2 $e^{p(z)} \equiv \psi(z)$. Then by (4.1) we have

$$f(z+c) = e^{p(z)}(f(z)-a(z)) + a(z), \tag{4.13}$$

and

$$N(r, \frac{1}{f(z+c)-b(z)}) = N(r, \frac{1}{f(z)-a(z)+\frac{a(z)-b(z)}{e^{p(z)}}}) = S(r,f). \tag{4.14}$$

If $b(z)$ is a periodic function of period c, then by (4.12) we can get $e^{p(z)} \equiv 1$, which implies $f(z) \equiv f(z+c)$, a contradiction. Obviously, $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \not\equiv a(z)$. Otherwise, we can deduce $a(z) \equiv b(z)$, a contradiction.

Next, we discuss three subcases.

Subcase 2.1 $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \not\equiv b(z)$ and $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \not\equiv b(z-c)$. Then according to (4.6), (4.7),(4.14) and Lemma 2.8, we can get

$$T(r, f(z)) \leq \overline{N}(r, \frac{1}{f(z) - a(z) - \frac{a(z)-b(z)}{e^{p(z)}}}) + \overline{N}(r, \frac{1}{f(z) - b(z)})$$

$$+ \overline{N}(r, \frac{1}{f(z) - b(z-c)}) + S(r, f) = S(r, f), \tag{4.15}$$

that is $T(r, f(z)) = S(r, f)$, a contradiction.

Subcase 2.2 $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \equiv b(z)$, but $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \not\equiv b(z-c)$. It follows that $e^{p(z)} \equiv 1$. Therefore by (4.1) we have $f(z) \equiv f(z+c)$, a contradiction.

Subcase 2.3 $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \equiv b(z)$, $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \equiv b(z-c)$. It follows that $e^{p(z)} \equiv 1$. Therefore by (4.1) we have $f(z) \equiv f(z+c)$, a contradiction.

Subcase 2.4 $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \not\equiv b(z)$ and $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \equiv b(z-c)$. It is easy to see that

$$\frac{a(z) - b(z)}{a(z-c) - b(z-c)} = e^{p(z)}. \tag{4.16}$$

Furthermore, (4.12) implies

$$\frac{a(z+c) - b(z+c)}{a(z) - b(z)} = e^{p(z)}, \tag{4.17}$$

$$\frac{a(z) - b(z)}{a(z-c) - b(z-c)} = e^{p(z-c)}. \tag{4.18}$$

It follows from (4.16) and (4.18) that

$$e^{p(z)} = e^{p(z+c)}. \tag{4.19}$$

By (4.1), (4.8) and (4.19), we know that $f(z)$ and $f(z+nc)$ share $a(z)$ and ∞ CM, so we set

$$F(z) = \frac{f(z) - a(z)}{b(z) - a(z)}, \quad G(z) = \frac{f(z+nc) - a(z)}{b(z+nc) - a(z+nc)}. \tag{4.20}$$

Since $f(z)$ and $f(z+nc)$ share $a(z)$ and ∞ CM, and $(b(z), b(z+nc)$ CM, so $F(z)$ and $G(z)$ share $0, \infty$ CM almost, and 1 CM almost. We claim that F is not a bilinear transform of G. Otherwise, we can see from Lemma 2.9 that if (i) occurs, we have $N(r, f(z)) = N(r, F(z)) + S(r, f) = S(r, f)$, then by Remark 1 and Theorem G, we get $f(z) \equiv f(z+c)$, a contradiction.

If (ii) occurs, we have $N(r, f(z)) = N(r, F(z)) + S(r, f) = S(r, f)$, then by Remark 1 and Theorem G, we get $f(z) \equiv f(z+c)$, a contradiction.

If (iii) occurs, we have

$$N(r,\frac{1}{f(z)-a(z)})=S(r,f),\quad N(r,\frac{1}{f(z)-b(z)})=S(r,f). \tag{4.21}$$

Then it follows from above, $a(z)-\frac{a(z)-b(z)}{e^{p(z)}}\not\equiv a(z)$, $a(z)-\frac{a(z)-b(z)}{e^{p(z)}}\not\equiv b(z)$ and Lemma 2.8 that $T(r,f)=S(r,f)$, a contradiction.

If (iv) occurs, we have $F(z)\equiv jG(z)$, that is

$$\frac{b(z+nc)-a(z+nc)}{b(z)-a(z)}=j(\frac{f(z+nc)-a(z)}{f(z)-a(z)}), \tag{4.22}$$

where $j\neq 0,1$ is a finite constant. Then it follows from above, (4.17) and (4.19) that $e^{np(z)}=je^{np(z)}$, therefore we have $j=1$, a contradiction.

If (v) occurs, we have

$$N(r,\frac{1}{f(z)-a(z)})=S(r,f). \tag{4.23}$$

Then by Lemma 2.8, (4.7), (4.14) and $b(z-c)\not\equiv a(z)$, we obtain $T(r,f)=S(r,f)$, a contradiction.

If (vi) occurs, we have

$$N(r,f(z))=N(r,F(z))+S(r,f)=S(r,f), \tag{4.24}$$

and hence we can see from Theorem G and Remark 1 that $f(z)\equiv f(z+c)$, a contradiction.

Therefore, $F(z)$ is not a linear fraction transformation of $G(z)$. If $b(z)$ is a small function with period nc, that is $b(z+(n-1)c)\equiv b(z-c)$, we can set

$$\begin{aligned} D(z)&=(f(z)-b(z))(b(z+nc)-b(z+(n-1)c))\\ &\quad-(f(z+nc)-b(z+nc))(b(z)-b(z-c))\\ &=(f(z)-b(z-c))(b(z+nc)-b(z+(n-1)c))\\ &\quad-(f(z+nc)-b(z+(n-1)c))(b(z)-b(z-c)) \end{aligned}$$

If $D(z)\equiv 0$, then we have $f(z+nc)-b(z-c)\equiv -(f(z)-b(z-c))$. And thus we know that $f(z)$ and $f(z+nc)$ share $a(z),b(z-c)$ and ∞ CM. We suppose

$$F_1(z)=\frac{f(z)-a(z)}{b(z-c)-a(z)}, G_1(z)=\frac{f(z+nc)-a(z)}{b(z-c)-a(z)}. \tag{4.25}$$

Then we know that $F_1(z)$ and $G_1(z)$ share $0,1,\infty$ CM almost and $G_1(z)=-F_1(z)$. So by Lemma 2.10, we will obtain either $N(r,f(z))=N(r,F_1)+S(r,f)=S(r,f)$, but in this case, according to Theorem G and Remark 1, we can deduce a contradiction. Or $F_1(z)=G_1(z)$, that is $f(z)\equiv f(z+nc)$. Therefore, we obtain $f(z)\equiv b(z-c)$, that is $T(r,f(z))=S(r,f)$, a contradiction.

Hence $D(z) \not\equiv 0$, and by (4.7)-(4.8), (4.14) and Lemma 2.1, we have

$$\begin{aligned}
2T(r, f(z)) &= m(r, \frac{1}{f(z)-b(z)}) + m(r, \frac{1}{f(z)-b(z-c)}) + S(r,f) \\
&= m(r, \frac{1}{f(z)-b(z)} + \frac{1}{f(z)-b(z-c)}) + S(r,f) \\
&\le m(r, \frac{D(z)}{f(z)-b(z)} + \frac{D(z)}{f(z)-b(z-c)}) + m(r, \frac{1}{D(z)}) + S(r,f) \\
&\le m(r, D) + N(r, D) \le m(r, f(z)) + N(r, f(z)) + S(r,f) \\
&= T(r,f) + S(r,f), \qquad (4.26)
\end{aligned}$$

which implies $T(r,f) = S(r,f)$, a contradiction.

By (4.16) we have

$$\frac{\Delta_c b(z)}{1-e^{p(z)}} + b(z) = a(z). \qquad (4.27)$$

Combining (4.18) and the fact that $a(z)$ is a small function with period c, we can get

$$\frac{\Delta_c b(z+c)}{1-e^{p(z)}} + b(z+c) = a(z). \qquad (4.28)$$

According to (4.27) and (4.28), we obtain

$$e^{p(z)} = \frac{b_{2c}(z) - b_c(z)}{\Delta_c b(z)}. \qquad (4.29)$$

So if $\rho(b(z)) < \rho(e^{p(z)})$, we can follows from (4.28) and Lemma 2.11 that

$$\rho(e^{p(z)}) = \rho(\frac{b_{2c}(z) - b_c(z)}{\Delta_c^2 b(z)}) \le \rho(b(z)) < \rho(e^{p(z)}), \qquad (4.30)$$

which is a contradiction.

If $\rho(b(z)) < 1$, we claim that $p(z) \equiv B$ is a non-zero constant. Otherwise, the order of right hand side of (4.28) is 0, but the left hand side is 1, which is impossible. Therefore, by (4.1) we know that $f(z+c) - a(z) = B(f(z) - a(z))$, and then by Lemma 2.10 we will get $N(r,f) = S(r,f)$, so by Theorem G and Remark 1 we can obtain $f(z) \equiv f(z+c)$, a contradiction.

This completes Theorem 1.2.

Acknowledgements The author would like to thank to referee for his helpful comments.

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Поступила 20 января 2023

После доработки 06 июня 2023

Принята к публикации 26 июня 2023

Известия НАН Армении, Математика, том 59, н. 2, 2024, стр. 56 – 66.

PORTFOLIO VALUE-AT-RISK APPROXIMATION FOR GEOMETRIC BROWNIAN MOTION

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Abstract. Value-at-risk (VaR) serves as a measure for assessing the risk associated with individual securities and portfolios. When calculating VaR for portfolios, the dimension of the covariance matrix increases as more securities are included. In this study, we present a solution to address the issue of dimensionality by directly computing the VaR of a portfolio using a single security, therefore requiring only one variance and one mean. Our results demonstrate that, under the assumption of Gaussian distribution, the deviation between the computed VaR and actual values is relatively small.

MSC2020 numbers: 60J65; 65C30; 60H35; 97M30; 91B24.

Keywords: Portfolio value-at-risk; sum of lognormal distribution; Fenton-Wilkinson approximation.

1. Introduction

The computation of portfolio value-at-risk (VaR) typically involves a strict algorithm that assumes a distribution close to normal and incorporates a correlation matrix of security returns.

However, a challenge arises due to the increasing dimensionality of the covariance matrix, resulting in exponential computational burden with the inclusion of each new security (see [1]). In this paper, we propose an alternative calculation algorithm that assumes a Gaussian distribution of returns, while significantly reducing the computational burden. We examine this straightforward method and explore variations, focusing primarily on the maximum absolute difference between the two approaches. Initially, we investigate the maximum deviation in the case of positively correlated securities, followed by a general analysis encompassing various scenarios.

Existing literature primarily emphasizes the reduction of computational burden through simplification of matrix-based calculations. As our primary concern is a single quantity, it is more feasible to approximate the VaR itself. Other authors

[1]The research of the second author is partially supported by the Mathematical Studies Center at Yerevan State University

concentrate on improved methods of estimating VaR when the underlying distribution deviates from the Gaussian assumption (as seen in [2]). While we do not delve into these alternative approaches for computing VaR in portfolios with a sum of lognormal distributions, it may prove effective to incorporate third and fourth moments (see [3]). Given that we primarily deal with the sum of lognormal distributions, we employ existing approximation methods. Specifically, we utilize the Fenton-Wilkinson approximation ([4, 5]) due to its simplicity, although more accurate approximations exist (see [6, 7]). Remarkably, our findings indicate that the simple Fenton-Wilkinson approximation sufficiently approximates VaR under the assumption of normality.

The paper is organized as follows. In Section 2, we introduce the general framework and the proposed method. Section 3 analyzes the maximum potential difference, and finally, we conclude with a discussion of our results.

2. VaR computation and approximation

We deal with only two subsequent periods of time. Here we consider mixture of n securities represented by geometric Brownian motions

$$S(t) = \sum_{i=1}^{n} w_i S_i(t) \tag{2.1}$$

with $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$, and $S_i(t)$, $i = \overline{1,n}$ are processes satisfying the following stochastic differential equations (SDE).

$$dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i(t)dW_i(t) \tag{2.2}$$

with $W_i(t) \sim N(0,t)$ not necessarily independent Brownian motions, i.e. for each i, $W_i(t+1) - W_i(t) \sim N(0,1)$ iid for each $t \in \{1, 2, ...\}$ [2] with correlation coefficients

$$\rho_{i,j}(t) = corr(W_i(t), W_j(t)) \tag{2.3}$$

As we deal with only two periods ($t = 0, 1$) and having no randomness in period 0, we take

$$\rho_{i,j}(t) = \rho_{ij} \tag{2.4}$$

Thus $S_i(t)$, $i = \overline{1,n}$ have log-normal distribution

$$\begin{aligned} S_i(t) &= S_i(0)e^{(\mu_i - \sigma_i^2/2)t + \sigma_i W_i(t)} \\ S_i(t) &\sim LogN\left(\ln S_i(0) + \left(\mu_i - \frac{\sigma_i^2}{2}\right)t, \sigma_i^2 t\right) \end{aligned} \tag{2.5}$$

[2] Note that we consider only discrete points of time. However, initially the Brownian motion should be defined on continuous domain. We are only interested in two periods $t = 0, 1$

The log-returns we denote by $X_i(t)$ for individual stock, and $X(t)$, for portfolio.

$$X_i(t) = \ln\left(\frac{S_i(t+1)}{S_i(t)}\right), \quad X_i(t) \sim N\left(\mu_i - \frac{\sigma_i^2}{2}, \sigma_i^2\right) \tag{2.6}$$

Note that correlation of log-returns is also $\rho(X_i(t), X_j(t)) =: \rho_{X_i,X_j} = \rho_{ij}$, thus yielding the following vector-distribution.

$$(X_1(t), ..., X_n(t)) \sim N\left(\mu = \begin{pmatrix} \mu_1 - \frac{\sigma_1^2}{2} \\ \vdots \\ \mu_n - \frac{\sigma_n^2}{2} \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_1^2 & \dots & \sigma_1\sigma_n\rho_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_1\sigma_n\rho_{1n} & \dots & \sigma_n^2 \end{pmatrix}\right), \tag{2.7}$$

where μ is vector of means and Σ is covariance matrix. From (2.1) and (2.7), the portfolio VaR (the quantile of portfolio return $X(t)$), have the following form

$$VaR_X = \sum_{i=1}^n w_i S_i(0)\left(\mu_i - \frac{1}{2}\sigma_i^2\right) + z_{\alpha/2}\sqrt{(w_1 S_1(0), ..., w_n S_n(0)) \cdot \Sigma \cdot \begin{pmatrix} w_1 S_1(0) \\ \vdots \\ w_n S_n(0), \end{pmatrix}}$$

where VaR_X is value at risk for given portfolio, and $z_{\alpha/2}$ is quantile of standard normal distribution (with probability $P(X \leq z_{\alpha/2}) = 1 - \frac{\alpha}{2}$) (see [1, 2]). Note that S(t) in (2.1), is distributed as sum of lognormal distributions, i.e.

$$S(t) \sim \sum_{i=1}^n w_i LogN\left(\ln S_i(0) + \left(\mu_i - \frac{\sigma_i^2}{2}\right)t, \sigma_i^2 t\right) \tag{2.8}$$

By Fenton - Wilkinson ([3, 4]) approximation we have

$$S(t) \sim_{approx} LogN(\mu_z(t); \sigma_z^2(t)) \sim: \widetilde{S(t)}, \tag{2.9}$$

where

$$\sigma_z^2(t) = \frac{1}{\left(\sum_{i=1}^n w_i\left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right)e^{\frac{\sigma_i^2 t}{2}}\right)^2} \cdot \left[t\sum_{i,j=1}^n \rho_{ij}^S \sigma_i\sigma_j w_i w_j\left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right)\left(S_j(0) + (\mu_j - \frac{1}{2}\sigma_j^2)\right)e^{\frac{\sigma_i^2 t + \sigma_j^2 t}{2}}\right]$$
$$\mu_z(t) = \ln\left(\sum_{i=1}^n w_i\left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right)e^{\frac{\sigma_i^2 t}{2}}\right) - \frac{\sigma_z^2(t)}{2} \tag{2.10}$$

Let's approximate $X(t)$ with return of $\widetilde{S(t)}$, $\widetilde{X_p(t)} := \ln\left(\frac{\widetilde{S(t+1)}}{\widetilde{S(t)}}\right)$, with

$$\widetilde{X_p(t)} \sim N(\mu_z(t+1), \sigma_z^2(t+1)) - N(\mu_z(t), \sigma_z^2(t)) \tag{2.11}$$

To completely determine the distribution, we additionally need covariance

$$C(t) = Cov\left(\ln\widetilde{S(t+1)}; \ln\widetilde{S(t)}\right) = E\left(\ln\widetilde{S(t+1)} \cdot \ln\widetilde{S(t)}\right) - E\left(\ln\widetilde{S(t+1)}\right) \cdot E\left(\ln\widetilde{S(t)}\right)$$

with $E(\ln \widetilde{S(t)}) = \mu_z(t)$. To find covariance $C(t)$ we use formula for exponential terms. So let's consider $E(e^{\ln \widetilde{S(t+1)}} \cdot e^{\ln \widetilde{S(t)}})$.

$$E\left(e^{\ln \widetilde{S(t+1)}} \cdot e^{\ln \widetilde{S(t)}}\right) = e^{\mu_z(t)+\mu_z(t+1)+\frac{\sigma_z^2(t)+\sigma_z^2(t)+2C(t)}{2}} \tag{2.12}$$

On the other hand

$$E\left(e^{\ln S(t+1)} \cdot e^{\ln S(t)}\right) = E\left(\sum_{i=1}^{n} w_i S_i(t) \cdot \sum_{i=1}^{n} w_i S_i(t)\right) \tag{2.13}$$

The covariance may be approximated by [3]

$$C(t) \approx \ln\left(E\left(\sum_{i=1}^{n} w_i S_i(t) \cdot \sum_{i=1}^{n} w_i S_i(t)\right)\right) - \mu_z(t) - \mu_z(t+1) - \frac{1}{2}\sigma_z^2(t) - \frac{1}{2}\sigma_z^2(t+1).$$

Hence we have the following approximate distribution

$$\widetilde{X_p(t)} \sim_{approx} N\left(\mu_z(t+1) - \mu_z(t), \sigma_z^2(t+1) + \sigma_z^2(t) - 2C(t)\right) \tag{2.14}$$

The idea is to approximate portfolio VaR using quantile of approximate distribution of $X_p(t)$ (for 2 consecutive days $t = 0; t+1 = 1$).

$$VaR_{X_p} = \mu_z(t+1) - \mu_z(t) + S(0) z_{\alpha/2} \sqrt{\sigma_z^2(t+1) + \sigma_z^2(t) - 2C(t)} \tag{2.15}$$

For $t = 0$, we have the following (by (2.10))

$$\begin{aligned} \sigma_z^2(0) &= \frac{1}{\left(\sum_{i=1}^{n} w_i \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right)\right)^2} \cdot 0 = 0 \\ \mu_z(0) &= \ln\left(\sum_{i=1}^{n} w_i \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right)\right) \end{aligned} \tag{2.16}$$

So for t=0 we have random variable with 0 variance, which is obvious as nothing is random in that period. For t+1=1 we have:

$$\begin{aligned} \sigma_z^2(1) &= \frac{1}{\left(\sum_{i=1}^{n} w_i \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right) e^{\frac{\sigma_i^2}{2}}\right)^2} \cdot \\ &\left[\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j w_i w_j \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right)\left(S_j(0) + (\mu_j - \frac{1}{2}\sigma_j^2)\right) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}\right] \\ \mu_z(1) &= \ln\left(\sum_{i=1}^{n} w_i \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right) e^{\frac{\sigma_i^2}{2}}\right) - \frac{\sigma_z^2(1)}{2} \end{aligned} \tag{2.17}$$

[3] It is indeed approximation, as in (2.12), $\widetilde{S(t)}$ is used, while in (2.13) $S(t)$.

Note that as one of our random variables has 0 variance the covariance can be taken to be 0. So we have:

$$VaR_{X_p} = \ln\left(\sum_{i=1}^{n} w_i \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right) e^{\frac{\sigma_i^2}{2}}\right) - \ln\left(\sum_{i=1}^{n} w_i \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right)\right) - \\ - \frac{1}{2}\sigma_z^2(1) + z_{\alpha/2}\left(\sum_{i=1}^{n} w_i S_i(0)\right)\sqrt{\sigma_z^2(1)}.$$

Hereafter, we will consider only risk neutral pricing, i.e. $\mu_i = \frac{1}{2}\sigma_i^2$.

3. Difference in methods

We claim that difference between VaR_{X_p} and VaR_X is not big, in sense that there exist C such that

$$\left|\frac{VaR_{X_p} - VaR_X}{VaR_X}\right| < C$$

for any correlation coefficients ρ_{ij} with $i \neq j$; $i, j = \overline{1, n}$ and any weights w_i in risk neutral setting. Or at least we attempt to prove similar result[4].

Remark 3.1. *We don't yet know if C depends on general structure of ρ-s and σ-s, or is there any absolute constant. At least we will try to show the existence of some bounds. Also note that while we may not come to theoretically small C, in practice C is quite small.*

For the risk-neutral pricing (i.e. taking $\mu_i = \frac{1}{2}\sigma_i^2$) we obtain

(3.1)

$$VaR_X = z_{\alpha/2}\sqrt{(w_1 S_1(0), ..., w_n S_n(0)) \cdot \Sigma \cdot \begin{pmatrix} w_1 S_1(0) \\ \vdots \\ w_n S_n(0) \end{pmatrix}}$$

$$VaR_{X_p} = \ln\left(\sum_{i=1}^{n} w_i S_i(0) e^{\frac{\sigma_i^2}{2}}\right) - \ln\left(\sum_{i=1}^{n} w_i S_i(0)\right) - \\ - \frac{1}{2}\frac{1}{\left(\sum_{i=1}^{n} w_i S_i(0) e^{\frac{\sigma_i^2}{2}}\right)^2}\left[\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}\right] + \\ z_{\alpha/2}\left(\sum_{i=1}^{n} w_i S_i(0)\right)\sqrt{\frac{1}{\left(\sum_{i=1}^{n} w_i S_i(0) e^{\frac{\sigma_i^2}{2}}\right)^2}\left[\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}\right]}$$

Let's first consider the case where we deal only with non-negative correlations, i.e. $\rho_{ij} \geq 0$.

[4]No formal derivations of approximation were given originally for log-normal approximation with Fenton-Wilkinson. So some computational comparisons had been done later, see [6].

3.1. **First bounds.** We have the following obvious (quite loose) bound for the fourth term of VaR_{X_p} in (3.1)

$$z_{\alpha/2}\frac{\sum_{i=1}^n w_iS_i(0)}{\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}}\sqrt{\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_jS_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}\right]}\leq \tag{3.2}$$
$$z_{\alpha/2}\frac{\sum_{i=1}^n w_iS_i(0)}{\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}}e^{\frac{\sigma_{max}^2}{2}}VaR_X\leq VaR_Xe^{\frac{\sigma_{max}^2}{2}-\frac{\sigma_{min}^2}{2}}$$

And similarly

(3.3)

$$z_{\alpha/2}\frac{\sum_{i=1}^n w_iS_i(0)}{\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}}\sqrt{\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_jS_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}\right]}\geq$$
$$z_{\alpha/2}e^{\frac{-\sigma_{max}^2}{2}}\sqrt{\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_jS_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}\right]}\geq VaR_Xe^{\frac{\sigma_{min}^2}{2}-\frac{\sigma_{max}^2}{2}}$$

Considering the first three terms of VaR_{X_p} in (3.1), and using the same argumentation we have the following bounds

$$\ln\left(\frac{\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}}{\sum_{i=1}^n w_iS_i(0)}\right)-\frac{1}{2}\frac{1}{\left(\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}\right)^2}\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_jS_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}\right]\geq$$
$$\frac{1}{2}\sigma_{min}^2-\frac{1}{2}e^{\sigma_{max}^2-\sigma_{min}^2}\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_j\right]$$

which in turn, using the following $\sigma_{max}^2\geq\sum_{i,j=1}^n\rho_{ij}\sigma_i\sigma_jw_iw_j\geq\sigma_{min}^2$, we give

(3.4)

$$\ln\left(\frac{\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}}{\sum_{i=1}^n w_iS_i(0)}\right)-\frac{1}{2}\frac{1}{\left(\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}\right)^2}\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_jS_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}\right]\geq$$
$$\frac{1}{2}\sigma_{min}^2-\frac{1}{2}\sigma_{max}^2e^{\sigma_{max}^2-\sigma_{min}^2}$$

Similarly one can derive

(3.5)

$$\ln\left(\frac{\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}}{\sum_{i=1}^n w_iS_i(0)}\right)-\frac{1}{2}\frac{1}{\left(\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}\right)^2}\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_jS_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}\right]\leq$$
$$\frac{1}{2}\sigma_{max}^2-\frac{1}{2}\sigma_{min}^2e^{\sigma_{min}^2-\sigma_{max}^2}$$

Combining (3.2), (3.3), (3.4) and (3.5), we obtain the following bound

$$
\begin{aligned}
&\frac{1}{2}\sigma_{min}^2 - \frac{1}{2}\sigma_{max}^2 e^{\sigma_{max}^2 - \sigma_{min}^2} + VaR_X e^{\frac{\sigma_{min}^2}{2} - \frac{\sigma_{max}^2}{2}} \leq VaR_{X_p} \\
\leq &\frac{1}{2}\sigma_{max}^2 - \frac{1}{2}\sigma_{min}^2 e^{\sigma_{min}^2 - \sigma_{max}^2} + VaR_X e^{\frac{\sigma_{max}^2}{2} - \frac{\sigma_{min}^2}{2}}
\end{aligned} \tag{3.6}
$$

Note that this bound is indeed loose, as right side can get quite big thanks to exponent, while the left side can be quite small. Also note that, if $\sigma_i = \sigma_{max} = \sigma_{min}$, we retrieve $VaR_{X_p} = VaR_X$.

3.2. **Bounds for positive correlations.** The better bound stated in the following proposition can be obtained. First let's make some notations value of portfolio $VP := \sum_{i=1}^n w_i S_i(0)$ and

$$
\sigma_{wS}^2 := \frac{\sum_{i,j=1}^n w_i w_j \sigma_i \sigma_j S_i(0) S_j(0)}{\sum_{i,j=1}^n w_i w_j S_i(0) S_j(0)} \tag{3.7}
$$

Proposition 3.1. *The following inequality holds if we assume non-negative correlations:*

$$
\begin{aligned}
&\frac{1}{2}\sigma_{min}^2 - \frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2 \left(\frac{1}{VP}\right)^2 VaR_X^2 \frac{2\sigma_{max}^2 - \sigma_{min}^2}{\sigma_{wS}^2} + VaR_X \leq VaR_{X_p} \leq \\
&\frac{1}{2}\sigma_{max}^2 - \frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2 \left(\frac{1}{VP}\right)^2 VaR_X^2 + VaR_X \sqrt{\frac{2\sigma_{max}^2 - \sigma_{min}^2}{\sigma_{wS}^2}}
\end{aligned} \tag{3.8}
$$

Proof. We make use of Holder's inequality for left part, and Abel's inequality for right part (3.8).

Lemma 3.1. *For positive values of x_i and w_i, the following inequality is true.*

$$
\frac{\sum_{i=1}^n x_i w_i e^{x_i}}{\sum_{i=1}^n w_i e^{x_i}} \geq \frac{\sum_{i=1}^n x_i w_i}{\sum_{i=1}^n w_i}. \tag{3.9}
$$

Proof. We define

$$
H(a) = \frac{\sum_{i=1}^n x_i w_i e^{a x_i}}{\sum_{i=1}^n w_i e^{a x_i}} \tag{3.10}
$$

and consider its derivative with respect to a.

$$
H'(a) = \frac{\left(\sum_{i=1}^n x_i^2 w_i e^{a x_i}\right) \cdot \left(\sum_{i=1}^n w_i e^{a x_i}\right) - \left(\sum_{i=1}^n x_i w_i e^{a x_i}\right)^2}{\left(\sum_{i=1}^n w_i e^{a x_i}\right)^2} \tag{3.11}
$$

Due to Holder's inequality the numerator is non-negative. Indeed, denote

$$
a_k = x_k e^{\frac{a x_k}{2}} \sqrt{w_k}; \quad b_k = e^{\frac{a x_k}{2}} \sqrt{w_k} \tag{3.12}
$$

Then the numerator is exactly

$$
\left(\sum_{k=1}^n a_k\right)^2 \left(\sum_{k=1}^n b_k\right)^2 - \left(\sum_{h=1}^n a_h b_h\right)^2 \geq 0.
$$

By exactly the same technique, one can show that the following lemma is also true.

Lemma 3.2. *For positive values of x_i and w_i and for some strictly increasing function $f(x)$, the following inequality holds.*

$$\frac{\sum_{i=1}^{n} x_i w_i e^{f(x_i)}}{\sum_{i=1}^{n} w_i e^{f(x_i)}} \geq \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i} \tag{3.13}$$

This inequality is enough to show the first part of (3.8).

Proof. Consider only the last part of Var_{X_p} with $z_{\alpha/2}$, in (3.1).

$$A := z_{\alpha/2}\left(\sum_{i=1}^{n} w_i S_i(0)\right)\sqrt{\frac{1}{\left(\sum_{i=1}^{n} w_i S_i(0) e^{\frac{\sigma_i^2}{2}}\right)^2}\left[\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}\right]}$$

$$= z_{\alpha/2}\left(\sum_{i=1}^{n} w_i S_i(0)\right)\left(\frac{\sum_{i,j=1}^{n} \rho_{ij} w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}}{\left(\sum_{i=1}^{n} w_i S_i(0) e^{\frac{\sigma_i^2}{2}}\right)^2}\left[\frac{\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}}{\sum_{i,j=1}^{n} \rho_{ij} w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}}\right]\right)^{1/2}$$

For which using the inequality once and as soon as ρ-s are positive, we have

$$A \geq z_{\alpha/2}\left(\sum_{i=1}^{n} w_i S_i(0)\right)\left(\frac{\sum_{i,j=1}^{n} \rho_{ij} w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}}{\left(\sum_{i=1}^{n} w_i S_i(0) e^{\frac{\sigma_i^2}{2}}\right)^2}\left[\frac{\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j w_i w_j S_i(0) S_j(0)}{\sum_{i,j=1}^{n} \rho_{ij} w_i w_j S_i(0) S_j(0)}\right]\right)^{1/2}$$

Note that here we have used the (3.13) twice [5]. Let's do it once more

$$A \geq z_{\alpha/2}\left(\sum_{i=1}^{n} w_i S_i(0)\right)\left(\frac{\sum_{i,j=1}^{n} w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}}{\left(\sum_{i=1}^{n} w_i S_i(0) e^{\frac{\sigma_i^2}{2}}\right)^2}\cdot \frac{\sum_{i,j=1}^{n} \rho_{ij} w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}}{\sum_{i,j=1}^{n} w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}}\left[\frac{\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j w_i w_j S_i(0) S_j(0)}{\sum_{i,j=1}^{n} \rho_{ij} w_i w_j S_i(0) S_j(0)}\right]\right)^{1/2}$$

[5] We used it once for sum with i-s and once for sum with j-s.

and using Lemma 3.2 once more (again two times)[6], for the first term in square root we have

$$A \geq VaR_X \left(\frac{\sum_{i,j=1}^{n} \rho_{ij} w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}}{\sum_{i,j=1}^{n} w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}} \cdot \frac{\sum_{i,j=1}^{n} \sigma_i \sigma_j w_i w_j S_i(0) S_j(0)}{\sum_{i,j=1}^{n} \rho_{ij} w_i w_j S_i(0) S_j(0)} \right)^{1/2} = VaR_X \tag{3.14}$$

where the equality can be easily checked, just by multiplying sums.

For the next part of inequality we will make use of Abel's inequality. Denoting $\overline{x}_w = \sum w_i x_i$, the following lemma holds.

Lemma 3.3. *For positive values w_i, the following inequality is true*

$$\frac{\sum_{i=1}^{n} x_i w_i e^{x_i}}{\sum_{i=1}^{n} w_i e^{x_i}} \cdot \frac{\sum_{i=1}^{n} w_i}{\sum_{i=1}^{n} x_i w_i} \leq \frac{|\max x_i| + R}{\overline{x}_w} \tag{3.15}$$

with $R = \max x_i - \min x_i$.

Without loss of generality, we can assume that x_i are in increasing order. Hence, by Abel's inequality (see [8]), we have

$$\sum_{i=1}^{n} x_i w_i e^{x_i} \leq (|x_n| + x_n - x_1) \max_j \sum_{i=1}^{j} w_i e^{x_i} = (|x_n| + x_n - x_1) \sum_{i=1}^{n} w_i e^{x_i} \tag{3.16}$$

Thus it will yield

$$\begin{aligned} \frac{\sum_{i=1}^{n} x_i w_i e^{x_i} \cdot \sum_{i=1}^{n} w_i}{\sum_{i=1}^{n} w_i e^{x_i} \cdot \sum_{i=1}^{n} x_i w_i} &\leq \frac{(|x_n| + x_n - x_1) \sum_{i=1}^{n} w_i e^{x_i} \cdot \sum_{i=1}^{n} w_i}{\sum_{i=1}^{n} w_i e^{x_i} \cdot \sum_{i=1}^{n} x_i w_i} \\ &= \frac{(|x_n| + x_n - x_1) \sum_{i=1}^{n} w_i}{\sum_{i=1}^{n} x_i w_i} = \frac{|\max x_i| + R}{\overline{x}_w} \end{aligned} \tag{3.17}$$

Using this inequality and considering A again, we obtain

$$A = z_{\alpha/2} \left(\sum_{i=1}^{n} w_i S_i(0) \right) \left(\frac{\sum_{i,j=1}^{n} \rho_{ij} \sigma_i \sigma_j w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}}{\left(\sum_{i=1}^{n} w_i S_i(0) e^{\frac{\sigma_i^2}{2}} \right)^2} \right)^{1/2}$$

$$\leq z_{\alpha/2} \left(\frac{\sum_{i,j=1}^{n} \rho_{ij} \frac{\sigma_i \sigma_j}{2} w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}}{\left(\sum_{i=1}^{n} w_i S_i(0) e^{\frac{\sigma_i^2}{2}} \right)^2} \cdot \frac{(\sum_{i=1}^{n} w_i S_i(0))^2}{\sum_{i,j=1}^{n} \rho_{ij} \frac{\sigma_i \sigma_j}{2} w_i w_j S_i(0) S_j(0)} \right.$$

$$\left. \left[\sum_{i,j=1}^{n} \rho_{ij} \frac{\sigma_i \sigma_j}{2} w_i w_j S_i(0) S_j(0) \right] \right)^{1/2}$$

[6]Note that numerator of expression in square brackets with $z_{\alpha/2}$ is VaR_X itself.

Using Lemma 3.3 for first two fractions, we come to the following result:

$$A \leq z_{\alpha/2}\sqrt{\left[\sum_{i,j=1}^{n}\rho_{ij}\sigma_i\sigma_j w_i w_j S_i(0)S_j(0)\right]\frac{\max(\sigma_i\sigma_j)+(\sigma_{max}^2-\sigma_{min}^2)}{2\sigma_{wS}^2}} \tag{3.18}$$

or

$$A \leq VaR_X\sqrt{\frac{\max(\sigma_i\sigma_j)+(\sigma_{max}^2-\sigma_{min}^2)}{2\sigma_{wS}^2}} \tag{3.19}$$

For the third term VaR_{X_p} in (3.1), note that expression in square brackets is bigger than $(\frac{1}{z_{\alpha/2}}VaR_X)^2$ for positive correlations, we obtain

$$\begin{aligned} VaR_{X_p} \geq{}& \ln\left(\frac{\sum_{i=1}^{n} w_i S_i(0)e^{\frac{\sigma_i^2}{2}}}{\sum_{i=1}^{n} w_i S_i(0)}\right) - \\ & \frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2\left(\frac{1}{VP}\right)^2 VaR_X^2\frac{\max(\sigma_i\sigma_j)+(\sigma_{max}^2-\sigma_{min}^2)}{2\sigma_{wS}^2}+VaR_X \\ \geq{}& \frac{1}{2}\sigma_{min}^2-\frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2\left(\frac{1}{VP}\right)^2 VaR_X^2+VaR_X \end{aligned} \tag{3.20}$$

and lastly the main formula can be derived.

$$\begin{aligned} &\frac{1}{2}\sigma_{min}^2-\frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2\left(\frac{1}{VP}\right)^2 VaR_X^2\frac{2\sigma_{max}^2-\sigma_{min}^2}{\sigma_{wS}^2}+VaR_X \leq VaR_{X_p} \leq \\ &\frac{1}{2}\sigma_{max}^2-\frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2\left(\frac{1}{VP}\right)^2 VaR_X^2+VaR_X\sqrt{\frac{2\sigma_{max}^2-\sigma_{min}^2}{\sigma_{wS}^2}} \end{aligned}$$

3.3. **Bounds for general case.** The following result is immediate consequence of above results. One can prove it by separating the stock considered into two groups: positively correlated and negatively, in the following sense. Taking $\rho_+ = \{ij|\rho_{ij} = \rho_{ji} > 0\}$ and $\rho_- = \{ij|\rho_{ij} = \rho_{ji} < 0\}$, other indices does not contribute to sum and using this grouping, we get the following result.

Proposition 3.2. *The following inequality holds:*

$$\begin{aligned} &\frac{1}{2}\sigma_{min}^2-\frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2\left(\frac{1}{VP}\right)^2 VaR_X^2\max\left(1,\frac{2\sigma_{max}^2-\sigma_{min}^2}{\sigma_{wS}^2}\right) \\ &+VaR_X\min\left(1,\sqrt{\frac{2\sigma_{max}^2-\sigma_{min}^2}{\sigma_{wS}^2}}\right) \leq VaR_{X_p} \leq \frac{1}{2}\sigma_{max}^2-\frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2\left(\frac{1}{VP}\right)^2\cdot \\ &\cdot VaR_X^2\min\left(1,\frac{2\sigma_{max}^2-\sigma_{min}^2}{\sigma_{wS}^2}\right)+VaR_X\max\left(1,\sqrt{\frac{2\sigma_{max}^2-\sigma_{min}^2}{\sigma_{wS}^2}}\right) \end{aligned}$$

4. Discussion and conclusion

Our calculations have revealed tighter bounds, in scenarios involving either solely positive correlation or both positive and negative correlations. This could be attributed to the relatively small magnitudes of the volatilities themselves, suggesting the potential for the derivation of improved bounds. Nevertheless, for portfolios with relatively confined volatility values, the current bounds prove sufficiently tight. It is worth noting that enhancing these bounds is primarily contingent on the theoretical justification of the proposed methodology. Computations indicate significantly tighter real bounds. For a three-stock portfolio, this translates to approximately $0.5 - 0.9\%$ of the Gaussian-VaR value, or roughly $0.1 - 0.2\%$ of the portfolio value. As the number of stocks increases, the disparity diminishes gradually. It is crucial to emphasize that the pursuit of better bounds is rooted in the theoretical validation of the proposed procedure.

For the above case compared to Gaussian-VaR, our lower bound deviates by no more than 0.00025%, showcasing its robustness. However, the upper bound exhibits a substantial discrepancy of up to 7.5%, a noteworthy disparity. From an empirical standpoint, particularly in domains where non-gaussian behavior may dominate, our methodology could yield significant differences. However, as of now, such disparities have not been observed in the context of stock portfolios.

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Поступила 03 июля 2023

После доработки 26 октября 2023

Принята к публикации 15 ноября 2023

Известия НАН Армении, Математика, том 59, н. 2, 2024, стр. 67 – 83.

POWER OF AN ENTIRE FUNCTION SHARING ONE VALUE PARTIALLY WITH ITS DERIVATIVE

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Abstract. In the paper, we investigate the uniqueness problem of a power of an entire function that share one value partially with its derivatives and obtain a result, which improve several previous results. Also in the paper we include some applications of our main result.

MSC2020 numbers: 30D35; 30D45.

Keywords: entire functions; derivative; Nevanlinna theory; uniqueness and normal family.

1. Introduction and main result

In this paper, a meromorphic function f always means it is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with standard notation and main results of Nevanlinna Theory (see, e.g., [3, 8]). By $S(r, f)$ we denote any quantity that satisfies the condition $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside of an exceptional set of finite linear measure. A meromorphic function a is said to be a small function of f if $T(r, a) = S(r, f)$. Moreover, we use notation $\rho(f)$ for the order of a meromorphic function f. As usual, the abbreviation CM means counting multiplicities, while IM means ignoring multiplicities. Let f and g be two non-constant meromorphic functions and $a \in \mathbb{C}$. If $g - a = 0$ whenever $f - a = 0$, we write $f = a \Rightarrow g = a$.

In 1996, Brück [1] discussed the possible relation between f and f' when an entire function f and it's derivative f' share only one finite value CM. In this direction an interesting problem still open is the following conjecture proposed by Brück [1].

Conjecture A. *Let f be a non-constant entire function such that*

$$\rho_1(f) := \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r} \notin \mathbb{N} \cup \{\infty\}.$$

If f and f' share one finite value a CM, then $f' - a = c(f - a)$, where $c \in \mathbb{C} \setminus \{0\}$.

The conjecture for the special cases (1) $a = 0$ and (2) $N(r, 0; f') = S(r, f)$ had been confirmed by Brück [1].

Though the conjecture is not settled in its full generality, it gives rise to a long course of research on the uniqueness of entire and meromorphic functions sharing a single value with its derivatives. Specially, it was observed by Yang and Zhang [9] that Brück's conjecture holds if instead of an entire function one considers its suitable power. They proved the following theorem.

Theorem A. [9] *Let f be a non-constant entire function and $n \in \mathbb{N}$ such that $n \geq 7$. If f^n and $(f^n)'$ share 1 CM, then $f^n \equiv (f^n)'$ and $f(z) = c\exp(\frac{z}{n})$, where $c \in \mathbb{C} \setminus \{0\}$.*

In 2010, Zhang and Yang [12] improved and generalised Theorem A by considering higher order derivatives and by lowering the power of the entire function. In one of their results they also considered IM sharing of values. We now state two results of Zhang and Yang [12].

Theorem B. [12] *Let f be a non-constant entire function and $k, n \in \mathbb{N}$ such that $n \geq k+1$. If f^n and $(f^n)^{(k)}$ share 1 CM, then $f^n \equiv (f^n)^{(k)}$ and $f(z) = c\exp(\frac{\lambda}{n}z)$, where $c, \lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda^k = 1$.*

Theorem C. [12] *Let f be a non-constant entire function and $k, n \in \mathbb{N}$ such that $n \geq k+2$. If f^n and $(f^n)^{(k)}$ share 1 IM, then the conclusion of Theorem B holds.*

In connection to Theorem C, Zhang and Yang [12] posed the problem of investigating the validity of the result for $n \geq k+1$. They could prove Theorem C for $n \geq k+1$ but only in the case when $k = 1$. We now recall the result.

Theorem D. [12] *Let f be a non-constant entire function and $n \in \mathbb{N} \setminus \{1\}$. If f^n and $(f^n)'$ share 1 IM, then $f^n \equiv (f^n)'$ and $f(z) = c\exp(z)$, where $c \in \mathbb{C} \setminus \{0\}$.*

In the paper, we have extended and improved above Theorems in the following directions:

(1) We relax the nature of sharing with the idea of “partially"sharing value.
(2) We replace the first derivative $(f^n)'$ in Theorem D by the general derivative $(f^n)^{(k)}$.

We now state our main result as follows.

Theorem 1.1. *Let f be a non-constant entire function and $k, n \in \mathbb{N}$ such that $n \geq k+1$. If $f^n = 1 \Rightarrow (f^n)^{(k)} = 1$, then only one of the following cases holds:*

(1) *$f^n \equiv (f^n)^{(k)}$ and $f(z) = c\exp\left(\frac{\lambda}{n}z\right)$, where $c, \lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda^k = 1$,*
(2) *$n = 2$ and $f(z) = c_0\exp\left(\frac{1}{4}z\right) + c_1$, where $c_0, c_1 \in \mathbb{C} \setminus \{0\}$ such that $c_1^2 = 1$.*

If $k \geq 2$, then from Theorem 1.1, we have the following corollary.

Corollary 1.1. *Let f be a non-constant entire function and $k, n \in \mathbb{N}$ such that $k \geq 2$ and $n \geq k+1$. If $f^n = 1 \Rightarrow (f^n)^{(k)} = 1$, then $f^n \equiv (f^n)^{(k)}$ and $f(z) = c\exp\left(\frac{\lambda}{n}z\right)$, where $c, \lambda \in \mathbb{C}\setminus\{0\}$ such that $\lambda^k = 1$.*

Clearly Corollary 1.1 improves Theorems A-D for the case when $k \geq 2$.

We now make the following observation on the conclusions of Theorem 1.1:

From the conclusion (2), we see that $k = 1$ and $n = 2$. Note that

$$f^2 - 1 = c_0^2\exp\big(\frac{1}{2}z\big) + 2c_0c_1\exp\big(\frac{1}{4}z\big)$$

and

$$(f^2)' - 1 = \frac{1}{2}\Big(c_0^2\exp\big(\frac{1}{2}z\big) + c_0c_1\exp\big(\frac{1}{4}z\big) - 2\Big).$$

It is easy to conclude that $(f^2)' = 1 \not\Rightarrow f^2 = 1$. Therefore if we add the condition that $(f^2)' = 1 \Rightarrow f^2 = 1$ in Theorem 1.1, then the conclusion (2) will be automatically ruled out.

As a result, from Theorem 1.1, we immediately have the following corollary.

Corollary 1.2. *Let f be a non-constant entire function and $k, n \in \mathbb{N}$ such that $n \geq k+1$. If f^n and $(f^n)^{(k)}$ share 1 IM, then the conclusion of Theorem B holds.*

Obviously Corollary 1.2 improves Theorem D.

Now we exhibit the following example to show that the condition "$n \geq k+1$" in Theorem 1.1 and Corollary 1.2 is sharp.

Example 1.1. *Let $f(z) = \exp(\frac{z}{2}) + 2\exp(\frac{z}{4}) + 1$ and $k = n = 1$. It is easy to verify that $f(z) = 1 \Rightarrow f'(z) = 1$, but $f(z)$ does not satisfy any case of Theorem 1.1.*

Example 1.2. *Let $f(z) = 2\exp(\frac{z}{2}) - 1$ and $k = n = 1$. It is easy to verify that f and f' share 1 IM, but $f \not\equiv f'$.*

2. **Auxiliary lemmas**

Lemma 2.1. *([5], [[4], Theorem 4.1]) Let f be a non-constant entire function such that $\rho(f) \leq 1$ and $k \in \mathbb{N}$. Then $m(r, \frac{f^{(k)}}{f}) = o(\log r)$ as $r \to \infty$.*

Lemma 2.2. [7] *Let f be a non-constant meromorphic function and let $a_n(\not\equiv 0), a_{n-1}, \ldots, a_0$ be small functions of f. Then $T(r, \sum_{i=0}^{n} a_i f^i) = nT(r,f) + S(r,f)$.*

Now we introduce some basic ideas about normal families.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of D (see [6]).

Now we introduce the notation of the spherical derivative. Let h be a non-constant meromorphic function. The spherical derivative of h at $z \in \mathbb{C}$ is given as

$$h^{\#}(z) = \frac{|h'(z)|}{1 + |h(z)|^2}.$$

We remember that h is called a normal function if there exists a positive real number M such that $h^{\#}(z) \leq M \ \forall \ z \in \mathbb{C}$.

Here we introduce some other results related to Zalcman's lemma. We also use Zalcman's lemma to prove our Lemma 2.5 which plays an important role in the proof of the main result of the paper.

The following lemma is the famous Marty's Criterion.

Lemma 2.3. [6] *A family $\mathcal{F}$ of meromorphic functions on a domain D is normal and only if for each compact subset $K \subseteq D$, there exists a constant M such that $f^{\#}(z) \leq M \ \forall \ f \in \mathcal{F}$ and $z \in K$.*

Zalcman's lemma.[[11]] *Let $\mathcal{F}$ be a family of functions holomorphic in a domain D. If $\mathcal{F}$ is not normal at $z_0 \in D$, then there exist a sequence of points $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers ρ_n, $\rho_n \to 0$ and a sequence of functions $f_n \in \mathcal{F}$ such that*

$$g_n(\zeta) = f_n(z_n + \rho_n \zeta) \to g(\zeta)$$

converges locally uniformly in $\mathbb{C}$, where g is a non-constant entire function. The function g may be taken to satisfy the normalization $g^{\#}(\zeta) \leq g^{\#}(0) = 1 \ \forall \ \zeta \in \mathbb{C}$.

Lemma 2.4. [2] *Let f be a non-constant entire function such that $N(r, f) = O(\log r)$ as $r \to \infty$. If f has bounded spherical derivative on $\mathbb{C}$, then $\rho(f) \leq 1$.*

It does not seem that Theorem 1.1 can be proved by using the methods in [12]. In order to prove Theorem 1.1, we need the following result related to normal families.

Lemma 2.5. *Let f be a non-constant entire function such that*

$$(f^{k+1})'(f^{k+1} - (f^{k+1})^{(k)}) = \varphi f^{k+1}(f^{k+1} - 1),$$

where $\varphi(\not\equiv 0)$ is an entire function and $k \in \mathbb{N}$. If

$$f = 0 \Rightarrow (f^{k+1})^{(k)} = 0 \ \text{ and } \ f^{k+1} = 1 \Rightarrow (f^{k+1})^{(k)} = 1,$$

then $\rho(f) \leq 1$.

Proof. Let $\mathcal{F} = \{F_\omega\}$, where $F_\omega(z) = F(\omega + z) = f^{k+1}(\omega + z)$, $z \in \mathbb{C}$. Clearly $\mathcal{F}$ is a family of entire functions defined on $\mathbb{C}$. By assumption, we have $F(\omega + z) = 0 \Rightarrow F^{(k)}(\omega + z) = 0$ and $F(\omega + z) = 1 \Rightarrow F^{(k)}(\omega + z) = 1$. If $k = 1$, then by Theorem 1.3 [**?**], $\mathcal{F}$ is normal in $\mathbb{C}$. Henceforth we assume that $k \geq 2$.

Since normality is a local property, it is enough to show that $\mathcal{F}$ is normal at each point $z_0 \in \mathbb{C}$. Suppose on the contrary that $\mathcal{F}$ is not normal at z_0. Again since normality is a local property, we may assume that $\mathcal{F}$ is a family of holomorphic functions in a domain $D = \{z : 0 < |z - z_0| < R\}$, where $R > 0$. Then by Zalcman's lemma, there exist a sequence of functions $F_n \in \mathcal{F}$, where $F_n(z) = f^{k+1}(\omega_n + z)$, a sequence of complex numbers, z_n, $z_n \to z_0$ and a sequence of positive numbers ρ_n, $\rho_n \to 0$ such that

$$H_n(\zeta) = F_n(z_n + \rho_n\zeta) \to H(\zeta) \tag{2.1}$$

locally uniformly in $\mathbb{C}$, where H is a non-constant entire function such that $H^{\#}(\zeta) \leq 1$, $\forall\, \zeta \in \mathbb{C}$. Then by Lemma 2.4, we deduce that $\rho(H) \leq 1$.

Also by Hurwitz's theorem we conclude that all the zeros of H have multiplicity at least $k+1$. Clearly $H^{(k)} \not\equiv 0$. It is easy to deduce from (2.1) that

$$H_n^{(i)}(\zeta) = \rho_n^i F_n^{(i)}(z_n + \rho_n\zeta) \to H^{(i)}(\zeta) \tag{2.2}$$

locally uniformly in $\mathbb{C}$ for all $i \in \mathbb{N}$.

Now we claim that 1 is not a Picard exceptional value of H. If not, suppose 1 is a Picard exceptional value of H. Then by the second fundamental theorem, we have

$$\begin{aligned} T(r,H) \leq \overline{N}(r,0;H) + \overline{N}(r,1;H) + S(r,H) &\leq \frac{1}{k+1}\,N(r,0;H) + S(r,H) \\ &\leq \frac{1}{k+1}T(r,H) + S(r,H), \end{aligned}$$

which is impossible. Hence 1 is not a Picard exceptional value of H.

Suppose $H(\zeta_0) = 1$. Hurwitz's theorem implies the existence of a sequence $\zeta_n \to \zeta_0$ with

$$H_n(\zeta_n) = F_n(z_n + \rho_n\zeta_n) = 1.$$

Since $F = 1 \Rightarrow F^{(k)} = 1$, we have $H_n^{(k)}(z_n + \rho_n\zeta_n) = 1$. Then from (2.2), we have

$$H^{(k)}(\zeta_0) = \lim_{n\to\infty} H_n^{(k)}(\zeta_n) = \lim_{n\to\infty} \rho_n^k = 0.$$

Hence $H = 1 \Rightarrow H^{(k)} = 0$. First we suppose 0 is a Picard exceptional value of H. Since H is a non-constant entire function of order at most one and H has no zeros, then by Hadamard's Factorization theorem, we get $H(\zeta) = A\exp(\lambda\zeta)$, where $A, \lambda \in \mathbb{C} \setminus \{0\}$. Since $H = 1 \Rightarrow H^{(k)} = 0$, we get a contradiction.

Next we suppose that 0 is not a Picard exceptional value of H. Since all the zeros of H have multiplicity at least $k+1$, one can easily conclude that $H = 0 \Rightarrow H^{(k)} = 0$.

Also by the given condition, we have

$$\varphi_n(z_n+\rho_n\zeta)F_n(z_n+\rho_n\zeta)(F_n(z_n+\rho_n\zeta)-1) = F_n'(z_n+\rho_n\zeta)\big(F_n(z_n+\rho_n\zeta)-F_n^{(k)}(z_n+\rho_n\zeta)\big)$$

and so

$$\begin{aligned} (2.3)\qquad & \rho_n^{k+1}\varphi_n(z_n+\rho_n\zeta)F_n(z_n+\rho_n\zeta)(F_n(z_n+\rho_n\zeta)-1) \\ = \ & \rho_n F_n'(z_n+\rho_n\zeta)\big(\rho_n^k F_n(z_n+\rho_n\zeta)-\rho_n^k F_n^{(k)}(z_n+\rho_n\zeta)\big). \end{aligned}$$

Then from (2.1), (2.2) and (2.3), we conclude that

$$\rho_n^{k+1}\varphi_n(z_n+\rho_n\zeta)\to\psi_1(\zeta) \tag{2.4}$$

locally uniformly in $\mathbb{C}$, where ψ_1 is an entire function. Again using (2.1), (2.2) and (2.4), we deduce from (2.3) that

$$\psi_1(\zeta)H(\zeta)(H(\zeta)-1)=-H'(\zeta)H^{(k)}(\zeta). \tag{2.5}$$

Since $\rho(H)\leq 1$, it follows from (2.5) that $\rho(\psi_1)\leq 1$. Therefore applying Lemma 2.1, we deduce from (2.5) that $m(r,\psi_1)=o(\log r)$ as $r\to\infty$. Since $N(r,\psi_1)=0$, we have $T(r,\psi_1)=o(\log r)$ as $r\to\infty$, which implies that ψ_1 is a constant. We can write $\psi_1=c_1$, where $c_1\in\mathbb{C}\setminus\{0\}$. Consequently from (2.5), we have

$$c_1H(\zeta)(H(\zeta)-1)=-H'(\zeta)H^{(k)}(\zeta). \tag{2.6}$$

Let ζ_0 be a zero of H of multiplicity $m(\geq k+1)$. Then from (2.6), we conclude that $m=k+1$ and so all the zeros of H have multiplicity exactly $k+1$.

We claim that H is a transcendental entire function. If not, suppose that H is a polynomial. Since zeros of H are of multiplicity exactly $k+1$, H is a polynomial of degree $k+1$. Consequently we may assume that $H(\zeta)=a\,(\zeta-\zeta_0)^{k+1}$, where $a\in\mathbb{C}\setminus\{0\}$. Therefore $H^{(k)}(\zeta)=(k+1)!a(\zeta-\zeta_0)$. Note that $H(\zeta)-1=a\,(\zeta-\zeta_0)^{k+1}-1$. Since $H=1\Rightarrow H^{(k)}=0$, we obtain a contradiction. Hence H is a transcendental entire function.

Therefore we may assume that

$$H=h^{k+1}, \tag{2.7}$$

where h is a transcendental entire function having only simple zeros. Now (2.7) yields

$$\begin{aligned} (2.8)\qquad & H^{(k)}=(h^{k+1})^{(k)}=((k+1)h^kh')^{(k-1)} \\ = \ & (k+1)\big(kg^{k-1}(h')^2+h^kh''\big)^{(k-2)} \\ = \ & k(k+1)\big((k-1)h^{k-2}(h')^3\big)^{(k-3)}+k(k+1)\big(2h^{k-1}h'h''\big)^{(k-3)} \\ & +(k+1)\big(kh^{k-1}h'h''\big)^{(k-3)}+(k+1)\big(h^kh'''\big)^{(k-3)} \\ = \ & \cdots\cdots\cdots\cdots \\ = \ & (k+1)!h(h')^k+\frac{k(k-1)}{4}(k+1)!h^2(h')^{k-2}h''+\cdots+(k+1)h^kh^{(k)} \\ = \ & (k+1)!h(h')^k+\frac{k(k-1)}{4}(k+1)!h^2(h')^{k-2}h''+R_1(h), \end{aligned}$$

where $R_1(h)$ is a differential polynomial in h with constant coefficients and each term of $R_1(h)$ contains $h^m(3 \leq m \leq k)$ as a factor.

Denote by $N(r,1;H \mid\geq 2)$ the counting function of multiple 1-points of H.

Now we divide the following two cases.

Case 1. Suppose $N(r,1;H \mid\geq 2) = 0$. Then from (2.6), we conclude that h' has no zeros and so $\frac{h}{h'}$ is an entire function. Again from (2.6), we have $\frac{h}{h'} = -\frac{k+1}{c_1}\frac{H^{(k)}}{H-1}$ and so by Lemma 2.1, we deduce that $m(r, \frac{h}{h'}) = o(\log r)$ as $r \to \infty$. Since $N(r, \frac{h}{h'}) = 0$, we have $T(r, \frac{h}{h'}) = o(\log r)$ as $r \to \infty$, which implies that $\frac{h}{h'}$ is a constant. We can write $\frac{h}{h'} = c_2$, where $c_2 \in \mathbb{C} \setminus \{0\}$. On integration, we have $h(\zeta) = c_3 \exp(\frac{1}{c_2}\zeta)$, where $c_3 \in \mathbb{C} \setminus \{0\}$. This shows that H has no zeros, which is impossible.

Case 2. Suppose $N(r,1;H \mid\geq 2) \neq 0$. Now from (2.6), (2.7) and (2.8), we have

$$c_1 h^{k+1}(h^{k+1}-1) = -(k+1)h^k h'\big((k+1)!h(h')^k + \frac{k(k-1)}{4}(k+1)!h^2(h')^{k-2}h'' + R_1(h)\big),$$

i.e.,

$$\begin{aligned}c_1(h^{k+1}-1) = &-(k+1)(k+1)!(h')^{k+1} - \\ &\frac{k(k-1)(k+1)}{4}(k+1)!h(h')^{k-1}h'' + R_1(h).\end{aligned} \tag{2.9}$$

Differentiating (2.9) once, we get

$$\begin{aligned}c_1(k+1)h^k h' = &-(k+1)!(k+1)^2(h')^k h'' - \\ &\frac{k(k-1)(k+1)(k+1)!}{4}\big((h')^k h'' + (k-1)h(h')^{k-2}(h'')^2 + h(h')^{k-1}h'''\big) + R_2(h),\end{aligned} \tag{2.10}$$

where $R_2(h)$ is a differential polynomial in h.

Let ζ_0 be a zero of h. Now from (2.9) and (2.10), we have respectively

$$c_1 = (k+1)(k+1)!(h'(\zeta_0))^{k+1} \tag{2.11}$$

and

$$\big(k+1+\frac{k(k-1)}{4}\big)(h'(\zeta_0))^k h''(\zeta_0) = 0. \tag{2.12}$$

If $h''(\zeta_0) \neq 0$, then from (2.11) and (2.12) we arrive at a contradiction. Hence $h''(\zeta_0) = 0$ and so $h = 0 \Rightarrow h'' = 0$. Let $H_1 = \frac{h''}{h}$. Clearly $H_1 \not\equiv 0$. One can easily prove that H_1 is a non-zero constant, say $\lambda \in \mathbb{C}$. Therefore

$$h'' = \lambda_1 h. \tag{2.13}$$

Solving (2.13), we get

$$h(\zeta) = A_1 \exp\big(\sqrt{\lambda_1}\zeta\big) + B_1 \exp\big(-\sqrt{\lambda_1}\zeta\big),$$

where $A_1, B_1 \in \mathbb{C} \setminus \{0\}$. Note that

$$h'(\zeta) = A_1\sqrt{\lambda_1}\exp\big(\sqrt{\lambda_1}\zeta\big) - \sqrt{\lambda_1}B_1\exp\big(-\sqrt{\lambda_1}\zeta\big).$$

Again differentiating (2.13) and using it repeatedly, we have

$$h^{(2i)} = {\lambda_1}^i h \ \text{ and } \ h^{(2i+1)} = {\lambda_1}^i h', \ \text{ where } \ i = 1, 2, \ldots. \tag{2.14}$$

Then from (2.7) and (2.14), one can easily deduce that

$$(h^{k+1})^{(k)} = \tilde{c}_1 h(h')^k + \tilde{c}_2 h^2 (h')^{k-1} + \tilde{c}_3 h^3 (h')^{k-2} + \ldots + \tilde{c}_k h^k h' + \tilde{c}_{k+1} h^{k+1}, \tag{2.15}$$

where $\tilde{c}_1 = (k+1)!$ and $\tilde{c}_i \in \mathbb{C}$ for $i \geq 2$.

First we suppose $\tilde{c}_{k+1} \neq 0$. Let ζ_1 be a multiple zero of $H - 1$. Then obviously $H(\zeta_1) = 1$, $H'(\zeta_1) = 0$ and $H^{(k)}(\zeta_1) = 0$. Note that $H' = (k+1)h^k h'$. Since $H'(\zeta_1) = 0$, it follows that $h'(\zeta_1) = 0$ and $h(\zeta_1) \neq 0$. Therefore from (2.15), we get $\tilde{c}_{k+1} = 0$, which is impossible.

Next we suppose $\tilde{c}_{k+1} = 0$. Let $g = \frac{h'}{h}$. Obviously both $g - \sqrt{\lambda_1}$ and $g + \sqrt{\lambda_1}$ have no zeros. Now from (2.6) and (2.15), we deduce that

$$\begin{aligned} &c_1 h^{k+1} + (k+1)\big(\tilde{c}_1 (h')^{k+1} + \tilde{c}_2 h (h')^k + \\ &\quad \tilde{c}_3 h^2 (h')^{k-1} + \ldots + \tilde{c}_k h^{k-1} (h')^2\big) = c_1. \end{aligned} \tag{2.16}$$

Putting $h' = gh$ into (2.16), we get

$$(k+1)\big(\tilde{c}_1 g^{k+1} + \tilde{c}_2 g^k + \tilde{c}_3 g^{k-1} + \ldots + \tilde{c}_k g^2\big) + c_1 = \frac{c_1}{h^{k+1}}. \tag{2.17}$$

Note that the right hand side of (2.17) has no zeros. Consequently the left hand side may not have no zeros. Since both $g - \sqrt{\lambda_1}$ and $g + \sqrt{\lambda_1}$ have no zeros, we conclude that the left hand side of (2.17) must be one of the forms (i) $(k+1)\tilde{c}_1 (g - \sqrt{\lambda_1})^{k+1}$, (ii) $(k+1)\tilde{c}_1 (g + \sqrt{\lambda_1})^{k+1}$ and (iii) $(k+1)\tilde{c}_1 (g - \sqrt{\lambda_1})^m (g + \sqrt{\lambda_1})^n$, where $m + n = k + 1$. Note that

$$\begin{aligned} (k+1)\tilde{c}_1 \big(g - \sqrt{\lambda_1}\big)^{k+1} &= (k+1)\tilde{c}_1 g^{k+1} - (k+1)^2 \tilde{c}_1 \sqrt{\lambda_1} g^k + \cdots \\ &\quad + (-1)^k (k+1)^2 \tilde{c}_1 (\sqrt{\lambda_1})^k g + (-1)^{k+1} (k+1)\tilde{c}_1 (\sqrt{\lambda_1})^{k+1}. \end{aligned} \tag{2.18}$$

If the left hand side of (2.17) $= (k+1)\tilde{c}_1 (g - \sqrt{\lambda_1})^{k+1}$, then the left hand side of (2.17) and the right hand side of (2.18) must be identical. Note that the coefficient of g of the right hand side of (2.18) is non-vanishing. Therefore from (2.17) and (2.18), we arrive at a contradiction. Similarly if the left hand side of (2.17) $=$ $(k+1)\tilde{c}_1 (g + \sqrt{\lambda_1})^{k+1}$, then we get a contradiction. Again if the left hand side of (2.17) $= (k+1)\tilde{c}_1 (g - \sqrt{\lambda_1})^m (g + \sqrt{\lambda_1})^n$, where $m + n = k + 1$, then by a simple calculation we deduce that $m = n$ and $(-1)^m (k+1)\tilde{c}_1 (\sqrt{\lambda_1})^{m+n} = c_1$, i.e., $2m = k+1$ and $(-1)^m (k+1)(k+1)!(\sqrt{\lambda_1})^{k+1} = c_1$. Clearly k is odd. Note that

$$\begin{aligned} H(\zeta) = (h(\zeta))^{k+1} = A_1^{k+1} \exp\big((k+1)\sqrt{\lambda}\zeta\big) + \\ \ldots + B_1^{k+1} \exp\big(-(k+1)\sqrt{\lambda}\zeta\big). \end{aligned} \tag{2.19}$$

Now from (2.6) we get $c_1 H^2(\zeta) + H'(\zeta)H^{(k)}(\zeta) = c_1 H(\zeta)$ and so from (2.19) we can easily conclude that $c_1 + (k+1)^{k+1}(\sqrt{\lambda})^{k+1} = 0$. Since $(-1)^m(k+1)(k+1)!(\sqrt{\lambda_1})^{k+1} = c_1$, we get $(-1)^{\frac{k+1}{2}}(k+1)! + (k+1)^k = 0$, which is impossible for $k \geq 2$.

Hence all the foregoing discussion shows that $\mathcal{F}$ is normal at z_0. Consequently $\mathcal{F}$ is normal in $\mathbb{C}$. Hence by Lemma 2.3, there exists $M > 0$ satisfying $F^{\#}(\omega) = F^{\#}_{\omega}(0) < M$ for all $\omega \in \mathbb{C}$. Consequently by Lemma 2.4, we conclude that $\rho(F) \leq 1$ and so $\rho(f) \leq 1$. This completes the proof. □

3. Proof of the main result

Proof of Theorem 1.1. Let $F = f^n$. We put

$$\varphi = \frac{F'(F - F^{(k)})}{F(F-1)}. \tag{3.1}$$

Differentiating (3.1) once, we get

$$F''(F - F^{(k)}) + F'(F' - F^{(k+1)}) = \varphi' F(F-1) + \varphi F'(2F-1). \tag{3.2}$$

Now we divide the following two cases.

Case 1. Suppose $\varphi \not\equiv 0$. Then $F \not\equiv F^{(k)}$ and from (3.1), we get

$$\varphi = \frac{F'}{F-1}\Big(1 - \frac{F^{(k)}}{F}\Big). \tag{3.3}$$

Let z_0 be a zero of f with multiplicity p_0. Then z_0 is a zero of F and $F^{(k)}$ of multiplicities np_0 and $np_0 - k$ respectively and so from (3.1), we get in some neighbourhood of z_0

$$\varphi(z) = O\big((z - z_0)^{np_0 - k - 1}\big). \tag{3.4}$$

Since $n \geq k+1$, it follows from (3.4) that φ is analytic at z_0. Let z_1 be a zero of $F - 1$ of multiplicity p_1. Since $F = 1 \Rightarrow F^{(k)} = 1$, it follows that z_1 is a zero of $F^{(k)} - 1$ with multiplicity q_1, say. By Taylor's theorem we get in some neighbourhood of z_1

$$F(z) = 1 + a_{p_1}(z - z_1)^{p_1} + O(z - z_1)^{p_1+1},$$

$$F^{(k)}(z) = 1 + b_{q_1}(z - z_1)^{q_1} + O(z - z_1)^{q_1+1}$$

and

$$F'(z) = p_1 a_{p_1}(z - z_1)^{p_1 - 1} + O(z - z_1)^{p_1},$$

where $a_{p_1} \neq 0$ and $b_{q_1} \neq 0$. Consequently in some neighbourhood of z_1

$$\begin{aligned} F(z) - F^{(k)}(z) &= a_{p_1}(z-z_1)^{p_1} + O(z-z_1)^{p_1+1} \text{ if } p_1 < q_1 \\ &= -b_{q_1}(z-z_1)^{q_1} + O(z-z_1)^{q_1+1} \text{ if } q_1 < p_1 \\ &= (a_{p_1} - b_{q_1})(z-z_1)^{p_1} + O(z-z_1)^{p_1+1} \text{ if } p_1 = q_1. \end{aligned}$$

Then in some neighbourhood of z_1, we get from (3.1) that $\varphi(z) = O\big((z-z_1)^{t-1}\big)$, where $t = \min\{p_1, q_1\} \geq 1$ if $p_1 \neq q_1$ and $t \geq p_1 = q_1 \geq 1$ otherwise. Therefore we conclude that φ is analytic at z_1.

Since f is an entire function, from the above discussion, we deduce that φ is an entire function. Also (3.3) gives $m(r, \varphi) = S(r, f)$ and so $T(r, \varphi) = S(r, f)$. Again from (3.1), we get

$$\frac{1}{F} = \frac{1}{\varphi}\Big(\frac{F'}{F-1} - \frac{F'}{F}\Big)\Big(1 - \frac{F^{(k)}}{F}\Big). \tag{3.5}$$

Therefore we have $m(r, \frac{1}{F}) = S(r, f)$ and so $m(r, \frac{1}{f}) = S(r, f)$.

First we suppose $n > k+1$. Then from (3.4) we get $N(r, 0; f) \leq N(r, 0; \varphi) = S(r, f)$. Since $m(r, \frac{1}{f}) = S(r, f)$, we conclude that $T(r, f) = T(r, \frac{1}{f}) + O(1) = S(r, f)$, which is impossible.

Next we suppose $n = k+1$. Let z_0 be a zero of f with multiplicity p_0. So z_0 is a zero of F and $F^{(k)}$ of multiplicities $(k+1)p_0$ and $(k+1)p_0 - k$ respectively. If $p_0 \geq 2$, then from (3.1), we see that z_0 is a zero of φ, i.e., $\varphi(z_0) = 0$. Next we suppose that $p_0 = 1$. Clearly from (3.1), we get $\varphi(z_0) \neq 0$. Then in some neighbourhood of z_0, we get by Taylor's expansion

$$F(z) = \tilde{a}_{k+1}(z-z_0)^{k+1} + \tilde{a}_{k+2}(z-z_0)^{k+2} + \ldots \; (\tilde{a}_{k+1} \neq 0). \tag{3.6}$$

Clearly

$$\begin{cases} F'(z) = (k+1)\tilde{a}_{k+1}(z-z_0)^{k} + (k+2)\tilde{a}_{k+2}(z-z_0)^{k+1} + \ldots, \\ F''(z) = (k+1)k\tilde{a}_{k+1}(z-z_0)^{k-1} + (k+2)(k+1)\tilde{a}_{k+2}(z-z_0)^{k} + \ldots, \\ \ldots\ldots, \\ F^{(k)}(z) = (k+1)!\tilde{a}_{k+1}(z-z_0) + \ldots, \\ F^{(k+1)}(z) = (k+1)!\tilde{a}_{k+1} + \ldots \end{cases} \tag{3.7}$$

Now from (3.2), (3.6) and (3.7), we deduce that

$$(\varphi(z_0) - (k+1)(k+1)!\tilde{a}_{k+1})\,(z-z_0)^k + O\big((z-z_0)^{k+1}\big) \equiv 0,$$

which implies that $(k+1)!\tilde{a}_{k+1} = \frac{\varphi(z_0)}{k+1}$, i.e., $F^{(k+1)}(z_0) = \frac{\varphi(z_0)}{k+1}$. Consequently we get

$$F = 0 \Rightarrow F^{(k+1)} = \frac{\varphi}{k+1}. \tag{3.8}$$

Now from Lemma 2.5, we conclude that $\rho(F) \leq 1$. Consequently using Lemma 2.1, we deduce from (3.3) that $m(r, \varphi) = o(\log r)$ as $r \to \infty$.

Since $N(r,\varphi)=0$, we have $T(r,\varphi)=o(\log r)$ as $r\to\infty$, which implies that φ is a constant. We can write $\varphi=c$, where $c\in\mathbb{C}\setminus\{0\}$. Then from (3.1), we have

$$F'(F-F^{(k)})=cF(F-1). \tag{3.9}$$

Also from (3.9), one can easily conclude that f has only simple zeros, i.e., all the zeros of F have multiplicity exactly $k+1$.

We claim that F is a transcendental entire function. If not, suppose that F is a polynomial. Since zeros of F are of multiplicity exactly $k+1$, F is a polynomial of degree $k+1$. Consequently we may assume that $F(z)=a\,(z-\hat{z}_0)^{k+1}$, where $a\in\mathbb{C}\setminus\{0\}$. Therefore $F^{(k)}(z)=(k+1)!a(z-\hat{z}_0)$. Note that $F(z)-1=a\,(z-\hat{z}_0)^{k+1}-1$ and $F^{(k)}(z)-1=(k+1)!a(z-\hat{z}_0)-1$. It is clear that $F-1$ has $k+1$ distinct zeros. Since $F=1\Rightarrow F^{(k)}=1$, we obtain a contradiction. Hence F is a transcendental entire function.

Now applying Lemma 2.1, we deduce from (3.5) that $m\left(r,\frac{1}{F}\right)=o(\log r)$ as $r\to\infty$.

Let 0 be a Picard exceptional value of F. Then $T(r,F)=T\big(r,\frac{1}{F}\big)+O(1)=m\big(r,\frac{1}{F}\big)+O(1)=o(\log r)$ as $r\to\infty$, which implies that F is a constant. Therefore we arrive at a contradiction. Hence 0 is not a Picard exceptional value of F.
If 1 is a Picard exceptional value of F, by the second fundamental theorem, we get

$$\begin{aligned}&(k+1)T(r,f)=T(r,F)+O(1)\le\overline{N}(r,0;F)+\overline{N}(r,1;F)+S(r,F)\\ \le\;&\frac{1}{k+1}\,N(r,0;f)+S(r,f)\le\frac{1}{k+1}T(r,f)+S(r,f),\end{aligned}$$

which is impossible. Hence 1 is not a Picard exceptional value of F. Also we have

$$F=f^{k+1}. \tag{3.10}$$

Therefore from (3.10), we deduce that

$$\begin{aligned}F^{(k)}&=(k+1)!f(f')^k+\frac{k(k-1)}{4}(k+1)!f^2(f')^{k-2}f''+\cdots+(k+1)f^kf^{(k)}\\ &=(k+1)!f(f')^k+\frac{k(k-1)}{4}(k+1)!f^2(f')^{k-2}f''+R_1(f),\end{aligned} \tag{3.11}$$

where $R_1(f)$ is a differential polynomial in f with constant coefficients and each term of $R_1(f)$ contains $f^m(3\le m\le k)$ as a factor. Differentiating (3.11) once, we get

$$\begin{aligned}F^{(k+1)}&=(f^{k+1})^{(k+1)}\\ &=(k+1)!(f')^{k+1}+\frac{k(k+1)}{2}(k+1)!f(f')^{k-1}f''+(k+1)f^kf^{(k+1)}\\ &=(k+1)!(f')^{k+1}+\frac{k(k+1)}{2}(k+1)!f(f')^{k-1}f''+S_1(f),\end{aligned} \tag{3.12}$$

where $S_1(f)$ is a differential polynomial in f and each term of $S_1(f)$ contains f and its higher powers as a factor. Again differentiating (3.12) once, we get

$$F^{(k+2)} = (f^{k+1})^{(k+2)} = \frac{(k+1)!(k+1)(k+2)}{2}(f')^k f'' + S_2(f), \tag{3.13}$$

where $S_2(f)$ is a differential polynomial in f and each term of $S_2(f)$ contains f and its higher powers as a factor.

Now from (3.9), we deduce that

$$\frac{(k+1)f'\left(f^{k+1} - (f^{k+1})^{(k)}\right)}{f\left(f^{k+1}-1\right)} = c, \tag{3.14}$$

i.e.,

$$(k+1)f^{k+1}f' - (k+1)f'(f^{k+1})^{(k)} - cf^{k+2} = -cf. \tag{3.15}$$

Denote by $N(r,1;F \mid\geq 2)$ the counting function of multiple 1-points of F.

Now we divide the following two sub-cases.

Sub-case 1.1. Suppose $N(r,1;F \mid\geq 2) = 0$. Then from (3.14), we conclude that $f' \neq 0$. Since f is a transcendental entire function and $\rho(f) \leq 1$, it follows that

$$f'(z) = d_0 \exp(\lambda z), \tag{3.16}$$

where $d_0, \lambda \in \mathbb{C} \setminus \{0\}$. On integration, we have

$$f(z) = \frac{d_0}{\lambda}\exp(\lambda z) + d_1, \tag{3.17}$$

where $d_1 \in \mathbb{C}$. Since 0 is not a Picard exceptional value of $F = f^{k+1}$, it follows that $d_1 \neq 0$.

Let z_0 be a zero of f. Then $f(z_0) = 0$ and $F(z_0) = 0$. Also (3.8) gives $F^{(k+1)}(z_0) = \frac{c}{k+1}$. Now from (3.12), we conclude that

$$\left(f'(z_0)\right)^{k+1} = \frac{c}{(k+1)(k+1)!}. \tag{3.18}$$

Also from (3.17), we have

$$d_0 \exp(\lambda z_0) = -\lambda d_1, \text{ i.e., } d_0^{k+1}\exp\left((k+1)\lambda z_0\right) = (-\lambda d_1)^{k+1}. \tag{3.19}$$

Again from (3.16) and (3.18), we deduce that

$$d_0^{k+1}\exp\left((k+1)\lambda z_0\right) = \frac{c}{(k+1)(k+1)!}. \tag{3.20}$$

Therefore from (3.19) and (3.20), we deduce that

$$(-\lambda d_1)^{k+1} = \frac{c}{(k+1)(k+1)!}. \tag{3.21}$$

Now from (3.17), we have

$$\begin{aligned} f^{k+1}(z) &= \left(\frac{d_0}{\lambda}\right)^{k+1}\exp((k+1)\lambda z)+\binom{k+1}{1}\left(\frac{d_0}{\lambda}\right)^{k} d_1\exp(k\lambda z)+\ldots \\ &\quad +\binom{k+1}{k-1}\left(\frac{d_0}{\lambda}\right)^{2} d_1^{k-1}\exp(2\lambda z)+\binom{k+1}{k}\frac{d_0}{\lambda}d_1^{k}\exp(\lambda z)+d_1^{k+1}. \end{aligned}$$

and

$$\begin{aligned} f^{k+2}(z) &= \left(\frac{d_0}{\lambda}\right)^{k+2}\exp((k+2)\lambda z)+\binom{k+2}{1}\left(\frac{d_0}{\lambda}\right)^{k+1} d_1\exp((k+1)\lambda z)+\ldots \\ &\quad +\binom{k+2}{k}\left(\frac{d_0}{\lambda}\right)^{2} d_1^{k}\exp(2\lambda z)+\binom{k+2}{k+1}\frac{d_0}{\lambda}d_1^{k+1}\exp(\lambda z)+d_1^{k+2}. \end{aligned}$$

Therefore

$$\begin{aligned} &(f^{k+1}(z))^{(k)} \\ &= \left(\frac{d_0}{\lambda}\right)^{k+1}((k+1)\lambda)^k\exp((k+1)\lambda z)+\binom{k+1}{1}\left(\frac{d_0}{\lambda}\right)^{k}(k\lambda)^k d_1\exp(k\lambda z)+\ldots \\ &\quad +\binom{k+1}{k-1}\left(\frac{d_0}{\lambda}\right)^{2}(2\lambda)^k d_1^{k-1}\exp(2\lambda z)+\binom{k+1}{k}\frac{d_0}{\lambda}\lambda^k d_1^{k}\exp(\lambda z). \end{aligned}$$

Now from (3.15), we deduce that

$$\begin{aligned} (3.22)\quad &\left\{\frac{k+1}{\lambda^k}-(k+1)^{k+1}-\frac{c}{\lambda^{k+1}}\right\}\frac{d_0^{k+2}}{\lambda}\exp((k+2)\lambda z) \\ &+\left\{\frac{(k+1)^2}{\lambda^k}-(k+1)^2k^k-\frac{c(k+2)}{\lambda^{k+1}}\right\}d_0^{k+1}d_1\exp((k+1)\lambda z) \\ &+\ldots+\left\{\frac{(k+1)^2}{\lambda}-(k+1)^2\lambda-c\binom{k+2}{k}\frac{1}{\lambda^2}\right\}d_0^2d_1^k\exp(2\lambda z) \\ &+\left\{(k+1)-\frac{c(k+2)}{\lambda}\right\}d_0d_1^{k+1}\exp(\lambda z)-cd_1^{k+2}=-\frac{cd_0}{\lambda}\exp(\lambda z)-cd_1. \end{aligned}$$

Clearly from (3.22), we have

$$(3.23)\quad d_1^{k+1}=1 \text{ and } \left\{(k+1)-\frac{c(k+2)}{\lambda}\right\}d_0d_1^{k+1}=-\frac{cd_0}{\lambda}, \text{ i.e., } c=\lambda.$$

Now from (3.21) and (3.23), we have

$$(3.24)\qquad \lambda^k=(-1)^{k+1}\frac{1}{(k+1)(k+1)!}.$$

First we suppose $k=1$. Then from (3.23) and (3.24), we have respectively $d_1^2=1$ and $c=\lambda=\frac{1}{4}$. Also from (3.17), we have $f(z)=c_0\exp\left(\frac{1}{4}z\right)+d_1$, where $c_0=4d_0$.

Next we suppose $k\geq 2$. Now from (3.23) and (3.24), we calculate that

$$(3.25)\quad \frac{k+1}{\lambda^k}-(k+1)^{k+1}-\frac{c}{\lambda^{k+1}}=(k+1)\big((-1)^{k+1}k(k+1)!-(k+1)^k\big)\neq 0$$

for $k\geq 2$. Therefore one can easily arrive at a contradiction from Lemma 2.2 and (3.22).

Sub-case 1.2. Suppose $N(r,1;F\mid\geq 2)\neq 0$. Now differentiating (3.15) once, we have

$$(k+1)^2 f^k (f')^2 + (k+1) f^{k+1} f'' - (k+1) f'' (f^{k+1})^{(k)} - (k+1) f' (f^{k+1})^{(k+1)} - c(k+2) f^{k+1} f' = -cf'. \tag{3.26}$$

Again differentiating (3.26) once, we have

$$(k+1)^2 k f^{k-1}(f')^3 + 3(k+1)^2 f^k f' f'' + (k+1) f^{k+1} f''' - (k+1) f''' (f^{k+1})^{(k)} - 2(k+1) f''(f^{k+1})^{(k+1)} - (k+1) f'(f^{k+1})^{(k+2)} - c(k+2)(k+1) f^k (f')^2 - c(k+2) f^{k+1} f'' = -cf''. \tag{3.27}$$

Now from (3.11) (3.12), (3.13) and (3.27), we get

$$-(k+1)(k+1)!\frac{k^2+3k+6}{2}(f')^{k+1} f'' + S_3(f) = -cf'', \tag{3.28}$$

where $S_3(f)$ is a differential polynomial in f and each term of $S_3(f)$ contains f and its higher powers as a factor. Let z_0 be a zero of f. Now from (3.18) and (3.28), we have respectively

$$(f'(z_0))^{k+1} = \frac{c}{(k+1)(k+1)!} \tag{3.29}$$

and

$$(k+1)(k+1)!\frac{k^2+3k+6}{2}(f'(z_0))^{k+1} f''(z_0) = cf''(z_0). \tag{3.30}$$

If $f''(z_0)\neq 0$, then from (3.29) and (3.30) we arrive at a contradiction. Hence $f''(z_0)=0$ and so $f=0\Rightarrow f''=0$. Let

$$H_1 = \frac{f''}{f}. \tag{3.31}$$

Clearly $H_1\not\equiv 0$. One can easily prove that H_1 is a non-zero constant. Let us suppose that $H_1=\tilde{\lambda}\in\mathbb{C}\setminus\{0\}$. Now from (3.31), we deduce that

$$f'' = \tilde{\lambda} f. \tag{3.32}$$

Differentiating (3.32) and using it repeatedly, we have

$$f^{(2i)} = \tilde{\lambda}^i f \text{ and } f^{(2i+1)} = \tilde{\lambda}^i f', \text{ where } i=1,2,\ldots. \tag{3.33}$$

First we suppose k is odd. Then from (3.11) and (3.33), one can easily deduce that

$$(f^{k+1})^{(k)} = c_1 f (f')^k + c_3 f^3 (f')^{k-2} + \ldots + c_k f^k f', \tag{3.34}$$

where $c_1=(k+1)!$ and $c_i\in\mathbb{C}$ for $i\geq 3$.

Denote by $\overline{N}(r,1;F,F^{(k)}\mid\geq 2)$ the reduced counting function of common multiple 1-points of F and $F^{(k)}$.

If z_1 is a zero of $F-1$ with multiplicity $p_1 \geq 2$ and a zero of $F^{(k)}-1$ with multiplicity $q_1 \geq 2$, then from (3.9), we deduce that

$$\overline{N}(r,1;F,F^{(k)} \mid\geq 2)=0. \tag{3.35}$$

Let z_1 be a zero of $F-1$ of multiplicity p_1. Then from (3.35), we conclude that z_1 is a simple zero of $F^{(k)}-1$. Obviously $F(z_1)=1$ and $F^{(k)}(z_1)=1$, i.e., $(f^{k+1})^{(k)}(z_1)=1$. Note that $F'=(k+1)f^kf'$. Since $F'(z_1)=0$, it follows that $f'(z_1)=0$ and $f(z_1)\neq 0$. Therefore from (3.34), we conclude that $1=0$, which is impossible.

Next we suppose k is even. Solving (3.32), we get

$$f(z)=A_1\exp\big(\sqrt{\tilde{\lambda}}z\big)+B_1\exp\big(-\sqrt{\tilde{\lambda}}z\big), \tag{3.36}$$

where $A_1,B_1\in\mathbb{C}\setminus\{0\}$. Note that

$$f'(z)=A_1\sqrt{\tilde{\lambda}}\exp\big(\sqrt{\tilde{\lambda}}z\big)-\sqrt{\tilde{\lambda}}B_1\exp\big(-\sqrt{\tilde{\lambda}}z\big) \tag{3.37}$$

$$(f(z))^{k+1}=A_1^{k+1}\exp\big((k+1)\sqrt{\tilde{\lambda}}z\big)+\ldots+B_1^{k+1}\exp\big(-(k+1)\sqrt{\tilde{\lambda}}z\big) \tag{3.38}$$

$$(f(z))^{k+2}=A_1^{k+2}\exp\big((k+2)\sqrt{\tilde{\lambda}}z\big)+\ldots+B_1^{k+2}\exp\big(-(k+2)\sqrt{\tilde{\lambda}}z\big) \tag{3.39}$$

and so

$$\begin{aligned}\big((f(z))^{k+1}\big)^{(k)} &= A_1^{k+1}\big((k+1)\sqrt{\tilde{\lambda}}\big)^k\exp\big((k+1)\sqrt{\tilde{\lambda}}z\big)+\ldots\\ &\quad +B_1^{k+1}(-1)^k\big((k+1)\sqrt{\tilde{\lambda}}\big)^k\exp\big(-(k+1)\sqrt{\tilde{\lambda}}z\big).\end{aligned} \tag{3.40}$$

Therefore from (3.15) and (3.36)-(3.40), we deduce that

$$\begin{aligned}& A_1^{k+2}\Big(\big((k+1)\sqrt{\tilde{\lambda}}\big)^{k+1}-(k+1)\sqrt{\tilde{\lambda}}+c\Big)\exp\big((k+2)\sqrt{\tilde{\lambda}}z\big)+\ldots\\ & +B_1^{k+2}\Big((-1)^{k+1}\big((k+1)\sqrt{\tilde{\lambda}}\big)^{k+1}+(k+1)\sqrt{\tilde{\lambda}}+c\Big)\exp\big(-(k+2)\sqrt{\tilde{\lambda}}z\big)\\ = \;& A_1c\exp\big(\sqrt{\tilde{\lambda}}z\big)+B_1c\exp\big(-\sqrt{\tilde{\lambda}}z\big).\end{aligned} \tag{3.41}$$

Now from (3.41), one can easily conclude that

$$\big((k+1)\sqrt{\tilde{\lambda}}\big)^{k+1}-(k+1)\sqrt{\tilde{\lambda}}+c=0 \;\text{ and }\; -\big((k+1)\sqrt{\tilde{\lambda}}\big)^{k+1}+(k+1)\sqrt{\tilde{\lambda}}+c=0.$$

Solving we get $c=0$, which is impossible.

Case 2. Suppose $\varphi\equiv 0$. Since $F'\not\equiv 0$, we get $F\equiv F^{(k)}$. Now $F\equiv F^{(k)}$ implies that $\rho(F)=1$, i.e., $\rho(f)=1$ and f has no zeros. Therefore we conclude that $f(z)=d\exp\big(\frac{\lambda}{n}z\big)$, where $d,\lambda\in\mathbb{C}\backslash\{0\}$ such that $\lambda^k=1$. This completes the proof. □

4. Some applications

From Theorem 1.1, we see that the problem of the entire function g and its k-th derivative sharing one value a is related to the problem of the non-linear differential equation $g'(g-g^{(k)})-\varphi g(g-a)=0$ having a non-constant entire solution, where φ is an entire function. In general, it is difficult to judge whether the differential equation has a non-constant solution. However for the very special case $g=f^n$, where $n\in\mathbb{N}$, we can solve the equation completely.

As the applications of Theorem 1.1, we now present the following results.

Theorem 4.1. *Let φ be an entire function and $k,n\in\mathbb{N}$. Suppose F is a non-constant meromorphic solution of the differential equation $F'(F-F^{(k)})-\varphi F(F-1)=0$, where $F=f^n$ and $n\geq k+1$. Then only one of the following cases holds:*

(1) $F\equiv F^{(k)}$ *and* $f(z)=c\exp\left(\frac{\lambda}{n}z\right)$, *where* $c,\lambda\in\mathbb{C}\setminus\{0\}$ *such that* $\lambda^k=1$,

(2) $n=2$ *and* $f(z)=c_0\exp\left(\frac{1}{4}z\right)+c_1$, *where* $c_0,c_1\in\mathbb{C}\setminus\{0\}$ *such that* $c_1^2=1$.

Theorem 4.2. *Let φ be a non-constant entire function and $k,n\in\mathbb{N}$. Suppose F is a non-constant meromorphic solution of the differential equation $F'\left(F-F^{(k)}\right)-\varphi F(F-1)=0$, where $F=f^n$ and $n\geq k+1$. Then $k=1$, $n=2$ and $f(z)=c_0\exp\left(\frac{1}{4}z\right)+c_1$, where $c_0,c_1\in\mathbb{C}\setminus\{0\}$ such that $c_1^2=1$.*

Proof of Theorem 4.1. We have

$$F'(F-F^{(k)})=\varphi F(F-1), \tag{4.1}$$

where $F=f^n$ and φ is a non-constant entire function. Let F be a non-constant meromorphic solution of the equation (4.1). Now we divide the following two cases.

Case 1. Suppose $\varphi\not\equiv 0$. Since φ is a non-constant entire function, from (4.1), one can easily conclude that F is a non-constant entire function. Now we prove that $F=1\Rightarrow F^{(k)}=1$. If 1 is a Picard exceptional value of F, then obviously $F=1\Rightarrow F^{(k)}=1$. Next we suppose that 1 is not a Picard exceptional value of F. Let z_0 be a zero of $F-1$ of multiplicity p_0. Clearly z_0 is a zero of F' of multiplicity p_0-1. Then from (4.1), we deduce that z_0 must be a zero of $F-F^{(k)}$. Since $F-F^{(k)}=(F-1)-(F^{(k)}-1)$, it follows that z_0 is a zero of $F^{(k)}-1$. So $F=1\Rightarrow F^{(k)}=1$. Since $\varphi\not\equiv 0$, we have $F\not\equiv F^{(k)}$. Now proceeding in the same way as done in the proof of Case 1 of Theorem 1.1, one can easily conclude that $k=1$, $n=2$ and $f(z)=c_0\exp\left(\frac{1}{4}z\right)+c_1$, where $c_0,c_1\in\mathbb{C}\setminus\{0\}$ such that $c_1^2=1$.

Case 2. Suppose $\varphi\equiv 0$. Since $F'\not\equiv 0$, it follows that $F\equiv F^{(k)}$. We now want to prove that F is an entire function. For this let z_1 be a pole of F of multiplicity p_1. Then z_1 is also a pole $F^{(k)}$ of multiplicity p_1+k. Since $F\equiv F^{(k)}$, we arrive at a contradiction. Hence F is an entire function. The fact $F \equiv F^{(k)}$ implies that $\rho(F) = 1$, i.e., $\rho(f) = 1$. Also $F \equiv F^{(k)}$ implies that f has no zeros. Therefore we conclude that $f(z) = d \exp\left(\frac{\lambda}{n} z\right)$, where $d, \lambda \in \mathbb{C} \backslash \{0\}$ such that $\lambda^k = 1$. This completes the proof. □

Proof of Theorem 4.2. Since φ is a non-constant entire function, it follows that $\varphi \not\equiv 0$. Now the proof of Theorem 4.2 follows directly from the proof of Theorem 4.1. So we omit the proof. □

Now from Theorems 4.1 and 4.2, we immediately obtain the following corollary.

Corollary 4.1. *Let φ be a non-constant entire function and $k, n \in \mathbb{N}$ such that $k \geq 2$. Then the differential equation $F'(F - F^{(k)}) - \varphi F(F-1) = 0$, where $F = f^n$ and $n \geq k+1$ has no solutions.*

Following example shows that the condition "$n \geq k+1$" in Theorem 4.1 is sharp.

Example 4.1. *Let $f(z) = \exp(z) + \exp(-z)$, $k = 2$, $n = 1$ and $\varphi = 0$. Clearly f satisfies the differential equation (4.1), but f does not satisfy any case of Theorem 4.1.*

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Поступила 11 марта 2023

После доработки 09 июня 2023

Принята к публикации 15 июня 2023

Известия НАН Армении, Математика, том 59, н. 2, 2024, стр. 84 – 90.

ON FRACTIONAL KIRCHHOFF PROBLEMS WITH LIOUVILLE-WEYL FRACTIONAL DERIVATIVES

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Abstract. In this paper, we study the following fractional Kirchhoff-type problem with Liouville-Weyl fractional derivatives:

$$\begin{cases} \left[a + b\Big(\int_{\mathbb{R}}(|u|^2 + |{}_{-\infty}D_x^{\beta}u|^2)dx\Big)^{\varrho-1}\right]({}_xD_{\infty}^{\beta}({}_{-\infty}D_x^{\beta}u) + u) = |u|^{2_\beta^*-2}u, \ in\ \mathbb{R}, \\ u \in \mathbb{I}_{-}^{\beta}(\mathbb{R}), \end{cases}$$

where $\beta \in (0, \frac{1}{2})$, ${}_{-\infty}D_x^{\beta}u(\cdot), {}_xD_{\infty}^{\beta}u(\cdot)$ denote the left and right Liouville-Weyl fractional derivatives, $2_\beta^* = \frac{2}{1-2\beta}$ is fractional critical Sobolev exponent $a \geq 0$ and $b > 0$. Under suitable values of the parameters ϱ, a and b, we obtain a non-existence result of nontrivial solutions of infinitely many nontrivial solutions for the above problem.

MSC2020 numbers: 35R11, 35A15, 35J60, 47G20, 35J20.

Keywords: Liouville-Weyl fractional derivatives; Kirchhoff-type problem; non-existence result; infinitely many nontrivial solutions.

1. Introduction

The purpose of this article is to study the non-existence results for the following fractional Kirchhoff-type equation with Liouville-Weyl fractional derivatives:

$$\begin{cases} \left[a + b\Big(\int_{\mathbb{R}}(|u|^2 + |{}_{-\infty}D_x^{\beta}u|^2)dx\Big)^{\varrho-1}\right]({}_xD_{\infty}^{\beta}({}_{-\infty}D_x^{\beta}u) + u) = |u|^{2_\beta^*-2}u, \ in\ \mathbb{R}, \\ u \in \mathbb{I}_{-}^{\beta}(\mathbb{R}), \end{cases} \tag{1.1}$$

where $\beta \in (0, \frac{1}{2})$, ${}_{-\infty}D_x^{\beta}u(\cdot), {}_xD_{\infty}^{\beta}u(\cdot)$ denote the left and right Liouville-Weyl fractional derivatives, $2_\beta^* = \frac{2}{1-2\beta}$ is fractional critical Sobolev exponent, $a \geq 0$ and $b > 0$.

The theory of fractional operators for a long time remained hidden from the scientific community, with its pioneering works involving the integrals and fractional derivatives of Liouville, Riemann, Grunwald-Letnikov and Riemann-Liouville [6, 10, 30]. Then, around 1974, at a conference at the University of New Haven, in the United States, the first international conference on fractional calculus took place [24]. From that moment on, fractional calculus began to be disseminated and disseminated and countless fractional derivatives have been introduced, each one with its importance and relevance in the field of fractional operators [1, 8, 9, 12, 14, 17, 18, 19, 22]. We highlight in a special way, when it comes to applications in: medicine, engineering, physics, biology among other areas [6, 10, 11, 13, 20, 23].

We note that when $a = 1$, $b = 0$, problem (1.1) boils down to a fractional differential equation of the type

$$ {}_xD^{\beta}_{+\infty}({}_{-\infty}D^{\beta}_x u) = g(u), \text{ in } \mathbb{R}, $$

which is a special case of the fractional advection-dispersion equation proposed by Benson et. all. [3, 4, 5]. When $\beta \in (\frac{1}{2}, 1)$ several existence and multiplicity results can be found in [25, 26] and the reference therein. Recently, the case $\beta \in (0, \frac{1}{2})$ was considered in [28, 29].

On the other hand, in these last years, the study of Kirchhoff problems with fractional derivatives have been attracted the attention from many mathematicians. For instance, Nyamoradi and Zhou [15] dealt with the existence of nontrivial solutions for a Kirchhoff type problem with Liouville-Weyl fractional derivatives by using minimal principle and Morse theory. Nyamoradi et. all. [16] studied a class of Schrödinger-Kirchhoff equation with Liouville-Weyl fractional derivatives and obtained the existence and multiplicity of solutions by using mountain pass theorem and the symmetric mountain pass theorem. Tayyebi and Nyamoradi [21] established the existence and multiplicity of nontrivial solutions for a Kirchhoff equation with Liouville-Weyl fractional derivatives by using symmetric mountain pass theorem, Morse theory combined with local linking arguments and the Clark's theorem. The authors in [2] by using local linking arguments and Morse theory studied the existence and multiplicity of solutions for a fractional Kirchhoff equation with Liouville-Weyl fractional derivatives.

Since we did not find in the literature any paper dealing with problems involving fractional derivatives and critical exponent, motivated by the previous works, in the present paper we intend to show the non-existence results for problem (1.1) by applying suitable variational arguments.

2. Preliminaries and main results

In this section, we recall some useful preliminaries which will play an important role to solve the problem (1.1), and we state the main results of this work.

Definition 2.1. *The left and right Liouville-Weyl fractional integrals of order* $0 < \beta < 1$ *on the whole axis* $\mathbb{R}$ *are defined by*

$$ {}_{-\infty}I^{\beta}_x \phi(x) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^{x} (x-\xi)^{\beta-1}\phi(\xi)d\xi, \tag{2.1} $$

$$ {}_xI^{\beta}_{\infty} \phi(x) = \frac{1}{\Gamma(\beta)} \int_{x}^{\infty} (\xi-x)^{\beta-1}\phi(\xi)d\xi. \tag{2.2} $$

respectively, where $x \in \mathbb{R}$.

The left and right Liouville-Weyl fractional derivatives of order $0 < \beta < 1$ *on the whole axis* $\mathbb{R}$ *are defined by*

$$ {}_{-\infty}D_x^{\beta}\phi(x) = \frac{d}{dx}{}_{-\infty}I_x^{1-\beta}\phi(x), \tag{2.3} $$

$$ {}_{x}D_{\infty}^{\beta}\phi(x) = -\frac{d}{dx}{}_{x}I_{\infty}^{1-\beta}\phi(x). \tag{2.4} $$

respectively, where $x \in \mathbb{R}$.

2.1. **Fractional space of Sobolev type.** By argument in [29], we will look for weak solutions of the problem (1.1) hence the natural setting involves the fractional space of Sobolev type $\mathbb{I}_{-}^{\beta}(\mathbb{R})$ defined as

$$ \mathbb{I}_{-}^{\beta}(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : {}_{-\infty}D_x^{\beta}u \in L^2(\mathbb{R})\} $$

endowed with the scalar product

$$ \langle u, v\rangle_{\beta} = \int_{\mathbb{R}} u(x)v(x)dx + \int_{\mathbb{R}} {}_{-\infty}D_x^{\beta}u(x)\cdot {}_{-\infty}D_x^{\beta}v(x)dx $$

and norm

$$ \|u\|_{\beta} = \Big(\int_{\mathbb{R}} u^2dx + \int_{\mathbb{R}} |{}_{-\infty}D_x^{\beta}u(x)|^2dx\Big)^{1/2}. $$

It is well known that $(\mathbb{I}_{-}^{\beta}(\mathbb{R}), \langle .,.\rangle_{\beta})$ is a Hilbert space. Moreover, for $\beta \in (0, \frac{1}{2})$ we have the continuous embedding

$$ \mathbb{I}_{-}^{\beta}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R}) \text{ for every } p \in [2, 2_{\beta}^*], \tag{2.5} $$

where $2_{\beta}^* = \frac{2}{1-2\beta}$ is the fractional critical Sobolev exponent.

In the case $a = 1$, $b = 0$, the problem (1.1) will be transformed into the following critical problem with Liouville-Weyl fractional derivatives:

$$ {}_{x}D_{\infty}^{\beta}({}_{-\infty}D_x^{\beta}u) + u = |u|^{2_{\beta}^*-2}u, \ \textit{in } \mathbb{R}. \tag{2.6} $$

Set

$$ S_{\beta} := \inf_{u\in\mathbb{I}_{-}^{\beta}(\mathbb{R})\setminus\{0\}} \frac{\int_{\mathbb{R}}(|u|^2 + |{}_{-\infty}D_x^{\beta}u|^2)dx}{\left(\int_{\mathbb{R}}|u(x)|^{2_{\beta}^*}dx\right)^{\frac{2}{2_{\beta}^*}}}. \tag{2.7} $$

For any $\varepsilon > 0$, we can define $\widetilde{u}(x)$ as $u_{\varepsilon}(x) = \sqrt{\varepsilon}\widetilde{u}\left(\frac{x}{\varepsilon}\right)$, where $\widetilde{u}(x)$ is a minimizer for S_{β}. Clearly, $u_{\varepsilon}(x)$ is also a minimizer for S_{β}, satisfying (2.6) and

$$ \int_{\mathbb{R}}(|u_{\varepsilon}|^2 + |{}_{-\infty}D_x^{\beta}u_{\varepsilon}|^2)dx = \int_{\mathbb{R}}|u_{\varepsilon}(x)|^{2_{\beta}^*}dx = S_{\beta}^{\frac{2_{\beta}^*}{2_{\beta}^*-2}}. \tag{2.8} $$

Now, under suitable values of the parameters a, b and ϱ, we state the main results of this paper as follow:

Theorem 2.1. *Suppose that* $\varrho > 1$ *and* $\beta \in (0, \frac{1}{2})$. *Then, problem (1.1) has no nontrivial solution under one of the following conditions:*

(i) $\varrho = \frac{2_{\beta}^*}{2}$, $a = 0$ *and* $b > S_{\beta}^{-\varrho}$;

(ii) $\varrho = \frac{2^*_\beta}{2}$, $a > 0$ *and* $b \geq S_\beta^{-\varrho}$;

(iii) $\varrho > \frac{2^*_\beta}{2}$, $a, b > 0$ *satisfy*

$$\frac{2a(\varrho-1)}{2\varrho - 2^*_\beta}\left(\frac{(2\varrho - 2^*_\beta)bS_\beta^{\frac{2^*_\beta(\varrho-1)}{2^*_\beta-2}}}{a(2^*_\beta - 2)}\right)^{\frac{2^*_s-2}{2(\varrho-1)}} > 1;$$

(iv) $\varrho = \frac{1+2\beta}{1-2\beta}$, $a, b > 0$ *satisfy* $1 < 4abS_\beta^{\varrho+1}$.

Theorem 2.2. *Suppose that* $\varrho > 1$ *and* $\beta \in (0, \frac{1}{2})$. *Then the following properties hold:*

(i) $\varrho \neq \frac{2^*_\beta}{2}$, $a = 0$ *and* $b > 0$, *then problem (1.1) has infinitely many positive solutions and these solutions are*

$$b^{\frac{1}{2^*_\beta - 2\varrho}} S_\beta^{\frac{2^*_\beta(\varrho-1)}{(2^*_\beta-2\varrho)(2^*_\beta-2)}} u_\varepsilon \quad \textit{for any } \varepsilon > 0.$$

(ii) $\varrho = \frac{2^*_\beta}{2}$, $a > 0$ *and* $b < S_\beta^{-\varrho}$, *then problem (1.1) has infinitely many positive solutions and these solutions are given by*

$$\left(\frac{a}{1 - bS_\beta^\varrho}\right) u_\varepsilon \quad \textit{for any } \varepsilon > 0.$$

(iii) $\varrho > \frac{2^*_\beta}{2}$, $a, b > 0$ *satisfy*

$$\frac{2a(\varrho-1)}{2\varrho - 2^*_\beta}\left(\frac{(2\varrho - 2^*_\beta)bS_\beta^{\frac{2^*_\beta(\varrho-1)}{2^*_\beta-2}}}{a(2^*_\beta - 2)}\right)^{\frac{2^*_\beta-2}{2(\varrho-1)}} = 1, \tag{2.9}$$

then problem (1.1) has infinitely many positive solutions and these solutions are

$$\left(\frac{a(2^*_\beta - 2)}{(2\varrho - 2^*_\beta)bS_\beta^{\frac{2^*_\beta(\varrho-1)}{2^*_\beta-2}}}\right)^{\frac{1}{2(\varrho-1)}} u_\varepsilon \quad \textit{for any } \varepsilon > 0.$$

3. Proof of the main results

In this section, we deal with the proof of Theorems 2.1 and 2.2. Let us introduce the energy functional associated with problem (1.1):

$$J(u) = \frac{a}{2}\|u\|_\beta^2 + \frac{b}{2\varrho}\|u\|_\beta^{2\varrho} - \frac{1}{2^*_\beta}\int_{\mathbb{R}} |u(x)|^{2^*_\beta} dx, \tag{3.1}$$

which is well-defined for each $u \in \mathbb{I}^\beta_-(\mathbb{R})$. We know that $J \in C^1(\mathbb{I}^\beta_-(\mathbb{R})$. Moreover, it is easy to see that a weak solution of problem (1.1) is a critical point of the functional J.

Firstly, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. Suppose that $u \in \mathbb{I}_{-}^{\beta}(\mathbb{R})\setminus\{0\}$ is a solution of (1.1). Hence,

(i) from (2.7), we have

$$S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2\varrho} = S_\beta^{-\varrho}\|u\|_\beta^{2\varrho} < b\|u\|_\beta^{2\varrho} = \int_{\mathbb{R}^N}|u(x)|^{2_\beta^*}dx \leq S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2_\beta^*} = S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2\varrho}.$$

which gives a contradiction. Then, (i) holds true.

(ii) In view of (2.7), one can get

$$S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2\varrho} = S_\beta^{-\varrho}\|u\|_\beta^{2\varrho} \leq b\|u\|_\beta^{2\varrho} < a\|u\|_\beta^2 + b\|u\|_\beta^{2\varrho} = \int_{\mathbb{R}^N}|u(x)|^{2_\beta^*}dx \leq S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2\varrho},$$

which is impossible. Then, (ii) is satisfied.

(iii) Using the Young's inequality and (2.7), we can get

$$\begin{aligned} S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2_\beta^*} &= S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{\frac{2\varrho-2_\beta^*}{\varrho-1}}\|u\|_\beta^{\frac{\varrho 2_\beta^*-2\varrho}{\varrho-1}} \\ &\leq a\|u\|_\beta^2 + \frac{2_\beta^*-2}{2(\varrho-1)}\left(\frac{2a(\varrho-1)}{2\varrho-2_\beta^*}\right)^{-\frac{2\varrho-2_\beta^*}{2_\beta^*-2}} S_\beta^{-\frac{(\varrho-1)2_\beta^*}{2_\beta^*-2}}\|u\|_\beta^{2\varrho} \\ &< a\|u\|_\beta^2 + b\|u\|_\beta^{2\varrho} \\ &= \int_{\mathbb{R}^N}|u(x)|^{2_\beta^*}dx \leq S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2_\beta^*}, \end{aligned}$$

which leads to a contradiction. So, (iii) is verified.

(iv) From geometric-arithmetic inequality and (2.7) one can get

$$\begin{aligned} \|u\|_\beta^{\varrho+1} &< 2\sqrt{ab}S_\beta^{\frac{\varrho+1}{2}}\|u\|_\beta^{\varrho+1} \leq (a\|u\|^2 + b\|u\|^{2\varrho})S_\beta^{\frac{\varrho+1}{2}} \\ &\leq S_\beta^{\frac{\varrho+1}{2}}\int_{\mathbb{R}}|u(x)|^{2_\beta^*}dx \leq S_\beta^{\frac{\varrho+1}{2}}S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2_\beta^*} = \|u\|_\beta^{\varrho+1} \end{aligned}$$

a contradiction. Hence, we get the result (iv). □

Secondly, we give the proof of Theorem 2.2. To this end, for any $\varepsilon > 0$, we set

$$v_{\varepsilon,\beta}(x) = \vartheta^{\frac{1}{2_\beta^*-2}}u_\varepsilon(x), \tag{3.2}$$

and it is a positive solution of (2.6). So, $v_{\varepsilon,s}$ satisfies

$$\vartheta({}_xD_\infty^\beta({}_{-\infty}D_x^\beta v_{\varepsilon,\beta}) + v_{\varepsilon,\beta}) = |v_{\varepsilon,\beta}|^{2_\beta^*-2}v_{\varepsilon,\beta},\ in\ \mathbb{R}. \tag{3.3}$$

Then, if

$$\vartheta = a + b\Big(\int_{\mathbb{R}}(|v_{\varepsilon,\beta}|^2 + |{}_{-\infty}D_x^\beta v_{\varepsilon,\beta}|^2)dx\Big)^{\varrho-1}, \tag{3.4}$$

we can deduce that $v_{\varepsilon,\beta}$ is a solution of (1.1). Since u_ε satisfies (2.8), then by inserting (3.2) into (3.4) we can infer that

$$\vartheta = a + bS_\beta^{\frac{2_\beta^*(\varrho-1)}{2_\beta^*-2}}\vartheta^{\frac{2(\varrho-1)}{2_\beta^*-2}}. \tag{3.5}$$

Furthermore, if $\vartheta \in (0,+\infty)$ is a solution of (3.5), then $v_{\varepsilon,\beta}$ is a solution of problem (1.1).

Proof of Theorem 2.2. (i) If $\varrho \neq \frac{2^*_\beta}{2}$, then $\frac{2(\varrho-1)}{2^*_\beta-2} \neq 1$. So, if $a=0$, (3.5) has solution

$$\vartheta = b^{\frac{2^*_\beta-2}{2^*_\beta-2\varrho}} S_\beta^{\frac{2^*_\beta(\varrho-1)(2^*_\beta-2)}{(2^*_\beta-2)(2^*_\beta-2\varrho)}}.$$

Hence, in view of (3.2) we get the result (i).

(ii) If $\varrho = \frac{2^*_\beta}{2}$, then $\frac{2(\varrho-1)}{2^*_\beta-2} = 1$. So, (3.5) is equivalent to

$$\vartheta = a + bS^\varrho_\beta \vartheta, \tag{3.6}$$

and then $\vartheta = \frac{1}{1-bS^\varrho_\beta} > 0$. Hence, by (3.2) it follows that (ii) holds true.

(iii) If $\varrho > \frac{2^*_\beta}{2}$, then $\frac{2(\varrho-1)}{2^*_\beta-2} > 1$. Define

$$\varphi(\vartheta) := a\vartheta^{-1} + bS_\beta^{\frac{2^*_\beta(\varrho-1)}{2^*_\beta-2}} \vartheta^{\frac{2\varrho-2^*_\beta}{2^*_\beta-2}}$$

which implies that

$$\varphi(\vartheta) = 1 \quad \text{iff} \ \ \vartheta \text{ solves (3.5)}. \tag{3.7}$$

We can easily see that $\varphi(\vartheta)$ achieves its minimum at

$$\vartheta_0 = \left(\frac{a(2^*_\beta - 2)}{(2\varrho - 2^*_\beta) bS_\beta^{\frac{2^*_\beta(\varrho-1)}{2^*_\beta-2}}} \right)^{\frac{2^*_\beta-2}{2(\varrho-1)}}$$

and

$$\min_{\vartheta>0} \varphi(\vartheta) = \varphi(\vartheta_0) = \frac{2a(\varrho-1)}{2\varrho - 2^*_\beta} \left(\frac{(2\varrho - 2^*_\beta) bS_\beta^{\frac{2^*_\beta(\varrho-1)}{2^*_\beta-2}}}{a(2^*_\beta-2)} \right)^{\frac{2^*_\beta-2}{2(\varrho-1)}}.$$

By condition (2.9) we have $\varphi(\vartheta_0) = 1$, and from (3.7) we get that ϱ_0 is a solution of (3.5). From (3.2), we have the result (iii).

Acknowledgments. The authors would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript.

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Поступила 17 февраля 2023
После доработки 23 июня 2023
Принята к публикации 15 июля 2023

Индекс 77735

ИЗВЕСТИЯ НАН АРМЕНИИ: МАТЕМАТИКА

том 59, номер 2, 2024

Содержание

IZVESTIYA NAN ARMENII: MATEMATIKA

Vol. 59, No. 2, 2024

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