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ON BENIGN SUBGROUPS CONSTRUCTED BY HIGMAN'S SEQUENCE BUILDING OPERATION

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Abstract. For Higman's sequence building operation ω_m and for any integer sequences set $\mathcal{B}$ the subgroup $A_{\omega_m \mathcal{B}}$ is benign in a free group G as soon as $A_{\mathcal{B}}$ is benign in G. Higman used this property as a key step to prove that a finitely generated group is embeddable into a finitely presented group if and only if it is recursively presented. We build the explicit analog of this fact, i.e., we explicitly give a finitely presented overgroup $K_{\omega_m \mathcal{B}}$ of G and its finitely generated subgroup $L_{\omega_m \mathcal{B}} \le K_{\omega_m \mathcal{B}}$ such that $G \cap L_{\omega_m \mathcal{B}} = A_{\omega_m \mathcal{B}}$ holds. Our construction can be used in explicit embeddings of finitely generated groups into finitely presented groups, which are theoretically possible by Higman's theorem. To build our construction we suggest some auxiliary "nested" free constructions based on free products with amalgamation and HNN-extensions.

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Keywords: recursive group; finitely presented group; embedding of group; benign subgroup; free product of groups with amalgamated subgroup; HNN-extension of group; sequence building operation

1. Introduction

Higman's fundamental result establishing connection between group theory and computability theory states: *a finitely generated group G can be embedded in a finitely presented group if and only if it is recursively presented* [9]. The requirement that G is finitely generated is not critical, and it can be replaced by the condition that G has an effectively enumerable countable set of generators, see the remark on p. 456 in [9].

Despite importance of this theorem, possibility of *explicit* embedding of any recursively presented group into some finitely presented group is a less intelligible issue, and it is open problem even for some well known groups. In particular, construction of an *explicit* embedding of the additive group $\mathbb{Q}$ of rationals into a finitely presented group was an open question mentioned by Bridson and de la Harpe as "Well-known problem" 14.10 (a) in Kourovka notebook [12] and also announced in [8]. Recently a direct solution to that problem was found by Belk, Hyde and Matucci in [6]; and an algorithm how to build such an explicit embedding was

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given in [17], *without* an explicit finitely presented group containing $\mathbb{Q}$ though. Also, based on recent work [2]–[5] it is possible to embed $\mathbb{Q}$ as a center for a continuum of non-isomorphic 2-generator groups. These along with some other remarks in the literature [19, 1] motivate research on explicit embeddings of recursively presented groups into finitely presented groups.

The key group-theoretic concept introduced in [9] is that of *benign subgroup*: a subgroup H is benign in a finitely generated group G, if there is a finitely presented overgroup K of G, and a finitely generated subgroup L of K such that $G \cap L = H$. Actually, the most part of [9] is dedicated to showing that if a subgroup H of a specific type is benign in the free group $G = \langle a, b, c\rangle$ of rank 3, then applying some specific kinds of operations to H (such as, the sequence building operation ω_m, see below) we again get a benign subgroup in G.

Denote by $\mathcal{E}$ the set of all functions $f : \mathbb{Z} \to \mathbb{Z}$ with finite supports. If $f(i) = 0$ for all $i < 0$ and $i \geq m$ (for a fixed $m = 1, 2, \ldots$), then f can be recorded as a sequence $f = (j_0, \ldots, j_{m-1})$ assuming $f(i) = j_i$ for $i = 0, \ldots, m-1$ [9]. Then the following words are defined in the free group $G = \langle a, b, c\rangle$ with respect to f:

$$b_f = b_0^{j_0} \cdots b_{m-1}^{j_{m-1}} \quad \text{and} \quad a_f = a^{b_f} = b_f^{-1} a\, b_f \tag{1.1}$$

where $b_i = b^{c^i}$ for $i = 1, \ldots, m-1$. Let $\mathcal{E}_m$ be the subset of all functions f of the above type. For any subset $\mathcal{B}$ of $\mathcal{E}$ denote $A_{\mathcal{B}} = \langle a_f \mid f \in \mathcal{B}\rangle$, in particular, $A_{\mathcal{E}_m} = \langle a_f \mid f \in \mathcal{E}_m\rangle$. See details and examples in [17].

For m and for any subset $\mathcal{B} \subseteq \mathcal{E}$ the *sequence building* operation ω_m is defined on $\mathcal{B}$ as follows: $\omega_m(\mathcal{B})$ consists of all $f \in \mathcal{E}$ for which for every $i \in \mathbb{Z}$ there exists a sequence $\big(f(mi+0), \ldots, f(mi+m-1)\big) \in \mathcal{B}$ [9]. In other words, this operation just constructs new sequences f by concatenation of some sequences of length m picked from $\mathcal{B}$. For details see [17], and also check Section 3 below where the new sequence (3.4) is built from the sequences $(6, 4, 5, 3), (7, 2, 4, 9) \in \mathcal{B}$ and from the zero sequence using ω_4. Having the subgroup $A_{\mathcal{B}} = \langle a_f \mid f \in \mathcal{B}\rangle$ of G one may construct the subgroup $A_{\omega_m \mathcal{B}} = \langle a_{\omega_m \mathcal{B}} \mid f \in \mathcal{B}\rangle$. And if $\mathcal{B} \subseteq \mathcal{E}_m$, then $A_{\mathcal{B}} \leq A_{\omega_m \mathcal{B}}$, see subsection 3.2 where samples of $A_{\mathcal{B}}$ and $A_{\omega_4 \mathcal{B}}$ are given.

If for some $\mathcal{B} \subseteq \mathcal{E}$ the group $A_{\mathcal{B}}$ is benign in G for a given finitely presented overgroup K holding G, and for the finitely generated subgroup $L \leq K$, we stress that by denoting $K = K_{\mathcal{B}}$ and $L = L_{\mathcal{B}}$, and writing $G \cap L_{\mathcal{B}} = A_{\mathcal{B}}$ in $K_{\mathcal{B}}$. Clearly, $K_{\mathcal{B}}$ and $L_{\mathcal{B}}$ may not be unique for a given $\mathcal{B}$.

A main strategy of [9] is to start from a set $\mathcal{B} \subseteq \mathcal{E}$ for which the subgroup $A_{\mathcal{B}}$ is benign in G, and to show that if a new set $\mathcal{B}'$ is obtained from $\mathcal{B}$ by means of certain operations, then $A_{\mathcal{B}'}$ also is benign in G. In this terms [9, Lemma 4.10] states that if for the given $\mathcal{B} \subseteq \mathcal{E}$ the subgroup $A_{\mathcal{B}}$ is benign in G, then $A_{\omega_m \mathcal{B}}$ also is benign in G for any m.

The objective of this note is to additionally show that if the respective groups $K_{\mathcal{B}}$ and $L_{\mathcal{B}}$ can be constructed *explicitly*, then $K_{\omega_m \mathcal{B}}$ and $L_{\omega_m \mathcal{B}}$ can also be constructed *explicitly*:

Theorem 1.1. *Let $\mathcal{B} \subseteq \mathcal{E}$ be a sequences set such that $A_{\mathcal{B}}$ is benign in G and, moreover, the respective finitely presented group $K_{\mathcal{B}}$ and its finitely generated subgroup $L_{\mathcal{B}}$ are given explicitly. Then for any $m = 1, 2, \ldots$ the subgroup $A_{\omega_m \mathcal{B}}$ also is benign in G, and the finitely presented group $K_{\omega_m \mathcal{B}}$ and its finitely generated subgroup $L_{\omega_m \mathcal{B}}$ can also be given explicitly.*

The promised explicit group $K_{\omega_m \mathcal{B}}$ is given in (5.9), $L_{\omega_m \mathcal{B}}$ is given in (5.10), while the components $\Psi, \bar{\Delta}, L'$, etc., used in those formulas all are defined in Section 5 using some free constructions. And under explicitly given $K_{\mathcal{B}}$ and $L_{\mathcal{B}}$ one may understand, say, their presentations with generators and defining relations.

The proof of this theorem occupies sections 3–5 below. In particular, in Section 3 we build an initial embedding construction in which $A_{\omega_m \mathcal{B}}$ is an intersection of G with certain subgroup $W_{\mathcal{B}}$. As this construction is not yet finitely presented, we in Section 4 suggest some auxiliary "nested" free constructions (such as (4.3)), and using them we obtain the finitely presented $K_{\omega_m \mathcal{B}}$ in Section 5.

In order to avoid any repetition of material already published in [16, 17] or elsewhere, we below often adopt constructions from other work. This makes parts of the current text dependant on other articles, but the provided exact references, we hope, alleviate any inconvenience.

2. Preliminary information

2.1. **Free constructions.** For background information on free products with amalgamation and on HNN-extensions we refer to [7] and [10]. Notations vary in the literature, and to maintain uniformity we are going to adopt notations we used in [16].

If any groups G and H have subgroups, respectively, A and B isomorphic under $\varphi : A \to B$, then the (generalized) free product of G and H with amalgamated subgroups A and B is denoted by $G *_{\varphi} H$ (an alternative notation in the literature being $G *_{A=B} H$). When G and H are overgroups of the same subgroup A, and φ is just the *identical* isomorphism on A, we write $\Gamma = G *_A H$.

If G has subgroups A and B isomorphic under $\varphi : A \to B$, then the HNN-extension of the base G by some stable letter t with respect to the isomorphism φ

is denoted by $G *_\varphi t$. In case when $A = B$ and φ is *identity* on A, we may write $\Gamma = G *_A t$. We also use HNN-extensions $G *_{\varphi_1,\varphi_2,\ldots} (t_1, t_2, \ldots)$ with more than one stable letters, see [16] for details.

Our usage of the *normal forms* in free constructions is close to [7].

2.2. **Benign subgroups and Higman operations.** For detailed informaton on bening subgroups we refer to Sections 3, 4 in [9], see also Section 3 in [16]. Higman operations and their basic properties can be found in Section 2 in [9], see also Section 3 in [16] and Section 2 in [17].

From definition of bening subgroup it is very easy to see that arbitrary finitely generated subgroup H in any finitely presented group G is benign in G, for, the group G itself acts as a finitely presented overgroup of G with a finitely generated subgroup H, such that $H \cap H = H$. We are going to often use this remark in the sequel.

2.3. **Subgroups in free constructions.** The following two auxiliary facts are adopted from [16], and they follow from more general Lemma 2.2 and Lemma 2.4 in [16].

Corollary 2.1 (Corollary 2.3 in [16])**.** *Let $\Gamma = G *_A H$, and let $G' \le G$, $H' \le H$ be subgroups such that $G' \cap A = H' \cap A$. Then for $\Gamma' = \langle G', H' \rangle$ and $A' = G' \cap A$ we have:*

(1) *$\Gamma' = G' *_{A'} H'$, in particular, if $A \le G', H'$, then $\Gamma' = G' *_A H'$;*
(2) *$\Gamma' \cap A = A'$, in particular, if $A \le G', H'$, then $\Gamma' \cap A = A$;*
(3) *$\Gamma' \cap G = G'$ and $\Gamma' \cap H = H'$.*

Corollary 2.2 (Corollary 2.5 in [16])**.** *Let $\Gamma = G *_A t$, and let $G' \le G$ be a subgroup. Then for $\Gamma' = \langle G', t \rangle$ and $A' = G' \cap A$ we have:*

(1) *$\Gamma' = G' *_{A'} t$, in particular, if $A \le G'$, then $\Gamma' = G' *_A t$;*
(2) *$\Gamma' \cap A = A'$, in particular, if $A \le G'$, then $\Gamma' \cap A = A$;*
(3) *$\Gamma' \cap G = G'$.*

Remark 2.1. It is easy to adapt Corollary 2.2 for the case of multiple stable letters $t_1, \ldots, t_k$ which fix the same subgroup A in G. In such a case point (1) in Corollary 2.2 will read: $\Gamma' = G' *_{A'} (t_1, \ldots, t_k)$ for $\Gamma' = \langle G', t_1, \ldots, t_k \rangle$ and $A' = G' \cap A$. We are going to use this fact only once, in the proof of Lemma 5.4.

2.4. **The "conjugates collecting" process.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be some disjoint subsets in any group G. Then any element $w \in \langle \mathfrak{X}, \mathfrak{Y} \rangle$ can be written as:

$$w = u \cdot v = x_1^{\pm v_1} x_2^{\pm v_2} \cdots x_k^{\pm v_k} \cdot v$$

with some $v_1, v_2, \ldots, v_k,\ v \in \langle \mathfrak{Y} \rangle$, and $x_1, x_2, \ldots, x_k \in \mathfrak{X}$. The proof, examples and variations of this fact can be found in Subsection 2.6 in [16]. We use the name "conjugates collecting" just because we heavily used it in [16], and we need a name to refer to (we were unable to find a conventional name to this in the literature).

3. The initial embedding construction

3.1. **Construction of Δ.** The free group $\langle b, c\rangle$ contains a free subgroup $\langle b_i \mid i \in \mathbb{Z}\rangle$ of infinite rank, which for any $m = 1, 2, \ldots$ decomposes into a free product $B_m * \bar{B}_m$ with $B_m = \langle \ldots b_{-2}, b_{-1};\ b_m, b_{m+1}, \ldots\rangle$ and $\bar{B}_m = \langle b_0, \ldots, b_{m-1}\rangle$.

Introducing three stable letters g, h, k, all fixing B_m, build the HNN-extension:

$$\Gamma = \langle b, c\rangle *_{B_m} (g, h, k). \tag{3.1}$$

Denote $\bar{G} = \langle g, h, k\rangle$, and in analogy with b_i, b_f, a_f of (1.1) introduce $h_i = h^{k^i}$, h_f, and $g_f = g^{h_f}$ in the free group $\bar{G}$. Fixing the subgroup $R = \langle g_f b_f^{-1} \mid f \in \mathcal{E}_m\rangle$ of Γ by means of a new stable letter a build the HNN-extension $\Gamma *_R a$.

The intersection $\langle b, c\rangle \cap R$ is trivial because the non-trivial words of type $g_f b_f^{-1}$ generate R freely, and so any non-trivial word they generate must involve at least one g, and hence it need to be outside $\langle b, c\rangle$. Then by (1) in Corollary 2.2 the subgroup generated in $\Gamma *_R a$ by $\langle b, c\rangle$ together with a is equal to $\langle b, c\rangle *_{\langle b,c\rangle \cap R} a = \langle b, c\rangle *_{\{1\}} a = \langle b, c\rangle * a$ which is the free group $G = \langle a, b, c\rangle$. So a, b, c generate a free subgroup in Γ, and hence the map sending a, b, c to a, b^{c^m}, c can be continued to an isomorphism $\rho : G \to \langle a, b^{c^m}, c\rangle$. Identifying this ρ to a further stable letter r we arrive to the final HNN-extension of this section:

$$\Delta = \big(\Gamma *_R a\big) *_\rho r = \Big(\big(\langle b, c\rangle *_{B_m} (g, h, k)\big) *_R a\Big) *_\rho r. \tag{3.2}$$

3.2. **Obtaining $G \cap W_{\mathcal{B}} = A_{\omega_m \mathcal{B}}$ in Δ.** For any subset $\mathcal{B}$ of $\mathcal{E}$ denote $W_{\mathcal{B}} = \langle g_f,\ a,\ r \mid f \in \mathcal{B}\rangle$, and show that in Δ we have

$$G \cap W_{\mathcal{B}} = A_{\omega_m \mathcal{B}}. \tag{3.3}$$

Firstly notice that if $\mathcal{B}_m = \mathcal{B} \cap \mathcal{E}_m$, then $\omega_m(\mathcal{B}) = \omega_m(\mathcal{B}_m)$. Hence we may without loss of generality suppose $\mathcal{B} \subseteq \mathcal{E}_m$ below (if a *short* sequence contains less than m integers, we can without loss of generality extend its length to m by adding some extra 0's at the end).

For arbitrary sequence $f \in \omega_m \mathcal{B}$ the element $a_f = a^{b_f}$ is inside $W_{\mathcal{B}}$. Let us display this uncomplicated fact by a routine step-by-step construction example. Let $m = 4$ and let $(6, 4, 5, 3), (7, 2, 4, 9) \in \mathcal{B}$. Then by sequence building operations $\omega_4 \mathcal{B}$ contains the sequence, say,

$$f = (0, 0, 0, 0,\ \ 7, 2, 4, 9,\ \ 0, 0, 0, 0,\ \ 0, 0, 0, 0,\ \ 6, 4, 5, 3,\ \ 7, 2, 4, 9). \tag{3.4}$$

To show that $a^{b_f} \in W_{\mathcal{B}}$ start by the initial functions $l_1 = (7,2,4,9)$ and $l_2 = (6,4,5,3)$ in $\mathcal{B}$, and then use them by a few steps to arrive to the function f above. We are going to use the evident fact that the relation $(g_f b_f^{-1})^a = g_f b_f^{-1}$ is equivalent to $a^{g_f} = a^{b_f}$.

Step 1. Since $l_1 = (7,2,4,9)$ is in $\mathcal{B}$, then $g_{l_1} \in W_{\mathcal{B}}$, and so $a^{g_{l_1}} = a^{b_{l_1}} = a^{b_0^7\, b_1^2 b_2^4 b_3^9} \in W_{\mathcal{B}}$.

Step 2. Since $b_i^r = b_i^\rho = (b^\rho)^{(c^i)^\rho} = (b^{c^4})^{c^i} = b^{c^{i+4}} = b_{i+4}$, then conjugating the above obtained element $a^{b_{l_1}}$ by r we get:

$$\big(a^{b_{l_1}}\big)^r = \big(a^r\big)^{(b_0^7\, b_1^2 b_2^4 b_3^9)^r} = a^{b_4^7\, b_5^2 b_6^4 b_7^9} = a^{b_0^0\, b_1^0 b_2^0 b_3^0 \cdot b_4^7\, b_5^2 b_6^4 b_7^9} = a^{b_{l_3}} \in W_{\mathcal{B}}$$

for the sequence $l_3 = (0,0,0,0,\ 7,2,4,9)$.

Next, conjugating $a^{b_{l_3}}$ by g_{l_2} we have:

$$\big(a^{b_{l_3}}\big)^{g_{l_2}} = a^{b_{l_3} \cdot g_{l_2}} = a^{b_4^7\, b_5^2 b_6^4 b_7^9 \cdot g_{l_2}}.$$

Step 3. Each of stable letters g, h, k commutes with any b_i for $i < 0$ or $i \geq m = 4$, and so g_{l_2} commutes with $b_4^7\, b_5^2 b_6^4 b_7^9$ and so:

$$a^{b_4^7\, b_5^2 b_6^4 b_7^9 \cdot g_{l_2}} = a^{g_{l_2} \cdot b_4^7\, b_5^2 b_6^4 b_7^9}.$$

Then once more applying step 1 to $a^{g_{l_2}}$ we transform the above to:

$$\big(a^{g_{l_2}}\big)^{b_4^7\, b_5^2 b_6^4 b_7^9} = a^{b_0^6\, b_1^4 b_2^5 b_3^3 \cdot b_4^7\, b_5^2 b_6^4 b_7^9} = a^{b_{l_4}}$$

for the sequence $l_4 = (6,4,5,3,\ \ 7,2,4,9)$. Then we repeat the above step 2 for *three times* i.e., conjugate the above by r^3 to get the element $a^{b_{l_5}}$ for the sequence:

$$l_5 = (0,0,0,0,\ \ 0,0,0,0,\ \ 0,0,0,0,\ \ 6,4,5,3,\ \ 7,2,4,9).$$

Next apply step 3 and step 1 again to conjugate $a^{b_{l_5}}$ by g_{l_1}. We get the element $a^{b_{l_6}}$ for the sequence:

$$l_6 = (7,2,4,9,\ \ 0,0,0,0,\ \ 0,0,0,0,\ \ 6,4,5,3,\ \ 7,2,4,9).$$

Then we again apply step 2, i.e., conjugate $a^{b_{l_6}}$ by r to discover in $W_{\mathcal{B}}$ the element $a^{b_f} = a_f$ with the sequence f promised in (3.4) above.

Since such a procedure can easily be performed for an *arbitrary* $f \in \omega_m \mathcal{B}$, we get that $A_{\omega_m \mathcal{B}} \leq W_{\mathcal{B}}$. And since also $A_{\omega_m \mathcal{B}} \leq G$, we have $A_{\omega_m \mathcal{B}} \leq G \cap W_{\mathcal{B}}$.

Next assume some word w from $W_{\mathcal{B}} = \langle g_f,\, a,\, r \mid f \in \mathcal{B} \rangle$ is in G. Since w also is in Δ, it can be brought to its normal form involving stable letter r and some elements from $\Gamma *_L a$. The latter elements, in turn, can be brought to normal forms involving stable letter a and some elements from Γ. Then the latters can further be brought to normal forms involving stable letters g, h, k and some elements from $\langle b, c \rangle$. That is,

w can be brought to a *unique "nested" normal form* reflecting three "nested" HNN-extensions in the right-hand side of (3.2). Let us detect the cases when it involves nothing but the letters a, b, c. The only relations of Γ involve g, h, k, and they are equivalent to $a^{g_f} = a^{b_f}$. Thus, the only way by which g, h, k may be eliminated in the normal form is to have in w subwords of type $g_f^{-1} a\, g_f = a^{g_f}$ which can be replaced by respective subwords $a^{b_f} \in G$. If after this procedure some subwords g_f still remain, then three scenario cases are possible:

Case 1. The word w may contain a subword of type $w' = g_f^{-1} a^{b_l} g_f$ for such an l that $l(i) = 0$ for $i = 0, \ldots, m{-}1$. Check the example of step 1, when this is achieved for $l = l_3 = (0, 0, 0, 0,\ 7, 2, 4, 9)$ and $f = l_2 = (6, 4, 5, 3)$. Then just replace w' by $a^{b_{l'}}$ for an $l' \in \omega_m \mathcal{B}$ (such as $l' = l_4 = (6, 4, 5, 3,\ \ 7, 2, 4, 9)$ in our example).

Case 2. If $w' = g_f^{-1} a^{b_l} g_f$, but the condition $l(i) = 0$ fails for an $i = 0, \ldots, m-1$, then g_f does *not* commute with b_l, so we cannot apply the relation $a^{g_f} = a^{b_f}$, and so $w \notin G$. Turning to example in steps 1–3, notice that for, say, $f = (7, 2, 4, 9) \in \mathcal{B}$ we may *never* get something like $a^{(g_f)^2} = \big(a^{b_0^7 b_1^2 b_2^4 b_3^9}\big)^{g_f} = a^{(b_0^7 b_1^2 b_2^4 b_3^9)^2}$ because g_f does not commute with b_0, b_1, b_2, b_3. That is, all the *new* functions l we get are from $\omega_m \mathcal{B}$ *only.*

Case 3. If g_f is in w, but is not in a subword $g_f^{-1} a^{b_l} g_f$, we again have $w \notin G$, unless all such g_f trivially cancel each other.

This means, if $w \in G$, then elimination of g, h, k turns w to a product of elements from $\langle r \rangle$ and of some a^{b_f} for some $f \in \omega_m \mathcal{B}$ (a also is of that type, as $(0) \in \mathcal{B}$). Now apply 2.4 for $\mathfrak{X} = \{a^{b_f} \mid f \in \omega_m \mathcal{B}\}$ and $\mathfrak{Y} = \{r\}$ to state that w is a product of some power r^i and of some elements each of which is an a^{b_f} conjugated by a power r^{n_i} of r. These conjugates certainly are in $\omega_m \mathcal{B}$ (see step 2 above), and so $w \in G$ if and only if $i = 0$, i.e., if $w \in A_{\omega_m \mathcal{B}}$.

Hence, equality (3.3) is established for any subset $\mathcal{B}$ of $\mathcal{E}$.

Remark 3.1. However, (3.3) cannot yet guarantee that $A_{\omega_m \mathcal{B}}$ is benign in G as soon as $A_{\mathcal{B}}$ is benign in G, because the group Δ in (3.2) is not finitely presented, and its subgroup $W_{\mathcal{B}} = \langle g_f,\, a,\, r \mid f \in \mathcal{B} \rangle$ may not be finitely generated, when $\mathcal{B}$ is infinite. The sections below will add these missing features replacing Δ by a much bulkier construction.

4. Auxiliary free constructions

In this section we generalize some of the results in Section 3 in [9]. Hence, the lemmas below may be of some independent interest also.

For any subgroup A of an arbitrary group G the well known equality $G \cap G^t = A$ holds in the HNN-extension $G *_A t$. It trivially follows, say, from uniqueness of the normal form in $G *_A t$. We need the following generalization of this fact:

Lemma 4.1. *Let $A_1, \dots, A_r$ be arbitrary subgroups in a group G. Then the following equality holds in the HNN-extension $G *_{A_1,\dots,A_r} (t_1, \dots, t_r)$:*

$$G \cap G^{t_1 \cdots t_r} = \textstyle\bigcap_{i=1}^r A_i. \tag{4.1}$$

Proof. Choose a transversal T_{A_i} to A_i in G, $i = 1, \dots, r$. Take any $g \in G$, and show that if $g^{t_1 \cdots t_r} \in G$, then g is inside each of A_i. Write $g = a_1 l_1$ where $a_1 \in A_1$ and $l_1 \in T_{A_1}$. In turn, a_1 can be written as $a_1 = a_2 l_2$ where $a_2 \in A_2$ and $l_2 \in T_{A_1}$. This process can be continued for $A_3, \dots, A_r$. (the case when some of a_i or l_i, $i = 1, \dots, r$, are trivial is *not* ruled out). Since the inverse t_i^{-1} of the stable letter t_i also fixes A_i, calculation of the normal form for $g^{t_1 \cdots t_r}$ can be started via the following steps:

$$\begin{aligned} g^{t_1 \cdots t_r} &= t_r^{-1} \cdots t_1^{-1} a_1 l_1 \, t_1 \cdots t_r \\ &= t_r^{-1} \cdots t_2^{-1} a_1 t_1^{-1} l_1 \, t_1 \cdots t_r \\ &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &= a_r \, t_r^{-1} l_r \, t_{r-1}^{-1} l_{r-1} \cdots \, l_3 \, t_2^{-1} l_2 \, t_1^{-1} l_1 \, t_1 \cdots t_r \end{aligned} \tag{4.2}$$

The above belongs to G only if it contains no stable letters t_i. But the last line of (4.2) does not contain t_1 only when $l_1 = 1$, hence $t_1^{-1} l_1 \, t_1 = 1$, and $t_2^{-1} l_2 t_1^{-1} l_1 \, t_1 \, t_2 = t_2^{-1} l_2 \, t_2$. Then to exclude t_2 we must have $l_2 = 1$, hence $t_2^{-1} l_2 \, t_2 = 1$. At the end we get (4.2) reduced to $a_r t_r^{-1} l_r t_r = a_r$ where $l_r = 1$, and therefore $a_r \in \bigcap_{i=1}^r A_i$.

On the other hand, any $g \in \bigcap_{i=1}^r A_i$ is fixed by each of t_i, and so $g^{t_1 \cdots t_r} = g \in G$, and thus, $\bigcap_{i=1}^r A_i \subseteq G \cap G^{t_1 \cdots t_r}$. □

Another proof of this lemma could be deduced from Corollary 2.1 and Corollary 2.2 (in a manner rather similar to the proof of Lemma 4.3 below), but we prefer this version as it follows from more basic properties already.

Later we are going to use a specific free construction built for a system of groups via HNN-extensions and free products with amalgamation. Namely, let $G \le K_1, \dots, K_r$ be arbitrary groups such that $K_i \cap K_j = G$ for any distinct indices $i, j = 1, \dots, r$. If in each K_i we pick a subgroup L_i, and denote $G \cap L_i = A_i$, $i = 1, \dots, r$, we can build the following "nested" free construction:

$$\Theta = \Big(\cdots \Big(\big((K_1 *_{L_1} t_1) *_G (K_2 *_{L_2} t_2) \big) *_G (K_3 *_{L_3} t_3) \Big) \cdots \Big) *_G (K_r *_{L_r} t_r). \tag{4.3}$$

By these notations:

Lemma 4.2. *In the free construction Θ the following equality holds:*

$$\langle G, t_1, \ldots, t_r \rangle = G *_{A_1, \ldots, A_r} (t_1, \ldots, t_r).$$

Proof. Applying induction over r we for $r = 2$ have to display $\langle G, t_1, t_2 \rangle = G *_{A_1, A_2} (t_1, t_2)$ in $\Theta = (K_1 *_{L_1} t_1) *_G (K_2 *_{L_2} t_2)$.

In $K_1 *_{L_1} t_1$ we by (1) in Corollary 2.2 have $\langle G, t_1 \rangle = G *_{G \cap L_1} t_1 = G *_{A_1} t_1$. Similarly, $\langle G, t_2 \rangle = G *_{A_2} t_2$ in $K_2 *_{L_2} t_2$. And since in Θ the intersection of both $\langle G, t_1 \rangle$ and $\langle G, t_2 \rangle$ with G clearly is G, we apply (1) in Corollary 2.1 to get:

$$\langle G, t_1, t_2 \rangle = \big\langle \langle G, t_1 \rangle, \langle G, t_2 \rangle \big\rangle = (G *_{A_1} t_1) *_G (G *_{A_2} t_2).$$

But the above amalgamated free prodcut is noting but $G *_{A_1, A_2} (t_1, t_2)$, which is trivial to see by listing all the defining relations of both constructions: relations of G followed by relations stating that t_1 fixes the A_1 and t_2 fixes A_2 (plus the relations identifying both copies of G, if we initially assume them to be disjoint).

Next assume the proof is done for $r-1$, i.e.,

$$\langle G, t_1, \ldots, t_{r-1} \rangle = G *_{A_1, \ldots, A_{r-1}} (t_1, \ldots, t_{r-1}).$$

Again by (1) in Corollary 2.2 write $\langle G, t_r \rangle = G *_{G \cap L_r} t_r = G *_{A_r} t_r$. We have $\langle G, t_1, \ldots, t_{r-1} \rangle$ and $\langle G, t_r \rangle$ both intersect with G in G, and we by (1) in Corollary 2.1 get:

$$\langle G, t_1, \ldots, t_r \rangle = \big(G *_{A_1, \ldots, A_{r-1}} (t_1, \ldots, t_{r-1})\big) *_G (G *_{A_r} t_r) = G *_{A_1, \ldots, A_r} (t_1, \ldots, t_r).$$

□

Remark 4.1. The reader familiar with more general interpretations of free products with amalgamation (see Neumann's fundamental survey [18]) would notice that Θ in (4.3) is noting but the free product of the HNN-extensions $K_i *_{L_i} t_i$ with an amalgamated subgroup G. Indeed, the defining relations of this product can well be listed in an order matching the sintaxis of (4.3). Using the terms of [18] would allow us to avoid the bulky formula of (4.3), but it would require to involve here some new elements from [18] which would make the construction more complicated.

An immediate consequence of the above lemmas is:

Corollary 4.1. *If the subgroups $A_1, \ldots, A_r$ are benign in a finitely generated group G, then their intersection $\bigcap_{i=1}^r A_i$ also is benign in G. Moreover, if the finitely presented groups K_i with their finitely generated subgroups L_i can be given for each A_i explicitly, then the finitely presented K with its finitely generated subgroup L can be given for this intersection explicitly.*

Proof. By hypothesis we have some finitely presented overgroups $K_1, \dots, K_r$ of G and finitely generated $L_1, \dots, L_r$ such that $L_i \le K_i$ and $G \cap L_i = A_i$ for each $i = 1, \dots, r$. Then the free construction Θ of (4.3) is finitely presented, since to the finitely many relations of K_i we only add the relations stating that t_i fixes the finitely many generators of L_i, plus (if needed) relations identifying the finitely many generators of all copies of G in $K_i *_{L_i} t_i$, $i = 1, \dots, r$.

By Lemma 4.2 Θ contains the finitely generated subgroup $\langle G, t_1, \dots, t_r\rangle = G *_{A_1,\dots,A_r} (t_1, \dots, t_r)$, and by Lemma 4.1 we in that group have $G \cap L = \bigcap_{i=1}^r A_i$ for the finitely generated subgroup $L = G^{t_1 \cdots t_r}$. □

Lemma 4.3. *Let $A_1, \dots, A_r$ be arbitrary subgroups in a group G. Then in the HNN-extension $G *_{A_1,\dots,A_r} (t_1, \dots, t_r)$ the following equality holds:*

$$G \cap \big\langle \textstyle\bigcup_{i=1}^r G^{t_i} \big\rangle = \big\langle \textstyle\bigcup_{i=1}^r A_i \big\rangle. \tag{4.4}$$

Proof. For simplicity write the proof for the case $r = 3$. Set $T = \langle A_1, A_2, A_3\rangle$. By Lemma 4.2:

$$G *_{A_1, A_2, A_3} (t_1, t_2, t_3) = \big((G *_{A_1} t_1) *_G (G *_{A_2} t_2)\big) *_G (G *_{A_3} t_3).$$

$G *_{A_1} t_1$ contains $G *_{A_1} G^{t_1}$, and in this subgroup we by (3) in Corollary 2.1 have $\langle T, G^{t_1}\rangle \cap G = T$. For the same reason $\langle T, G^{t_2}\rangle \cap G = T$. Noticing $\langle T, G^{t_1}, G^{t_2}\rangle = \big\langle \langle T, G^{t_1}\rangle, \langle T, G^{t_2}\rangle \big\rangle$ and applying to it (2) of Corollary 2.1 inside the group $(G *_{A_1} t_1) *_G (G *_{A_2} t_2)$ we have $\langle T, G^{t_1}, G^{t_2}\rangle \cap G = T$. Since also $\langle T, G^{t_3}\rangle \cap G = T$, we again by (2) have

$$\big\langle \langle T, G^{t_1}, G^{t_2}\rangle, \langle T, G^{t_3}\rangle \big\rangle \cap G = T.$$

But since

$$T \le \langle G^{t_1}, G^{t_2}, G^{t_3}\rangle,$$

it remains to notice

$$\big\langle \langle T, G^{t_1}, G^{t_2}\rangle, \langle T, G^{t_3}\rangle \big\rangle = \langle G^{t_1}, G^{t_2}, G^{t_3}\rangle. \qquad □$$

Corollary 4.2. *If the subgroups $A_1, \dots, A_r$ are benign in a finitely generated group G, then their join $\big\langle \bigcup_{i=1}^r A_i \big\rangle$ also is benign in G. Moreover, if the finitely presented groups K_i with their finitely generated subgroups L_i can be given for each A_i explicitly, then the finitely presented K with its finitely generated subgroup L can be given for this join explicitly.*

Proof. Using the same constructions and notations as in the proof of Corollary 4.1 just notice that Θ is finitely presented, the join $L = \big\langle \bigcup_{i=1}^r G^{t_i} \big\rangle$ is finitely generated, and $G \cap L = \big\langle \bigcup_{i=1}^r A_i \big\rangle$ by Lemma 4.3. □

5. Adding finite presentation to the construction

5.1. **The HNN-extension Ξ_m.** In free group $\langle b,c\rangle$ we for any integer m can define a pair of isomorphisms ξ_m and ξ'_m via:

$$\xi_m(b) = b_{-m+1}, \quad \xi'_m(b) = b_{-m} \quad \text{and} \quad \xi_m(c) = \xi'_m(c) = c^2. \tag{5.1}$$

It is easy to verify that $\xi_m(b_i) = b_{2i-m+1}$ and $\xi'_m(b_i) = b_{2i-m}$. The pair ξ_m, ξ'_m can be used to define the HNN-extension

$$\Xi_m = \langle b,c\rangle *_{\xi_m,\xi'_m} (t_m, t'_m).$$

Here t_m, t'_m are any stable letters, and the subscript m is used to stress the correlation with ξ_m, ξ'_m, as below we are going to use this construction for multiple values of m.

Lemma 5.1. *In the above notations we for any m have:*

$$\langle b,c\rangle \cap \langle b_m, t_m, t'_m\rangle = \langle b_m, b_{m+1}, \ldots\rangle,$$

$$\langle b,c\rangle \cap \langle b_{m-1}, t_m, t'_m\rangle = \langle b_{m-1}, b_{m-2}, \ldots\rangle.$$

Proof. For any integer m and i we have $b_i^{t_m} = \xi_m(b_i) = b_{2i-m+1}$ and $b_i^{t'_m} = \xi'_m(b_i) = b_{2i-m}$ from where we collect:

$$\begin{aligned} &\ldots b_{m-2}^{t_m} = b_{m-3}, \quad b_{m-1}^{t_m} = b_{m-1}, \quad b_m^{t_m} = b_{m+1}, \quad b_{m+1}^{t_m} = b_{m+3}, \quad b_{m+2}^{t_m} = b_{m+5}, \ldots \\ &\ldots b_{m-2}^{t'_m} = b_{m-4}, \quad b_{m-1}^{t'_m} = b_{m-2}, \quad b_m^{t'_m} = b_m, \quad b_{m+1}^{t'_m} = b_{m+2}, \quad b_{m+2}^{t'_m} = b_{m+4}, \ldots \end{aligned} \tag{5.2}$$

The action of t_m^{-1} and ${t'_m}^{-1}$ can be deduced from the list above. From (5.2) it is straightforward that each of $b_m, b_{m+1}, \ldots$ indeed is in $\langle b_m, t_m, t'_m\rangle$. Say, $b_{m+8} = b_{m+4}^{t'_m} = b_{m+2}^{{t'_m}^2} = b_{m+1}^{{t'_m}^3} = b_m^{t_m \cdot {t'_m}^3} \in \langle b_m, t_m, t'_m\rangle$.

And on the other hand, bringing any word w on letters b_m, t_m, t'_m to the normal form in HNN-extension Ξ_m we first have to do cancellations like $t_m^{-1} b_m t_m = b_{m+1}$, and ${t'_m}^{-1} b_m t'_m = b_m$. Repeated applications of such steps may create in w some new letters $b_m, b_{m+1}, \ldots$ so that we may also have to do "reverse" cancellations like $t_m b_{m+1} t_m^{-1} = b_m$, $\;t_m b_{m+3} t_m^{-1} = b_{m+1}$, etc... or $\;t'_m b_m {t'}_m^{-1} = b_m$, $\;t'_m b_{m+2} {t'}_m^{-1} = b_{m+1}$, etc... That is, bringing w to normal form we *never* get a b_i outside $\langle b_m, b_{m+1}, \ldots\rangle$. If, in addition, w is in $\langle b,c\rangle$, then the normal form we obtained should contain no letters $t_m^{\pm 1}$ or ${t'_m}^{\pm 1}$. That is, if w is in $\langle b,c\rangle$, it in fact is in $\langle b_m, b_{m+1}, \ldots\rangle$, and we have $\langle b,c\rangle \cap \langle b_m, t_m, t'_m\rangle = \langle b_m, b_{m+1}, \ldots\rangle$.

The second equality stated by the lemma is proved analogously. □

Rules (5.1) define isomorphisms inside the free group $G = \langle a,b,c\rangle$ of rank 3 also, and we can define the HNN-extension $G *_{\xi_m,\xi'_m} (t_m, t'_m)$ which is noting but the ordinary free product $\langle a\rangle * \Xi_m$. Since $G = \langle a\rangle * \langle b, c\rangle$ and the subgroup $\langle b, c\rangle \cap \langle b_m, t_m, t'_m\rangle = \langle b_m, b_{m+1}, \ldots\rangle$ involves *no* occurrence of the letter a, we from Lemma 5.1 deduce:

Lemma 5.2. *In the above notations we for any m have:*

$$G \cap \langle b_m, t_m, t'_m\rangle = \langle b_m, b_{m+1}, \ldots\rangle,$$
$$G \cap \langle b_{m-1}, t_m, t'_m\rangle = \langle b_{m-1}, b_{m-2}, \ldots\rangle.$$

5.2. **Some special benign subgroups.** With above information we obtain three types of benign subgroups:

Corollary 5.1. *In the above notations for any integer m:*

(1) $\langle b_m, b_{m+1}, \ldots\rangle$ *is benign in* $\langle b, c\rangle$ *for the finitely presented group* Ξ_m *and its* 3*-generator subgroup* $\langle b_m, t_m, t'_m\rangle$,
(2) $\langle b_{m-1}, b_{m-2}, \ldots\rangle$ *is benign in* $\langle b, c\rangle$ *for the finitely presented group* Ξ_m *and its* 3*-generator subgroup* $\langle b_{m-1}, t_m, t'_m\rangle$,
(3) $B_m = \langle \ldots b_{-2}, b_{-1};\ b_m, b_{m+1}, \ldots\rangle$ *is benign in* $\langle b, c\rangle$ *for the finitely presented group*
$$\Theta_m = \big(\Xi_m *_{\langle b_m, t_m, t'_m\rangle} x\big) *_{\langle b,c\rangle} \big(\Xi_0 *_{\langle b_{-1}, t_0, t'_0\rangle} x'\big)$$
and its 4*-generator subgroup* $P_m = \Big\langle \langle b, c\rangle^x,\ \langle b, c\rangle^{x'} \Big\rangle$.

Proof. Points (1) and (2) directly follow Lemma 5.1.

B_m is the (free) product of $\langle b_m, b_{m+1}, \ldots\rangle$ and $\langle b_{-1}, b_{-2}, \ldots\rangle$. Hence, point (3) follows from Lemma 4.3 and Corollary 4.2 for $r = 2$, $G = \langle b, c\rangle$, $K_1 = \Xi_m$, $K_2 = \Xi_0$, $L_1 = \langle b_m, t_m, t'_m\rangle$, $L_2 = \langle b_{-1}, t_0, t'_0\rangle$, $A_1 = \langle b_m, b_{m+1}, \ldots\rangle$, $A_2 = \langle b_{-1}, b_{-2}, \ldots\rangle$. Then Θ_m is noting but the group Θ from (4.3). $\square$

Now we are able to replace our initial Γ from (3.1) by a *finitely presented* alternative:

$$\bar\Gamma = \Theta_m *_{P_m} (g, h, k) \tag{5.3}$$

with fixing action for g, h, k on the finitely generated subgroup P_m.

Lemma 5.3. *In above notations the following equalities hold in* $\bar\Gamma$:

(1) $\langle b, c\rangle \cap P_m = B_m$,
(2) $\langle b, c, g, h, k\rangle = \Gamma$.

Proof. The first point follows fom (3) in Corollary 5.1 (it holds in Θ_m, as it holds in $\bar\Gamma$). Next $\langle b, c, g, h, k\rangle = \langle b, c\rangle *_{\langle b,c\rangle \cap P_m} (g, h, k) = \Gamma$ by (1) in Corollary 2.2 and Remark 2.1. $\square$

5.3. **Presenting R as a join of benign subgroups.** Following the original steps in subsection 3.1 we now would have to build the HNN-extension $\bar{\Gamma} *_R a$ by fixing the subgroup $R = \langle g_f b_f^{-1} \mid f \in \mathcal{E}_m \rangle$ of $\bar{\Gamma}$ by some stable letter a. But since R is *not finitely generated*, that HNN-extension would *not* be finitely presented, and we need some extra complications to arrive to a finitely presented free construction.

Denote the subgroup $\Phi_m = \langle b_0, \ldots, b_{m-1}, g, h_0, \ldots, h_{m-1} \rangle$ in $\bar{\Gamma}$, and notice that:

Lemma 5.4. *Φ_m is freely generated by $2m{+}1$ elements $b_0, \ldots, b_{m-1}, g, h_0, \ldots, h_{m-1}$ in $\bar{\Gamma}$.*

Proof. Firstly, $\bar{B}_m = \langle b_0, \ldots, b_{m-1} \rangle$ has trivial intersection with P_m because (1) in Lemma 5.3 implies $\bar{B}_m \cap P_m \le \big(\bar{B}_m \cap \langle b, c \rangle\big) \cap P_m = \bar{B}_m \cap \big(\langle b, c \rangle \cap P_m\big) = \bar{B}_m \cap B_m = frm[o]{-}{-}$ due to $\langle B_m,\, \bar{B}_m \rangle = B_m * \bar{B}_m$. Therefore, we in $\bar{\Gamma}$ by (1) in Corollary 2.2 and by Remark 2.1 have:

$$\langle b_0, \ldots, b_{m-1},\ g, h, k \rangle = \bar{B}_m *_{\bar{B}_m \,\cap\, P_m} (g, h, k) = \bar{B}_m *_{frm[o]--} (g, h, k) = \bar{B}_m * \langle g, h, k \rangle$$

which simply is a free group of rank $m + 3$. Since $h_0, \ldots, h_{m-1}$ generate a free subgroup inside $\langle g, h, k \rangle$, they together with $b_0, \ldots, b_{m-1}$ generate a free subgroup (of rank $2m + 1$) inside $\langle b_0, \ldots, b_{m-1},\ g, h, k \rangle$. $\square$

Next we need a series of auxiliary benign subgroups inside $\bar{\Gamma}$. For an $s = 1, \ldots, m$ and for a sequence $f = (j_0, \ldots, j_{s-2}, j_{s-1}) \in \mathcal{E}_s$ denote $f^+ = (j_0, \ldots, j_{s-2},\ j_{s-1}{+}1)$ in $\mathcal{E}_s$, i.e., to get f^+ we just add 1 to the last coordinate of f. In these notations for any f the group $\bar{\Gamma}$ contains the elements $g_{f^+} \cdot b_{s-1}^{-1} \cdot g_f^{-1}$, such as, $g^{h_0^2 h_1^5 h_2^3 h_3^{\mathbf{8}}} \cdot b_3^{-1} \cdot g^{-\,h_0^2 h_1^5 h_2^3 h_3^7}$ for $f = (2, 5, 3, 7)$ with $s = 4$. Denote:

$$V_{\mathcal{E}_s} = \Big\langle g_{f^+} \cdot b_{s-1}^{-1} \cdot g_f^{-1} \mid f \in \mathcal{E}_s \Big\rangle =$$

$$= \Big\langle g^{h_0^{i_0} \cdots\, h_{s-2}^{i_{s-2}} h_{s-1}^{i_{s-1}\mathbf{+1}}} \cdot b_{s-1}^{-1} \cdot g^{-h_0^{i_0} \cdots\, h_{s-2}^{i_{s-2}} h_{s-1}^{i_{s-1}}} \mid i_0 \ldots, i_{s-2}, i_{s-1} \in \mathbb{Z} \Big\rangle,$$

and establish a property for $V_{\mathcal{E}_s}$:

Lemma 5.5. *$V_{\mathcal{E}_s}$ is a benign subgroup in $\bar{\Gamma}$ for the some explicitly given finitely presented group and its finitely generated subgroup.*

Proof. By Lemma 5.4 for any $s = 1, \ldots, m$ the elements $b_{s-1}, g, h_0, \ldots, h_{s-1}$ are *free* generators for the $(s+2)$-generator subgroup $\langle b_{s-1}, g, h_0, \ldots, h_{s-1} \rangle$ of Φ_m. Hence, each of the following maps $\lambda_{i,\,j}$ can be continued to some isomorphism on

$\langle b_{s-1}, g, h_0, \ldots, h_{s-1}\rangle$:

(5.4)

$$\begin{array}{llllll}
\lambda_{s-1,\,0} & \text{sends} & b_{s-1}, g, h_0, \ldots, h_{s-2}, h_{s-1} & \text{to} & b_{s-1}, g^{h_0}, h_0, \ldots, h_{s-2}, h_{s-1}; \\
\lambda_{s-1,\,1} & \text{sends} & b_{s-1}, g, h_0, \ldots, h_{s-2}, h_{s-1} & \text{to} & b_{s-1}, g^{h_1}, h_0^{h_1}, \ldots, h_{s-2}, h_{s-1}; \\
\vdots & & \vdots & & \vdots \\
\lambda_{s-1,\,s-1} & \text{sends} & b_{s-1}, g, h_0, \ldots, h_{s-2}, h_{s-1} & \text{to} & b_{s-1}, g^{h_{s-1}}, h_0^{h_{s-1}}, \ldots, h_{s-2}^{h_{s-1}},\ h_{s-1}.
\end{array}$$

In particular, for $m = 1$ the map $\lambda_{0,0}$ sends b_0, g, h_0 to b_0, g^{h_0}, h_0; for $m = 2$ the map $\lambda_{1,0}$ sends b_1, g, h_0, h_1 to b_1, g^{h_0}, h_0, h_1 and $\lambda_{1,1}$ sends b_1, g, h_0, h_1 to $b_1, g^{h_1}, h_0^{h_1}, h_1$, etc...

Introducing for each isomorphism $\lambda_{i,\,j}$ a respective stable letter $l_{i,\,j}$ we construct the HNN-extension:

$$\Lambda_s = \bar{\Gamma} *_{\lambda_{s-1,\,0},\,\ldots\,,\lambda_{s-1,\,s-1}} (l_{s-1,\,0}, \ldots, l_{s-1,\,s-1})$$

for each of $s = 1, \ldots, m$.

The effects of conjugation by elements $l_{s-1,\,0}, \ldots, l_{s-1,\,s-1}$ on the products $g_{f^+} \cdot b_{s-1}^{-1} \cdot g_f^{-1}$ is very easy to understand: $l_{s-1,\,i}$ just adds 1 to the i'th coordinate of f, say, for $s = 4,\ f = (2, 5, 3, 7)$ and $l_{3,\,2} = l_{4-1,\,3-1}$ we have:

(5.5)

$$\begin{aligned}
\left(g_{f^+} \cdot b_3^{-1} \cdot g_f^{-1}\right)^{l_{3,\,2}} &= \left(g^{h_2}\right)^{\left(h_0^{h_2}\right)^2 \left(h_1^{h_2}\right)^5 h_2^3\, h_3^{\mathbf{8}}} \cdot b_3^{-1} \cdot \left(g^{h_2}\right)^{-\left(h_0^{h_2}\right)^2 \left(h_1^{h_2}\right)^5 h_2^3\, h_3^7} \\
&= g^{h_2 \cdot\, h_2^{-1} h_0^2 h_2 h_2^{-1} h_1^5 h_2 h_2^3 h_3^{\mathbf{8}}} \cdot b_3^{-1} \cdot g^{-\, h_2 \cdot\, h_2^{-1} h_0^2 h_2 h_2^{-1} h_1^5 h_2 h_2^3 h_3^7} \\
&= g^{h_0^2 h_1^5 h_2^{\mathbf{4}} h_3^{\mathbf{8}}} \cdot b_3^{-1} \cdot g^{-\, h_0^2 h_1^5 h_2^{\mathbf{4}} h_3^7} = g_{f'^+} \cdot b_3^{-1} \cdot g_{f'}^{-1} \in V_{\mathcal{E}_4}
\end{aligned}$$

where $f' = (2, 5, 3{+}1, 7) = (2, 5, \mathbf{4}, 7)$. In particular, actions of the above letters $l_{i,\,j}$ keep the elements from $V_{\mathcal{E}_s}$ inside $V_{\mathcal{E}_s}$.

For the sequence $f_0 = (0, \ldots, 0) \in \mathcal{E}_s$ we have $g_{f_0^+} \cdot b_{s-1}^{-1} \cdot g_{f_0}^{-1} = g^{h_{s-1}} \cdot b_{s-1}^{-1} \cdot g^{-1}$. Applying the conjugate collection process of subsection 2.4 for $\mathfrak{X} = \{g^{h_{s-1}} \cdot b_{s-1}^{-1} \cdot g^{-1}\}$ and for $\mathfrak{Y} = \{l_{s-1,\,0}, \ldots, l_{s-1,\,s-1}\}$ we see that any element w from $\langle \mathfrak{X}, \mathfrak{Y}\rangle \le \Lambda_s$ is a product of elements of $g_{f^+} \cdot b_{s-1}^{-1} \cdot g_f^{-1}$ (for certain sequences $f \in \mathcal{E}_s$) and of certain powers of the stable letters $l_{s-1,\,0}, \ldots, l_{s-1,\,s-1}$. And w is inside $\bar{\Gamma}$ if and only if all those powers are cancelled out in the normal form, and w in fact is in $V_{\mathcal{E}_s}$, that is, denoting $L_s = \langle g^{h_{s-1}} \cdot b_{s-1}^{-1} \cdot g^{-1};\ \ l_{s-1,\,0}, \ldots, l_{s-1,\,s-1}, \rangle$ we have:

$$\bar{\Gamma} \cap L_s = V_{\mathcal{E}_s},$$

i.e., $V_{\mathcal{E}_s}$ is benign in $\bar{\Gamma}$ for the above finitely presented group Λ_s and for its $(s+1)$-generator subgroup L_s. □

Lemma 5.6. *$R = \langle g_f b_f^{-1} \mid f \in \mathcal{E}_m\rangle$ is a benign subgroup in $\bar{\Gamma}$ for some explicitly given finitely presented group and its finitely generated subgroup.*

Proof. First show that R is generated by its $m{+}1$ subgroups $\langle g\rangle, V_{\mathcal{E}_1}, \ldots, V_{\mathcal{E}_m}$. For each $s = 1, \ldots, m$ denote $Z_{\mathcal{E}_s} = \langle g_f b_f^{-1} \mid f \in \mathcal{E}_s\rangle$. In these notation R is noting but $Z_{\mathcal{E}_m}$ for $s = m$. It is easy to see that $\langle Z_{\mathcal{E}_{s-1}}, V_{\mathcal{E}_s}\rangle = Z_{\mathcal{E}_s}$ for each s (when $s = 1$, then take $\langle g\rangle$ as $Z_{\mathcal{E}_0}$), see details in [17] based on an original idea from [9]. Then:

$$Z_{\mathcal{E}_m} = \langle Z_{\mathcal{E}_{m-1}}, V_{\mathcal{E}_m}\rangle = \langle Z_{\mathcal{E}_{m-2}}, V_{\mathcal{E}_{m-1}}, V_{\mathcal{E}_m}\rangle = \cdots = \big\langle \langle g\rangle, V_{\mathcal{E}_1}, \ldots V_{\mathcal{E}_{m-1}}, V_{\mathcal{E}_m}\big\rangle.$$

By Lemma 5.5 each $V_{\mathcal{E}_s}$ is benign $\bar\Gamma$ for an explicitly given finitely presented group Λ_s and its finitely generated subgroup L_s. And the finitely generated $\langle g\rangle$ is clearly benign in $\bar\Gamma$ for the finitely presented group $\Lambda_0 = \bar\Gamma$ and its finitely generated subgroup $L_0 = \langle g\rangle$.

It remains to load these components into Corollary 4.2 and into (4.3) to get the following finitely presented overgroup holding $\bar\Gamma$:

$$\bar\Theta = \Big(\cdots\Big(\big((\Lambda_0 *_{L_0} t_0) *_{\bar\Gamma} (\Lambda_1 *_{L_1} t_1)\big) *_{\bar\Gamma} (\Lambda_2 *_{L_2} t_2)\Big)\cdots\Big) *_{\bar\Gamma} (\Lambda_m *_{L_m} t_m), \tag{5.6}$$

and its finitely generated subgroup $Q = \big\langle \bar\Gamma^{\,t_0}, \ldots, \bar\Gamma^{\,t_m}\big\rangle$. □

5.4. **Proof for Theorem 1.1.** Now we can use the above constructions to finish the main proof. The last two steps of the construction in Section 3 are effortless to mimic. As $\bar\Theta$ of (5.6) is finitely presented, and Q is finitely generated, the HNN-extension $\bar\Theta *_Q a$ is finitely presented. Inside $\bar\Theta *_Q a$ the elements a, b, c generate the same free subgroup discussed in Section 3, and we can again define an isomorphism ρ sending a, b, c to a, b^{c^m}, c together with the *finitely presented* analog of Δ from (3.2):

$$\bar\Delta = \big(\bar\Theta *_Q a\big) *_\rho r. \tag{5.7}$$

For any $\mathcal{B} \subseteq \mathcal{E}_m$ we in analogy with Section 3 can denote $W_\mathcal{B} = \langle g_f,\, a,\, r \mid f \in \mathcal{B}\rangle$ in $\bar\Delta$. Since each g_f, a, r from $\bar\Delta$, in fact, is from Δ already, we in $\bar\Delta$ have the analog of (3.3) also:

$$G \cap W_\mathcal{B} = A_{\omega_m \mathcal{B}}. \tag{5.8}$$

Since $A_\mathcal{B}$ is benign in G, by Theorem 1.1 hypothesis there is a finitely presented (explicitly given) overgroup $K_\mathcal{B}$ of G with a finitely generated subgroup $L_\mathcal{B}$ so that $G \cap L_\mathcal{B} = A_\mathcal{B}$ in $K_\mathcal{B}$.

As $\bar\Delta$ was built purely via free constructions in which we are in position to control which new elements (such as, stable letters) to adjoin, we can make sure no element of $\bar\Delta$ outside G is contained in $K_\mathcal{B}$, and hence, we can construct the finitely presented amalgamated product $\bar\Delta *_G K_\mathcal{B}$.

The subgroup $A_{\mathcal{B}}$ is benign also in $\bar{\Delta}$. Indeed, the group $\bar{\Delta} *_G K_{\mathcal{B}}$ is finitely presented, and to its subgroup $L_{\mathcal{B}} = \langle A_{\mathcal{B}}, L_{\mathcal{B}} \rangle$ we may apply (3) in Corollary 2.1 to get $\bar{\Delta} \cap L_{\mathcal{B}} = A_{\mathcal{B}}$, because $A_{\mathcal{B}} \cap G = A_{\mathcal{B}}$ and $L_{\mathcal{B}} \cap G = A_{\mathcal{B}}$.

Next, being finitely generated $\langle b, c\rangle$ is benign in $\bar{\Delta}$ for the finitely presented $\bar{\Delta}$ and for the finitely generated $\langle b, c\rangle$, see remark in subsection 2.2.

Hence by Corollary 4.2 the join $\big\langle A_{\mathcal{B}}, \langle b, c\rangle \big\rangle = W_{\mathcal{B}}$ is benign in $\bar{\Delta}$. As its finitely presented overgroup we may take:

$$\Psi = \big((\bar{\Delta} *_G K_{\mathcal{B}}) *_{L_{\mathcal{B}}} y\big) \; *_{\bar{\Delta}} \; \big(\bar{\Delta} *_{\langle b,c\rangle} y'\big)$$

(see (4.3)), and as a finitely generated subgroup we may take $L' = \big\langle \bar{\Delta}^{y},\ \bar{\Delta}^{y'} \big\rangle$.

G clearly is benign in $\bar{\Delta}$. Hence by (5.8) and by Corollary 4.1 the intersection $G \cap W_{\mathcal{B}} = A_{\omega_m \mathcal{B}}$ is benign in $\bar{\Delta}$ for the finitely presented group:

$$K_{\omega_m \mathcal{B}} = (\Psi *_{L'} z) \; *_{\bar{\Delta}} \; (\bar{\Delta} *_G z'), \tag{5.9}$$

and its finitely generated subgroup:

$$L_{\omega_m \mathcal{B}} = \bar{\Delta}^{z\, z'}, \tag{5.10}$$

i.e., $\bar{\Delta} \cap L_{\omega_m \mathcal{B}} = A_{\omega_m \mathcal{B}}$ in $K_{\omega_m \mathcal{B}}$. But since $G \le \bar{\Delta}$ and $A_{\omega_m \mathcal{B}} \le G$, we conclude that $G \cap L_{\omega_m \mathcal{B}} = A_{\omega_m \mathcal{B}}$ also holds in $K_{\omega_m \mathcal{B}}$.

This completes the proof of Theorem 1.1.

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SIMPLIFIED WHITTLE ESTIMATORS FOR SPECTRAL PARAMETERS OF STATIONARY LINEAR MODELS WITH TAPERED DATA

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Abstract. The paper is concerned with the statistical estimation of the spectral parameters of stationary models with tapered data. As estimators of the unknown parameters we consider the tapered Whittle estimator and the simplified tapered Whittle estimators. We show that under broad regularity conditions on the spectral density of the model these estimators are asymptotically statistically equivalent, in the sense that these estimators possess the same asymptotic properties. The processes considered will be discrete-time and continuous-time Gaussian, linear or Lévy-driven linear processes with memory.

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Keywords: tapered data; stationary processes; spectral density; parametric estimation; simplified Whittle estimator.

1. Introduction

The present paper is concerned with the following general parametric estimation problem. Let $\theta := (\theta_1, \ldots, \theta_p) \in \Theta \subset \mathbb{R}^p$ be an unknown vector parameter appearing (a) in the probability density of some random variable X, or (b) in the finite-dimensional probability densities of a random process $\{X(t),\ t \in \mathbb{U}\}$, where $\mathbb{U} = \mathbb{R}$ in the continuous-time (c.t.) case and $\mathbb{U} = \mathbb{Z}$ in the discrete-time (d.t.) case. The problem of interest is to estimate the value of the parameter θ based on the sample $\mathbf{X}_T$, where in case (a) $\mathbf{X}_T := \{X_1, \ldots, X_T\}$, $X_1, \ldots, X_T$ being T independent observations of the random variable X, and in case (b) $\mathbf{X}_T$ is an observed finite realization of the process $X(t)$: $\mathbf{X}_T := \{X(t), t \in D_T\}$, where $D_T := [0, T]$ in the c.t. case and $D_T := \{1, \ldots, T\}$ in the d.t. case. The usual methods of constructing estimators of the unknown parameter θ used in mathematical statistics (for example, the method of moments, the maximum likelihood method, the least-squares method, the Whittle method, etc.), as a rule, require finding the roots of some system of (possibly non-linear) estimating equations with respect to the unknown $\theta = (\theta_1, \ldots, \theta_p)$ of the form:

$$F_i(\mathbf{X}_T, \theta) = 0, \quad i = 1, \ldots p, \tag{1.1}$$

where $F_i(\theta) := F_i(\mathbf{X}_T, \theta)$ are certain functionals of $\mathbf{X}_T$ depending on θ.

The classical estimation methods often lead to estimators with good asymptotic properties. For example, in many cases one can prove that for sufficiently large values T there exist, with probability near 1, a root $\hat{\theta}_T$ of the system of estimating equations (1.1) which is a consistent estimator of θ, that is, $p - lim_{T\to\infty}\hat{\theta}_T = \theta_0$, where $p - lim$ denotes the limit in probability, and $\theta_0 \in \Theta$ is the unknown true value of the parameter θ. Moreover, under broad regularity conditions, the classical estimation methods often lead to τ_T-consistent and asymptotically normal estimators, where τ_T is a comparatively rapidly increasing function. (Recall that for a non-random function $\tau_T = \tau(T)$ increasing without bound as $T \to \infty$, we say that the statistic $\hat{\theta}_T$ is a τ_T-consistent estimator for θ if the distribution of the random vector $\tau_T(\hat{\theta}_T - \theta_0)$ converges (as $T \to \infty$) to a non-degenerate distribution.

These classical estimation methods, however, have two disadvantages. First, it is only for relatively simple situations that the system of estimating equations (1.1) has an explicit solution, and finding the roots of the system (1.1) often turns out to be very hard problem. Second, for the roots to be consistent, the estimating equations need to behave well throughout the parameter set. Another issue that arise in the statistical analysis of stationary models is that the data are frequently tapered before calculating the statistic of interest, and the statistical inference procedure, instead of the original data $\mathbf{X}_T$, is based on the *tapered data*: $\mathbf{X}_T^h := \{h_T(t)X(t),\, t \in D_T\}$, where $h_T(t) := h(t/T)$ with $h(t)$, $t \in \mathbb{R}$ being a *taper function*.

Therefore it is of considerable interest to find more easily constructed (simplified) estimators $\check{\theta}_T$ that are asymptotically statistically equivalent to $\hat{\theta}_T$, that is, having the same asymptotic (as $T \to \infty$) properties as the estimator $\hat{\theta}_T$. The problem of constructing simplified estimators with good asymptotic properties based on the standard (non-tapered) data $\mathbf{X}_T$ goes back to the classical work of Le Cam [16], and then it was developed by Dzhaparidze [8, 9] (see also Beinicke and Dzhaparidze [1] and Dzhaparidze [10]).

In this paper we focus on the Whittle estimation method of the spectral parameters of stationary models with tapered data. We provide sufficient conditions for the tapered Whittle estimator to be $\sqrt{T}$-consistent and asymptotically normal. Then we construct simplified Whittle estimators based on the tapered data, and show that under broad regularity conditions on the spectral density of the model the Whittle estimator and the simplified Whittle estimator are asymptotically statistically equivalent, in the sense that these estimators possess the same asymptotic properties.

The processes considered will be discrete-time and continuous-time Gaussian, linear or Lévy-driven linear processes with memory.

2. The model

We will consider here stationary processes possessing spectral density functions, and will distinguish the following three models.

(a) Discrete-time linear model. The process $\{X(t),\, t\in\mathbb{Z}\}$ is a discrete-time linear process of the form:

$$X(t)=\sum_{k=-\infty}^{\infty}a(t-k)\xi(k),\qquad \sum_{k=-\infty}^{\infty}|a(k)|^2<\infty, \tag{2.1}$$

where $\{\xi(k),k\in\mathbb{Z}\}\sim$ WN(0,1) is a standard white-noise, that is, a sequence of orthonormal random variables. The spectral density $f(\lambda)$ of $X(t)$ is given by formula:

$$f(\lambda)=\frac{1}{2\pi}\left|\sum_{k=-\infty}^{\infty}a(k)e^{-ik\lambda}\right|^2=\frac{1}{2\pi}|\widehat{a}(\lambda)|^2,\quad \lambda\in[-\pi,\pi]. \tag{2.2}$$

In the case where $\xi(k)$ is a sequence of Gaussian random variables, the process $X(t)$ is Gaussian.

(b) Continuous-time linear model. The process $\{X(t),\, t\in\mathbb{R}\}$ is a continuous-time linear process of the form:

$$X(t)=\int_{\mathbb{R}}a(t-s)d\xi(s),\qquad \int_{\mathbb{R}}|a(s)|^2ds<\infty, \tag{2.3}$$

where $\{\xi(s),s\in\mathbb{R}\}$ is a process with orthogonal increments and $\mathbb{E}|d\,\xi(s)|^2=ds$. The spectral density $f(\lambda)$ of $X(t)$ is given by formula:

$$f(\lambda)=\frac{1}{2\pi}\left|\int_{\mathbb{R}}e^{-i\lambda t}a(t)dt\right|^2=\frac{1}{2\pi}|\widehat{a}(\lambda)|^2,\quad \lambda\in\mathbb{R}. \tag{2.4}$$

In the case where $\xi(s)$ is a Gaussian process, the process $X(t)$ is Gaussian.

(c) Lévy-driven linear model. We first recall that a Lévy process, $\{\xi(s),\ s\in\mathbb{R}\}$ is a process with independent and stationary increments, continuous in probability, with sample-paths which are right-continuous with left limits and $\xi(0)=\xi(0-)=0$. The Wiener process $\{B(s),\ s\ge 0\}$ is a typical example of centered Lévy processes. A Lévy-driven linear process $\{X(t),\ t\in\mathbb{R}\}$ is a real-valued c.t. stationary process defined by (2.3), where $\xi(s)$ is a Lévy process satisfying the conditions: $\mathbb{E}\xi(s)=0$, $\mathbb{E}\xi^2(1)=1$ and $\mathbb{E}\xi^4(1)<\infty$. In the case where $\xi(s)=B(s)$, $X(t)$ is a Gaussian process.

The function $a(\cdot)$ in representations (2.1) and (2.3) plays the role of a *time-invariant filter*, and the linear processes defined by (2.1) and (2.3) can be viewed as the output of a linear filter $a(\cdot)$ applied to the process $\{\xi(u),\ u \in \mathbb{U}\}$, called the innovation or driving process of $X(t)$.

3. Data tapers and the tapered periodogram

In this section we introduce the data tapers and tapered periodogram. Our inference procedures will be based on the tapered data $\mathbf{X}_T^h$:

$$\mathbf{X}_T^h := \{h_T(t)X(t),\ t \in D_T\}, \tag{3.1}$$

where $D_T := [0, T]$ in the c.t. case and $D_T := \{1, \ldots, T\}$ in the d.t. case, and

$$h_T(t) := h(t/T) \tag{3.2}$$

with $h(t)$, $t \in \mathbb{R}$ being a *taper function* to be specified below.

For $k \in \mathbb{N} := \{1, 2, \ldots\}$, denote by $H_{k,T}(\lambda)$ the *tapered Dirichlet type kernel*, defined by

$$H_{k,T}(\lambda) := \begin{cases} \sum_{t=1}^{T} h_T^k(t)e^{-i\lambda t} & \text{in the d.t. case,} \\ \int_0^T h_T^k(t)e^{-i\lambda t}dt & \text{in the c.t. case,} \end{cases} \tag{3.3}$$

and put

$$H_{k,T} := H_{k,T}(0). \tag{3.4}$$

Define the finite Fourier transform of the tapered data (3.1):

$$d_T^h(\lambda) := \begin{cases} \sum_{t=1}^{T} h_T(t)X(t)e^{-i\lambda t} & \text{in the d.t. case,} \\ \int_0^T h_T(t)X(t)e^{-i\lambda t}dt & \text{in the c.t. case.} \end{cases} \tag{3.5}$$

and the tapered periodogram $I_T^h(\lambda)$ of the process $X(t)$:

$$I_T^h(\lambda) := \frac{1}{C_T}\, d_T^h(\lambda)d_T^h(-\lambda), \tag{3.6}$$

where

$$C_T := 2\pi H_{2,T}(0) = 2\pi H_{2,T} \neq 0. \tag{3.7}$$

Notice that for non-tapered case ($h(t) = \mathbb{I}_{[0,1]}(t)$), we have $C_T = 2\pi T$.

Throughout the paper, we will assume that the taper function $h(\cdot)$ satisfies the following assumption.

Assumption 3.1. The taper $h : \mathbb{R} \to \mathbb{R}$ is a continuous nonnegative function of bounded variation and of bounded support $[0, 1]$, such that $H_k \neq 0$, where

$$H_k := \lim_{T\to\infty} (1/T)H_{k,T}, \quad \textit{and } H_{k,T} \textit{ is as in } (3.4). \tag{3.8}$$

Observe that in the c.t. case we have $H_k = \int_0^1 h^k(t)dt$.

Remark 3.1. The data taper $h(t)$ normally has a maximum at $t = 1/2$ and decreases smoothly to zero as t tends to 0 or 1. For the d.t. case, an example of a taper function $h(t)$ satisfying Assumption 3.1 is the Tukey-Hanning taper function $h(t) = 0.5(1 - \cos(\pi t))$ for $t \in [0, 1]$. For the c.t. case, a simple example of a taper function $h(t)$ satisfying Assumption 3.1 is the function $h(t) = 1 - t$ for $t \in [0, 1]$.

The benefits of tapering the data have been widely reported in the literature (see, e.g., Brillinger [2], Dahlhaus [3]–[6], Dahlhaus and Künsch [7], Ginovyan and Sahakyan [13, 14], Guyon [15], and references therein). For example, data-tapers are introduced to reduce the so-called 'leakage effects', that is, to obtain better estimation of the spectrum of the model in the case where it contains high peaks. Tapering also can be used to reduce the so-called 'trough effects', that is, to obtain better estimator of the spectrum in the case where it contains strong troughs. Other application of data-tapers is in situations in which some of the data values are missing. Also, the use of tapers leads to bias reduction, which is especially important when dealing with spatial data. In this case, the tapers can be used to fight the so-called 'edge effects' (for details see Dahlhaus [5, 6], and Ginovyan and Sahakyan [14]).

4. Estimation of linear spectral functionals

Linear and non-linear functionals of the periodogram play a key role in the parametric estimation of the spectrum of stationary processes, when using the minimum contrast estimation method with various contrast functionals (see, e.g., Ginovyan and Sahakyan [14], and references therein). The result that follow is used to prove consistency and asymptotic normality of the minimum contrast estimators based on the Whittle functionals for linear models with tapered data. Specifically, we are interested in the nonparametric estimation problem, based on the tapered data (3.1), of the following linear spectral functional:

$$J = J(f, g) := \int_\Lambda f(\lambda) g(\lambda) d\lambda, \tag{4.1}$$

where $g(\lambda) \in L^q(\Lambda)$, $1/p + 1/q = 1$. Here, and in what follows, $\Lambda = \mathbb{R}$ in the c.t. case, and $\Lambda = [-\pi.\pi]$ in the d.t. case.

As an estimator J_T^h for functional $J(f)$, given by (4.1), based on the tapered data (3.1), we consider the averaged tapered periodogram (or a simple 'plug-in' statistic), defined by

$$J_T^h = J(I_T^h, g) := \int_\Lambda I_T^h(\lambda) g(\lambda) d\lambda, \tag{4.2}$$

where $I_T^h(\lambda)$ is the tapered periodogram of the process $X(t)$ given by (3.6). We will refer to $g(\lambda)$ and to its Fourier transform $\widehat{g}(t)$ as a *generating function* and *generating kernel* for the functional J_T^h, respectively. To state the corresponding results we first introduce the following assumptions.

Assumption 4.1. The spectral density f and the generating function g are such that $f, g \in L^1(\Lambda) \cap L^2(\Lambda)$ ($f, g \in L^2(\Lambda)$ in the d.t. case) and g is of bounded variation.

Assumption 4.2. (A) (d.t. case). The spectral density f and the generating function g are such that $f \in L^p(\Lambda)$ $(p \geq 1)$ and $g \in L^q(\Lambda)$ $(q \geq 1)$ with $1/p + 1/q \leq 1/2$.
(B) (c.t. case). The spectral density f and the generating function g are such that $f \in L^1(\Lambda) \cap L^p(\Lambda)$ $(p \geq 1)$ and $g \in L^1(\Lambda) \cap L^q(\Lambda)$ $(q \geq 1)$ with $1/p + 1/q \leq 1/2$.
(C) (c.t. Lévy-driven case). The filter a and the generating kernel $\widehat{g}$ are such that $a \in L^2(\Lambda) \cap L^p(\Lambda)$ and $\widehat{g} \in L^q(\Lambda)$ with $1 \leq p, q \leq 2$ and $2/p + 1/q \geq 5/2$.

Denote

$$e(h) := \lim_{T\to\infty} \frac{TH_{4,T}}{H_{2,T}^2}, \tag{4.3}$$

where $H_{k,T}$ is as in (3.4), and

$$\sigma_h^2(J) := 4\pi e(h) \int_\Lambda f^2(\lambda) g^2(\lambda) d\lambda + \kappa_4 e(h) \left[\int_\Lambda f(\lambda) g(\lambda) d\lambda \right]^2, \tag{4.4}$$

where κ_4 is the fourth cumulant of $\xi(1)$.

The proof of the next theorem can be found in Ginovyan and Sahakyan [13] (see also Ginovyan [11]).

Theorem 4.1. *Let the functionals $J := J(f, g)$ and $J_T^h := J(I_T^h, g)$ be defined by (4.1) and (4.2), respectively. Then under Assumptions 3.1, 4.1 and 4.2 the following asymptotic relation holds:*

$$(a) \quad \mathbb{E}(J_T^h) - J \to 0 \quad \text{as} \quad T \to \infty.$$

$$(b) \quad T^{1/2}\left[\mathbb{E}(J_T^h) - J\right] \to 0 \quad \text{as} \quad T \to \infty.$$

$$(c) \quad \lim_{T\to\infty} T\mathrm{Var}(J_T^h) = \sigma_h^2(J),$$

$$(d) \quad T^{1/2}\left[J_T^h - J\right] \xrightarrow{d} \eta \quad \text{as} \quad T \to \infty,$$

where $\mathbb{E}[\cdot]$ is the expectation operator, the symbol $\xrightarrow{d}$ stands for convergence in distribution, and η is a normally distributed random variable with mean zero and variance $\sigma_h^2(J)$ given by (4.4).

5. The Whittle estimation procedure

We assume here that the spectral density $f(\lambda)$ belongs to a given parametric family of spectral densities $\mathcal{F} := \{f(\lambda, \theta) : \theta \in \Theta\}$, where $\theta := (\theta_1, \ldots, \theta_p)$ is an unknown parameter and Θ is a subset of the Euclidean space $\mathbb{R}^p$. The problem of interest is to estimate θ on the basis of the tapered data (3.1), and investigate the asymptotic (as $T \to \infty$) properties of the suggested estimators. We use here the Whittle estimation method to estimate θ. This method, originally devised by P. Whittle for d.t. stationary processes (see Whittle [17]), is based on the smoothed periodogram analysis on a frequency domain, involving approximation of the likelihood function and asymptotic properties of empirical spectral functionals. The Whittle procedure of estimation of a spectral parameter θ based on the tapered sample (3.1) is to choose the estimator $\widehat{\theta}_{T,h}$ to minimize the weighted tapered Whittle functional:

$$U_{T,h}(\theta) := \frac{1}{4\pi} \int_\Lambda \left[\log f(\lambda, \theta) + \frac{I_T^h(\lambda)}{f(\lambda, \theta)}\right] \cdot w(\lambda)\, d\lambda, \tag{5.1}$$

where $I_T^h(\lambda)$ is the tapered periodogram of $X(t)$, given by (3.6), and $w(\lambda)$ is a weight function (that is, $w(-\lambda) = w(\lambda)$, $w(\lambda) \geq 0$, $w(\lambda) \in L^1(\mathbb{R})$) for which the integral in (5.1) is well defined. In the d.t. case as a weight function we take $w(\lambda) \equiv 1$. In the c.t. case, an example of common used weight function is $w(\lambda) = 1/(1+\lambda^2)$. So, the Whittle estimator $\widehat{\theta}_{T,h}$ of θ based on the tapered sample (3.1) is defined by

$$\widehat{\theta}_{T,h} := \operatorname*{Arg\,min}_{\theta \in \Theta} U_{T,h}(\theta), \tag{5.2}$$

where $U_{T,h}(\theta)$ is given by (5.1). Thus, the tapered Whittle estimator $\widehat{\theta}_{T,h}$ of θ is the root of the following system of estimating equations:

$$\begin{aligned} F_{h,i}(\theta) = F_{T,h,i}(\theta) &:= (\partial/\partial\theta_i) U_{T,h}(\theta) \\ &= \frac{1}{4\pi} \int_\Lambda \left[(\partial/\partial\theta_i) \log f(\lambda, \theta) + I_T^h(\lambda)(\partial/\partial\theta_i) f^{-1}(\lambda, \theta)\right] \cdot w(\lambda)\, d\lambda = 0, \quad i = 1, \ldots, p. \end{aligned} \tag{5.3}$$

The tapered Whittle estimator $\widehat{\theta}_{T,h}$ of θ possesses good asymptotic properties. To state these properties of $\widehat{\theta}_{T,h}$, we first introduce the following set of assumptions.

Assumption 5.1. The true value θ_0 of the parameter θ belongs to a compact set Θ in the p-dimensional Euclidean space $\mathbb{R}^p$, and $f(\lambda, \theta_1) \neq f(\lambda, \theta_2)$ whenever $\theta_1 \neq \theta_2$ almost everywhere in Λ with respect to the Lebesgue measure.

Assumption 5.2. The functions $f(\lambda, \theta)$, $f^{-1}(\lambda, \theta)$ and $(\partial/\partial\theta_k) f^{-1}(\lambda, \theta)$, $k = 1, \ldots, p$, are continuous in (λ, θ).

Assumption 5.3. The functions $f := f(\lambda, \theta)$ and $g := w(\lambda)(\partial/\partial\theta_k)f^{-1}(\lambda, \theta)$ satisfy Assumption 4.1 for all $k = 1, \ldots, p$ and $\theta \in \Theta$.

Assumption 5.4. The functions f, g, $a := a(\lambda, \theta)$ and $b := \widehat{g}$, where g is as in Assumption 5.3, satisfy Assumption 4.2.

Assumption 5.5. The functions $(\partial^2/\partial\theta_k\partial\theta_j)f^{-1}(\lambda, \theta)$ and $(\partial^3/\partial\theta_k\partial\theta_j\partial\theta_l)f^{-1}(\lambda, \theta)$, $k, j, l = 1, \ldots, p$, are continuous in (λ, θ) for $\lambda \in \Lambda$, $\theta \in N_\delta(\theta_0)$, where $N_\delta(\theta_0) := \{\theta : |\theta - \theta_0| < \delta\}$ is some neighborhood of θ_0.

Assumption 5.6. The matrices

$$W(\theta) := \|w_{ij}(\theta)\|, \; A(\theta) := \|a_{ij}(\theta)\|, \; B(\theta) := \|b_{ij}(\theta)\|, \quad i, j = 1, \ldots, p \tag{5.4}$$

are positive definite, where

$$w_{ij}(\theta) = \frac{1}{4\pi}\int_\Lambda \frac{\partial}{\partial\theta_i}\ln f(\lambda, \theta)\frac{\partial}{\partial\theta_j}\ln f(\lambda, \theta)w(\lambda)d\lambda, \tag{5.5}$$

$$a_{ij}(\theta) = \frac{1}{4\pi}\int_\Lambda \frac{\partial}{\partial\theta_i}\ln f(\lambda, \theta)\frac{\partial}{\partial\theta_j}\ln f(\lambda, \theta)w^2(\lambda)d\lambda, \tag{5.6}$$

$$b_{ij}(\theta) = \frac{\kappa_4}{16\pi^2}\int_\Lambda \frac{\partial}{\partial\theta_i}\ln f(\lambda, \theta)w(\lambda)d\lambda\int_\Lambda \frac{\partial}{\partial\theta_j}\ln f(\lambda, \theta)w(\lambda)d\lambda, \tag{5.7}$$

and κ_4 is the fourth cumulant of $\xi(1)$.

The next theorem, which was proved in Ginovyan [12], contains sufficient conditions for the tapered Whittle estimator $\widehat{\theta}_{T,h}$ to be $\sqrt{T}$-consistent and asymptotically normal.

Theorem 5.1. *Suppose that Assumptions 3.1 and 5.1–5.6 are satisfied. Then the Whittle estimator $\widehat{\theta}_{T,h}$ of an unknown spectral parameter θ based on the tapered data* (3.1) *is $\sqrt{T}$-consistent and asymptotically normal, that is,*

$$T^{1/2}\left(\widehat{\theta}_{T,h} - \theta_0\right) \xrightarrow{d} N_p\left(0, e(h)\Gamma(\theta_0)\right) \quad \text{as} \quad T \to \infty, \tag{5.8}$$

where $N_p(\cdot, \cdot)$ denotes the p-dimensional normal law, $\xrightarrow{d}$ stands for convergence in distribution, and

$$\Gamma(\theta_0) = W^{-1}(\theta_0)\left(A(\theta_0) + B(\theta_0)\right)W^{-1}(\theta_0). \tag{5.9}$$

Here the matrices W, A and B are defined in (5.4)-(5.7), and the tapering factor $e(h)$ is given by formula (4.3).

Remark 5.1 (The variance effect)**.** Since tapering of the data, roughly speaking, reduces the effective length of the data, it is not surprising that the corresponding tapered estimators, generally, will have larger variances than their non-tapered counterparts. Specifically, using the Cauchy-Schwartz inequality for the tapering factor $e(h)$ (defined by formula (4.3)) we have $e(h) \geq 1$, and the equality is attained in the non-tapered case, that is, for $h(t) = \mathbb{I}_{[0,1]}(t)$. Thus, the use of tapers, generally, will result in an efficiency loss. However, as it was observed by Dahlhaus (see [6], p.161), *'it is not correct to conclude from this that tapering always increases the variance of the estimators'*, because a taper function h can be chosen to satisfy $e(h) = 1$. Moreover, in the classical asymptotic setting, for d.t. Gaussian processes it is possible to choose the taper function $h(t)$ so that the corresponding tapered estimator will be asymptotically Fisher-efficient (for details see Dahlhaus [4, 6], Ginovyan and Sahakyan [14]).

6. The Le Cam-Dzhaparidze simplified estimators

We describe here the Le Cam-Dzhaparidze approach of constructing simplified estimators in the general setting (see Le Cam [16] and Dzhaparidze [8, 9]).

We first introduce the following set of assumptions (see Dzhaparidze [8]). In what follows, $\tau_T = \tau(T)$ stands for a non-random function increasing without bound as $T \to \infty$.

Assumption 6.1. The system of estimating equations (1.1) has a root $\hat{\theta}_T$ which is a consistent estimator of θ, that is, $p - lim_{T\to\infty}\hat{\theta}_T = \theta_0$.

Assumption 6.2. For $\theta \in \Theta$ the derivatives $F_i^{(k)}(\theta) = (\partial/\partial\theta_k)F_i(\theta)$, $i, k = 1, \dots p$, exist, and for any arbitrarily small $\varepsilon > 0$ and $\delta > 0$

$$\mathbb{P}\left(|F_i^{(k)}(\theta_0) - w_{ik}(\theta_0)| < \varepsilon\right) \geq 1 - \delta, \tag{6.1}$$

where $F_i(\theta)$ is as in (1.1) and $W(\theta) := \|w_{ik}(\theta), i, k = 1, \dots, p\|$ is a non-random matrix, which is non-degenerate for $\theta = \theta_0$.

Assumption 6.3. The second derivatives $F_i^{(k,j)}(\theta) = (\partial^2/\partial\theta_k\partial\theta_j)F_i(\theta)$ exist, which are continuous for $\theta \in \Theta$ and $i, k, j = 1, \dots, p$, and such that for any arbitrarily small $\delta > 0$ and some $M < \infty$,

$$\mathbb{P}\left(|F_i^{(k,j)}(\theta)| < M\right) \geq 1 - \delta. \tag{6.2}$$

Assumption 6.4. Along with (6.1), for sufficiently large T, the following stronger inequality holds:

$$\mathbb{P}\left(\sqrt{\tau_T}|F_i^{(k)}(\theta_0) - w_{ik}(\theta_0)| < \varepsilon\right) \geq 1 - \delta. \tag{6.3}$$

Assumption 6.5. There exists a random matrix $D_* := \|d_{ik}^*\|$, $i, k = 1, \dots, p$, such that for any arbitrarily small $\varepsilon > 0$ and $\delta > 0$, the inequality

$$\mathbb{P}\left(\sqrt{\tau_T}|d_{ik}^* - d_{ik}(\theta_0)| < \varepsilon\right) \geq 1 - \delta \tag{6.4}$$

holds for sufficiently large T and all $i, k = 1, \dots, p$, where $d_{ik}(\theta_0)$ are the elements of the matrix $D(\theta_0) := W^{-1}(\theta_0)$, and $W(\theta)$ is as in Assumption 6.2.

Theorem 6.1 (Dzhaparidze [8]). *Let $\mathbf{F}(\theta)$ be a p-dimensional vector with elements $F_i(\theta)$, $i = 1, \dots p$, $\mathcal{F}(\theta)$ be a matrix with elements $F_i^{(k)}(\theta)$ $i, k = 1, \dots p$, and θ_T^* be an arbitrary τ_T^*-consistent estimator of θ, where $\sqrt{\tau_T}/\tau_T^* \to 0$ as $T \to \infty$. The following assertions hold:*

(a) *Under Assumptions 6.1-6.3 the estimator $\check{\theta}_{1,T} := \theta_T^* - \mathcal{F}^{-1}(\theta_T^*)F(\theta_T^*)$ is asymptotically equivalent to $\hat{\theta}_T$ in the sense that*

$$p - \lim_{T\to\infty} \tau_T \left(\hat{\theta}_T - \check{\theta}_{1,T}\right) = 0.$$

(b) *Under Assumptions 6.1-6.5 the estimators of the form $\check{\theta}_T := \theta_T^* - D_* F(\theta_T^*)$ are asymptotically equivalent to $\hat{\theta}_T$ in the sense that*

$$p - \lim_{T\to\infty} \tau_T \left(\hat{\theta}_T - \check{\theta}_T\right) = 0.$$

Remark 6.1. Comparing assertions (a) and (b) of Theorem 6.1 one easily sees that if $D_* = \mathcal{F}^{-1}(\theta_T^*)$, then the estimator $\check{\theta}_{1,T}$ coincides with $\check{\theta}_T$.

7. Simplified Whittle estimators for spectral parameters with tapered data

As it was stated above (see Theorem 5.1), the tapered Whittle estimator $\widehat{\theta}_{T,h}$ of θ possesses good asymptotic properties, that is, the estimator $\widehat{\theta}_{T,h}$ is $\sqrt{T}$-consistent and asymptotically normal. Moreover, for d.t. Gaussian models it is also asymptotically Fisher-efficient (see Remark 5.1).

However, generally, the estimating equations (5.3) are non-linear, and it is a challenging problem to find the estimator $\widehat{\theta}_{T,h}$. So, it is important finding simpler estimators of the parameter θ having the same asymptotic properties as $\widehat{\theta}_{T,h}$. The estimators proposed here are asymptotically equivalent to the estimator $\widehat{\theta}_{T,h}$ under rather broad regularity conditions on the spectral density function $f(\lambda, \theta)$.

Theorem 7.1. *Let $\mathbf{F}_h(\theta)$ be a p-dimensional vector with elements $F_{i,h}(\theta)$ ($i = 1, \dots p$) given by* (5.3)*, $\theta_{T,h}^*$ be an arbitrary τ_T^*-consistent estimator of θ, where $\sqrt[4]{T}/\tau_T^* \to 0$ as $T \to \infty$, and let $D_* := \|d_{ik}^*\|$, $i, k = 1, \dots, p$, be a random matrix whose elements d_{ik}^* satisfy the condition* (6.4)*. Then under assumptions of Theorem 5.1 the estimators of the form*

$$\check{\theta}_{T,h} := \theta_{T,h}^* - D_* \mathbf{F}_h(\theta_{T,h}^*) \tag{7.1}$$

are asymptotically equivalent to the tapered Whittle estimator $\hat{\theta}_{T,h}$ in the sense that

$$p - \lim_{T\to\infty} \sqrt{T} \left(\hat{\theta}_{T,h} - \check{\theta}_{T,h}\right) = 0.$$

Proof. The result we deduce from Theorem 6.1 by using Theorem 4.1. We show that the Assumptions 6.2–6.4 are satisfied for functions $F_{h,i}(\theta)$ $(i = 1, \dots p)$.

First, applying Theorem 4.1(a) we easily conclude that for $k, j = 1, \dots, p$,

$$\lim_{T\to\infty} \mathbb{E}_0 \left[U^{(kj)}_{T,h}(\theta_0) \right] = w_{kj}(\theta_0) =$$

$$= \frac{1}{4\pi} \int_\Lambda \frac{\partial}{\partial\theta_k} \ln f(\lambda, \theta_0) \frac{\partial}{\partial\theta_j} \ln f(\lambda, \theta_0) w(\lambda) d\lambda, \tag{7.2}$$

where $\mathbb{E}_0[\xi]$ stands for expectation with respect to probability P_0, corresponding to spectral density $f(\lambda, \theta_0)$, and $U^{(kj)}_{T,h}(\theta) = (\partial^2/\partial\theta_k\partial\theta_j)U_{T,h}(\theta)$ with $U_{T,h}(\theta)$ as in (5.1) (for details see Ginovyan [12]).

Next, by applying Theorem 4.1(c), for the variance of $U^{(kj)}_{T,h}(\theta_0)$ $(k, j = 1, \dots, p)$, we have

$$\lim_{T\to\infty} \sqrt{T}\mathrm{Var}\left(U^{(kj)}_{T,h}(\theta_0) \right) = 0. \tag{7.3}$$

Therefore, by Chebyshev's inequality it follows that, for sufficiently large T,

$$\mathbb{P}\left(\sqrt[4]{T} | F^{(k)}_{h,i}(\theta_0) - w_{ik}(\theta_0) | < \varepsilon \right) \geq 1 - \delta, \tag{7.4}$$

where $\varepsilon > 0$ and $\delta > 0$ are arbitrary small numbers. Hence, Assumption 6.4 is satisfied with $\tau_T = \sqrt{T}$. Since the matrix $W(\theta) := \| w_{ik}(\theta),\, i, k = 1, \dots, p \|$ is assumed to be non-degenerate for $\theta = \theta_0$, Assumption 6.2 also holds. Finally, using Theorem 4.1(a) and (c), we easily infer that the function

$$F^{(kj)}_{T,h,i}(\theta) = \frac{1}{4\pi} \int_\Lambda I^h_T(\lambda) \frac{\partial^3}{\partial\theta_i\partial\theta_k\partial\theta_j} f^{-1}(\lambda, \theta) w(\lambda) d\lambda$$

satisfies Assumption 6.3. Thus, the result follows from Theorem 6.1(b). □

Corollary 7.1. *Let $\mathcal{F}(\theta)$ be a matrix with elements $F^{(k)}_{h,i}(\theta)$ $(i, k = 1, \dots p)$. Then under the conditions of Theorem 7.1 the estimator*

$$\check{\theta}_{1,T,h} := \theta^*_{T,h} - \mathcal{F}^{-1}(\theta^*_{T,h}) \mathbf{F}_h(\theta^*_{T,h}), \tag{7.5}$$

is also asymptotically equivalent to the estimator $\hat{\theta}_{T,h}$.

Corollary 7.2. *Assume that for $\theta \in \Theta$ there exist continuous derivatives $(\partial/\partial\theta_i) w_{k,j}(\theta)$ $(i, k, j = 1, \dots, p)$ satisfying $|(\partial/\partial\theta_i) w_{i,k}(\theta)| < C$, where the constant C does not depend on θ. Then under the conditions of Theorem 7.1, the estimator*

$$\check{\theta}_{2,T,h} := \theta^*_{T,h} - W^{-1}(\theta^*_{T,h}) \mathbf{F}_h(\theta^*_{T,h}), \tag{7.6}$$

is also asymptotically equivalent to the estimator $\hat{\theta}_{T,h}$.

Indeed, applying the theorem on the mean we easily conclude that the elements of the matrix $D(\theta^*_{T,h}) = W^{-1}(\theta^*_{T,h})$ satisfy the condition (6.4), and hence may be chosen as the $d^*_{i,k}$.

Remark 7.1. It is easy to see that, similar to the non-tapered case (see Dzhaparidze [8]), the estimators $\check{\theta}_{1,T,h}$ and $\check{\theta}_{2,T,h}$ can be constructed comparatively easily. In fact, to find them it is necessary to have available some τ_T^*-consistent estimator with $\sqrt[4]{T}/\tau_T^* \to 0$ as $T \to \infty$ and to determine the matrices $\mathcal{F}^{-1}(\theta)$ and $W^{-1}(\theta)$, respectively. Observe also that the estimators $\check{\theta}_{T,h}$, $\check{\theta}_{1,T,h}$ and $\check{\theta}_{2,T,h}$ are of interest only if it is too difficult to solve the system of estimating equation (5.3) directly for practical use. In the cases where the equations in (5.3) are linear (and so easily solved), then clearly the estimator $\check{\theta}_{1,T,h}$ coincides with $\hat{\theta}_{T,h}$.

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MOVABILITY OF MORPHISMS IN AN ENRICHED PRO-CATEGORY AND IN A J-SHAPE CATEGORY

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Abstract. Various types of movability for abstract classical pro-morphisms or coherent mappings, and for abstract classical or strong shape morphisms was given by the same authors in some previous paper [10], [11], [12]. In the present paper we introduce and study the notions of (uniform) movability, and (uniform) co-movability for a new type of pro-morphisms and shape morphisms belonging to the so called *enriched pro-category* pro^J-$\mathcal{C}$ and to the corresponding shape category $Sh^J_{(\mathcal{C},\mathcal{D})}$, which were introduced by N. Uglešić [27].

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1. Introduction

The notion of movability for metric compacta was introduced by K. Borsuk [2] as an important shape invariant. The movable spaces are a generalization of spaces having the shape of ANR's. The movability assumption allows a series of important results in algebraic topology (like the Whitehead and Hurewicz theorems) to remain valid when the homotopy pro-groups are replaced by the corresponding shape groups. The term "movability"comes from the geometric interpretation of the definition in the compact case: if X is a compactum lying in a space $M \in AR$, one says that X is movable if for every neighborhood U of X in M there exists a neighborhood $V \subset U$ of X such that for every neighborhood $W \subset U$ of X there is a homotopy $H : V \times [0,1] \to U$ such that $H(x,0) = x$ and $H(x,1) \in W$ for every $x \in V$. One shows that the choice of $M \in AR$ is irrelevant [2]. After the notion of movability had been expressed in terms of ANR-systems for arbitrary topological spaces [16], [17], it became clear that one could define it in arbitrary pro-categories. The definition of a movable object in an arbitrary pro-category and that of uniform movability were both given by Maria Moszyńska [20]. Uniform movability is important in the study of mono- and epi-morphisms in pro-categories and in the study of the shape of pointed spaces. In the book of Sibe Mardešić and

Jack Segal [17] all these approaches and applications of various types of movability are discussed.

Besides the classic case of movability pro-objects and shape objects, some notions of movability for some morphisms appear in the papers of T. Yagasaki [28] and [29], Z.Čerin [3], and D. A. Edwards and P. Tulley McAuley [6]. Unfortunately, these approaches are just particular cases and they do not deal with the movability of shape morphisms in the general case of an abstract shape theory.

Some categorical approaches to movability in shape theory were given by P.S. Gevorgyan [7], [8], P.S. Gevorgyan and I. Pop [9], Avakyan and Gevorgyan [1], and I. Pop [21], [23].

The idea of considering the notions of movability for abstract pro-morphisms and shape morphisms came from the article [22] of the second author, in which the notion of movability is defined for a covariant functor and for a natural transformation (functorial morphism). Then, considering the inverse systems as functors and the pro-morphisms as natural transformations, various types of movability can be obtained, for pro-morphisms and shape morphisms, which is done in the papers of P.S. Gevorgyan and I. Pop [10], [11], [12]. But what is achieved by introducing this property? In short: if $m : X \to Y$ is a pro-morphism or a shape morphism and if X or Y is a movable pro-object or a shape-object then m is a movable morphism. And if $Y = X$ and $m = 1_X$, then X is movable if and only if the morphism 1_X is movable. We see that the movability of morphisms (pro- or shape-) is a generalization of the movability of objects in that category. And then, to obtain a theorem on the morphism m, assuming that X or Y is movable, it may happen that the same result should be obtained with the weaker condition that m be movable.

In the present paper we introduce and study the notions of movability for a new type of pro-morphisms and shape morphisms associated with a category $\mathcal{C}$ and a pair $(\mathcal{C}, \mathcal{D})$ respectively, namely belonging to a so called *enriched pro-category* pro^J-$\mathcal{C}$, and respectively to the corresponding shape category $Sh^J_{(\mathcal{C},\mathcal{D})}$ having as the realizing subcategory the category pro^J-$\mathcal{D}$ for $(J, \leq)$ a directed partially ordered set, according to the article [27] by N. Uglešić.

Because by particularization of the set $(J, \leq)$ one can obtain the classical abstract shape theory and the so-called coarse shape theory, the results of this article can be considered as generalizations of the corresponding results from the papers [9], [10], and [12].

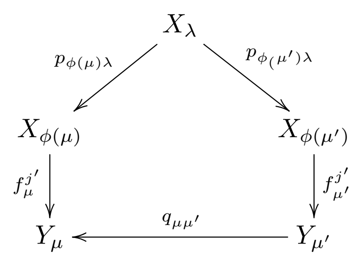
2. Enriched pro-category and J-shape category

In this section are given the notions and results from [27] necessary for the approach of our paper. Other notions and necessary results from shape theory can be found in the books [17] and [4].

Definition 2.1. Let $\mathcal{C}$ be a category, let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in $\mathcal{C}$ and let $J = (J, \leq)$ be a directed partially ordered set. A *J-morphism* (of $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{C}$) is every triple $(\mathbf{X}, ((f^j_\mu), \phi), \mathbf{Y})$, denoted $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$, where $((f^j_\mu), \phi)$ is an ordered pair consisting of a function $\phi : M \to \Lambda$, called the *index function*, and, for each $\mu \in M$, of a family (f^j_μ) of $\mathcal{C}$-morphisms $f^j_\mu : X_{\phi(\mu)} \to Y_\mu, j \in J$, such that, for every related pair $\mu' \geq \mu$ in M, there exists a $\lambda \in \Lambda, \lambda \geq \phi(\mu), \phi(\mu')$, and there exists a $j \in J$ so that for every $j' \geq j$,

$$f^{j'}_\mu p_{\phi(\mu)\lambda} = q_{\mu\mu'} f^{j'}_{\mu'} p_{\phi(\mu')\lambda}, \tag{2.1}$$

i.e., makes the following diagram commutative



If the index function ϕ is increasing and, for every pair $\mu \leq \mu'$, one may put $\lambda = \phi(\mu')$, then (f^j_μ, ϕ) is said to be a *simple* J-morphism.

If, in addition, $M = \Lambda$ and $\phi = 1_\Lambda$, then $(f^j_\lambda, 1_\Lambda)$ is said to be a *level* J-morphism.

Further, if the equality (2.1) holds for every $j \in J$, then $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ is said to be a *commutative* J-morphism.

Remark 2.1. a) The *composition* of two J-morphisms $(f^j_\mu, \phi) : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ and $(g^j_\nu, \psi) : \mathbf{Y} \to \mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ is defined as $(h^j_\nu, \chi) : \mathbf{X} \to \mathbf{Z}$, with $\chi = \phi \circ \psi$ and $h^j_\nu = g^j_\nu \circ f^j_{\psi(\nu)}$, $j \in J$, $\nu \in N$. This composition is associative.

b) The *identity J-morphism* of the inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is $(1^j_{X_\lambda}, 1_\Lambda) : \mathbf{X} \to \mathbf{X}$ with $1^j_{X_\lambda} = 1_{X_\lambda}$ for any $j \in J$, where 1_{X_λ} is the identity morphism of X_λ in the category $\mathcal{C}$.

c) For a category $\mathcal{C}$ and a directed partially ordered set J there exists a category inv^J-$\mathcal{C}$ having the object class $Ob(inv^J\text{-}\mathcal{C}) = Ob(inv\text{-}\mathcal{C})$ and the morphism class $Mor(inv^J\text{-}\mathcal{C})$ of all sets $(inv^J\text{-}\mathcal{C})(\mathbf{X}, \mathbf{Y})$ of all J-morphisms (f^j_μ, ϕ) of $\mathbf{X}$ to $\mathbf{Y}$, endowed with the composition and identities described in a) and b).

Definition 2.2. A J-morphism $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ of inverse systems in $\mathcal{C}$ is said to be *equivalent to* a J-morphism $({f'}^j_\mu), \phi') : \mathbf{X} \to \mathbf{Y}$, denoted by $(f^j_\mu, \phi) \sim ({f'}^j_\mu, \phi')$, if every $\mu \in M$ admits a $\lambda \in \Lambda$, $\lambda \geq \phi(\mu), \phi'(\mu)$, and a $j \in J$ such that, for every $j' \geq j$,

$$f^{j'}_\mu p_{\phi(\mu)\lambda} = {f'}^{j'}_\mu p_{\phi'(\mu)\lambda}, \tag{2.2}$$

i.e., makes the following diagram commutative

$$\begin{array}{ccc} X_\lambda & \xrightarrow{p_{\phi(\mu)\lambda}} & X_{\phi(\mu)} \\ {\scriptstyle p_{\phi'(\mu)\lambda}}\downarrow & & \downarrow{\scriptstyle f^{j'}_\mu} \\ X_{\phi'(\mu)} & \xrightarrow[{f'}^{j'}_\mu]{} & Y_\mu \end{array}$$

Remark 2.2. a) The defining equality (2.2) holds for every $\lambda' \geq \lambda$;

b) The relation $\sim$ is an equivalence relation on each set $(inv^J\text{-}\mathcal{C})(\mathbf{X}, \mathbf{Y})$;

c) The equivalence class $[(f^j_\mu, \phi)]$ of a J-morphism is denoted by $\mathbf{f}$;

d) Let $(f^j, \phi), ({f'}^j_\mu, \phi') : \mathbf{X} \to \mathbf{Y}$ and $(g^j_\nu, \psi), ({g'}^j_\nu, \psi') : \mathbf{Y} \to \mathbf{Z}$ be J-morphisms of inverse systems in $\mathcal{C}$. If $(f^j_\mu, \phi) \sim ({f'}^j_\mu, \phi')$ and $(g^j_\nu, \psi) \sim ({g'}^j_\nu, \psi')$, then $(g^j_\nu, \psi)(f^j_\mu, \phi) \sim ({g'}^j_\nu, \phi')({f'}^j_\mu, \phi')$;

e) By the above remarks one may compose the equivalence classes of J-morphisms of inverse systems in $\mathcal{C}$ by means of any pair of their representatives, i.e., $\mathbf{gf} = \mathbf{h}$, where $\mathbf{h}$ is the equivalence class of $(h^j_\nu, h) = (g^j_\nu, \psi)(f^j_\mu, \phi) = (g^j_\nu f^j_{g(\nu)}, \phi\psi)$. The corresponding quotient category $(inv^J\text{-}\mathcal{C})/\sim$ is denoted by $pro^J\text{-}\mathcal{C}$. The morphisms of this category are called *J-pro-morphisms*. There exists a subcategory $(pro^J\text{-}\mathcal{C})_c \subseteq pro^J\text{-}\mathcal{C}$ determined by all equivalence classes having commutative representatives. This category is isomorphic to the quotient category $(inv^J\text{-}C)_c/\sim$. Also $pro\text{-}\mathcal{C} = (inv\text{-}\mathcal{C})/\sim$ can be considered as a subcategory of $(pro^J\text{-}\mathcal{C})_c$ and, consequently as a subcategory of $pro^J\text{-}\mathcal{C}$

f) Now using the fact that if $(\Lambda, \leq)$ is a directed set and $(M, \leq)$ is a cofinite directed set, then every function $\phi : M \to \Lambda$ admits an increasing function $\phi' : M \to \Lambda$ such that $\phi \leq \phi'$ (see [17], Ch.I, §1.2, Lemma 1), it can be proved that: if $\mathbf{f} : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to Y = (Y_\mu, q_{\mu\mu'}, M)$ is a morphism in $pro^J\text{-}\mathcal{C}$, with $(M, \leq)$ cofinite, then $\mathbf{f}$ admits a simple representative $({f'}^j_\mu, \phi') : \mathbf{X} \to \mathbf{Y}$ ([27], Lemma 6).

g) There exists a covariant functor $\underline{I} \equiv \underline{I}^J_{\mathcal{C}} : pro\text{-}\mathcal{C} \to pro^J\text{-}\mathcal{C}$, by: $\underline{I}(\mathbf{X}) = \mathbf{X}$, for every inverse system $\mathbf{X}$ in $\mathcal{C}$, and if $\mathbf{f} \in pro\text{-}\mathcal{C}(\mathbf{X}, \mathbf{Y})$ and $\mathbf{f} = [(f_\mu, \phi)]$, then $\underline{I}(\mathbf{f}) = [(f^j_\mu, \phi)] \in (pro^J\text{-}\mathcal{C})(\mathbf{X}, \mathbf{Y})$, where for each $\mu \in M$, $f^j_\mu = f_\mu$ for all $j \in J$. Thus, every induced J-morphism is commutative, and therefore $\underline{I}^J_{\mathcal{C}} : pro\text{-}\mathcal{C} \to (pro^J\text{-}\mathcal{C})_c \subseteq$

pro^J-$\mathcal{C}$. It is easy to see that this functor is faithful ([27], Theorem 1), but it is not full ([27], Remark 2).

h) Every inverse system $\mathbf{X}$ in $\mathcal{C}$ is isomorphic in pro^J-$\mathcal{C}$ to a cofinite inverse system $\mathbf{X}'$.

An important theorem is the following ([27], Theorem 2; [17], Ch.I, §1.3, Theorem 3):

Theorem 2.1. *Let* $\mathbf{f} : \mathbf{X} \to \mathbf{Y} \in (pro^J\text{-}\mathcal{C})(\mathbf{X}, \mathbf{Y})$. *Then there exist inverse systems* $\mathbf{X}'$ *and* $\mathbf{Y}'$ *in* $\mathcal{C}$ *having the same cofinite index set* $(N, \leq)$, *there exists a morphism* $\mathbf{f}' : \mathbf{X}' \to \mathbf{Y}'$ *having a level representative* $(f'^j_\nu, 1_N)$ *and there exists isomorphisms* $\mathbf{i} : \mathbf{X} \to \mathbf{X}'$ *and* $j : \mathbf{Y} \to \mathbf{Y}'$ *of* pro^J-$\mathcal{C}$ *such that the following diagram in* pro^J-$\mathcal{C}$ *commutes*

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\mathbf{f}} & \mathbf{Y} \\ \downarrow{\mathbf{i}} & & \downarrow{\mathbf{j}} \\ \mathbf{X}' & \xrightarrow[\mathbf{f}']{} & \mathbf{Y}' \end{array}$$

Remark 2.3. a) If $J = \{1\}$, then $pro^{(1)}$-$\mathcal{C} = pro$-$\mathcal{C}$;

b) If $(J, \leq) = (\mathbb{N}, \leq)$, then $pro^{\mathbb{N}}$-$\mathcal{C} = pro^*$-$\mathcal{C}$ is the pro-category obtained from the category $(inv^*$-$\mathcal{C})$ with so-called, $*$-*morphisms* [14];

c) If J is a directed partially ordered set having $maxJ$, then pro^J-$\mathcal{C} \cong pro$-$\mathcal{C}$. The "inclusion"functor $I : pro$-$C \to pro^J$-C is a category isomorphism;

d) If J and K are finite directed partially ordered sets, then there exist the isomorphisms: pro^J-$\mathcal{C} \cong pro^K$-$\mathcal{C} \cong pro$-$\mathcal{C}$;

e) If there exists $maxJ$, then for every L there exists the canonical inclusion functor $\underline{I} : pro^J$-$\mathcal{C} \to pro^L$-$\mathcal{C}$ keeping the objects fixed;

f) Let J be a well ordered set and let K be a directed partially ordered set, both without maximal elements, such that there exists an increasing function $\phi : J \to K$ such that $\phi[J]$ is cofinal in K. Then there exists a functor $\underline{T} : pro^J$-$\mathcal{C} \to pro^K$-$\mathcal{C}$ which keeps the objects fixed and does not depend on ϕ. Furthermore, for every pair $\mathbf{X}$ and $\mathbf{Y}$ of inverse systems in $\mathcal{C}$, $\mathbf{X} \cong \mathbf{Y}$ in pro^J-$\mathcal{C}$ iff $\mathbf{X} \cong \mathbf{Y}$ in pro^K-$\mathcal{C}$.

Remark 2.4. A pro^J-C category is called an *enriched pro-category.* An enriched pro-category is interesting and useful by itself because, in general, it divides (classifies) the objects into larger classes (isomorphisms types) than the underling pro-category pro-$\mathcal{C}$. In addition, with the help of such an enriched pro-category one can construct in the usual way a corresponding J*-shape theory.*

Definition 2.3. A J-pro-morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is said to be pro^J-$\mathcal{D}$ *equivalent* to a J-pro-morphism $\mathbf{f}' : \mathbf{X}' \to \mathbf{Y}'$, denoted by $\mathbf{f} \sim \mathbf{f}'$, if there exist two canonical isomorphisms $\mathbf{i} : \mathbf{X} \to \mathbf{X}'$ and $\mathbf{j} : \mathbf{Y} \to \mathbf{Y}'$ of *pro*-$\mathcal{D}$ such that the following diagram in *pro*-$\mathcal{D}$ commutes:

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\mathbf{i}} & \mathbf{X}' \\ {\scriptstyle \mathbf{f}}\downarrow & & \downarrow{\scriptstyle \mathbf{f}'} \\ \mathbf{Y} & \xrightarrow[\mathbf{j}]{} & \mathbf{Y}' \end{array}$$

The equivalence class of a J-pro-morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is denoted by $\langle \mathbf{f} \rangle$.

Remark 2.5. If $\mathbf{f} \sim \mathbf{f}'$ and $\mathbf{g} \sim \mathbf{g}'$, then $\mathbf{gf} \sim \mathbf{g}'\mathbf{f}'$, so the composition $\langle \mathbf{g} \rangle \langle \mathbf{f} \rangle = \langle \mathbf{gf} \rangle$ is well defined.

Definition 2.4. For a pair of categories $(\mathcal{C}, \mathcal{D})$ with $\mathcal{D}$ a dense full (equivalent, full and pro-reflective, [26]) subcategory of $\mathcal{C}$, the (*abstract*) *J-shape category* $Sh^J_{(\mathcal{C},\mathcal{D})}$ is defined as follows. The objects of this category are all the objects of $\mathcal{C}$. A morphism $F \in Sh^J_{(C,D)}(X, Y)$ is the (*pro*^{J}-$\mathcal{D}$)-equivalence class $\langle \mathbf{f} \rangle$ of a J-morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ of *pro*^{J}-$\mathcal{D}$ for an arbitrary choice of $\mathcal{D}$-expansions $\mathbf{p} : X \to \mathbf{X}$, $\mathbf{q} : Y \to \mathbf{Y}$. In other words, a J-shape morphism $F : X \to Y$ is given by a diagram

$$\begin{array}{ccc} \mathbf{X} & \xleftarrow{\mathbf{p}} & X \\ {\scriptstyle \mathbf{f}}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbf{Y} & \xleftarrow[\mathbf{q}]{} & Y \end{array}$$

The composition of two J-shape morphisms $F : X \to Y$, $F = \langle \mathbf{f} \rangle$ and $G : Y \to Z$, $G = \langle \mathbf{g} \rangle$, is defined by representatives, i.e., $GF : X \to Z$, $GF = \langle \mathbf{gf} \rangle$.

The identity J-shape morphism on an object X, $1_X : X \to X$, is the (*pro*^{J}-D)-equivalence class $\langle 1_{\mathbf{X}} \rangle$ of the identity morphism $1_{\mathbf{X}}$ of $\mathbf{X}$ in *pro*^{J}-$\mathcal{D}$.

Since $Sh^J_{(C,D)}(X, Y) \approx$ *pro*^{J}-$\mathcal{D}(\mathbf{X}, \mathbf{Y})$ is a set, the J-shape category $Sh^J_{(C,D)}$ is well defined, and that its realizing category is *pro*^{J}-$\mathcal{D}$.

An interesting particular case of J-shape morphism is the following: If $f : X \to Y$ is a morphism in the category $\mathcal{C}$ and $\mathbf{p} : X \to \mathbf{X}$, $\mathbf{q} : Y \to \mathbf{Y}$ are $\mathcal{D}$-expansions, then there exists a morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in *pro*^{J}-$\mathcal{D}$, such that the following diagram in *pro*^{J}-$\mathcal{C}$ commutes:

$$\begin{array}{ccc} \mathbf{X} & \xleftarrow{\mathbf{p}} & X \\ {\scriptstyle \mathbf{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{Y} & \xleftarrow[\mathbf{q}]{} & Y \end{array}$$

This is a result of the definition of an expansion, [17] (Ch. I, §2.1). If we take other $\mathcal{D}$-expansions $\mathbf{p}' : X \to \mathbf{X}'$, $\mathbf{q}' : Y \to \mathbf{Y}'$, we obtain another morphism $\mathbf{f}' : \mathbf{X}' \to \mathbf{Y}'$ in *pro*^{J}-$\mathcal{D}$, such that $\mathbf{f}'\mathbf{p}' = \mathbf{q}'f$. And because $(\mathbf{f}'\mathbf{i})\mathbf{p} = \mathbf{f}'\mathbf{p}' = \mathbf{q}'f = \mathbf{jq}f = (\mathbf{jf})\mathbf{p}$, which implies $\mathbf{f}'\mathbf{i} = \mathbf{j}\mathbf{f}$, such that $\mathbf{f} \sim \mathbf{f}'$ in $pro^J\text{-}\mathcal{D}$, and in this way we can associate with every $f \in \mathcal{C}(X, Y)$ a $pro^J\text{-}\mathcal{D}$-equivalence class $\langle \mathbf{f} \rangle$, i.e., a J-shape morphism $F \in Sh^J_{(\mathcal{C},\mathcal{D})}(X, Y)$.

If one defines $S^J(X) = X$, $X \in Ob\mathcal{C}$, and $S^J(f) = F = \langle \mathbf{f} \rangle$, $f \in \mathcal{C}(X, Y)$, we obtain a covariant functor $S^J \equiv S^J_{(\mathcal{C},\mathcal{D})} : \mathcal{C} \to Sh^J_{(\mathcal{C},\mathcal{D})}$, called (*abstract*) *J-shape functor*.

Theorem 2.2 ([27], Theorem 5)**.** *Let $\mathcal{D}$ be a full and pro-reflective subcategory of $\mathcal{C}$ and J a directed partially ordered set. Then, for every pair $P, Q \in Ob\mathcal{D}$, the following statements are equivalent:*

(i) P and Q are isomorphic objects of $\mathcal{D}$, $P \cong Q$ in $\mathcal{D} \subseteq \mathcal{C}$;

(ii) P and Q have the same shape, $Sh(P) = Sh(Q)$, i.e., $P \cong Q$ in $Sh_{(\mathcal{C},\mathcal{D})}$;

(iii) P and Q have the same J-shape, $Sh^J_{(\mathcal{C},\mathcal{D})}(P) = Sh^J_{(\mathcal{C},\mathcal{D})}(Q)$, i.e., $P \cong Q$ in $Sh^J_{(\mathcal{C},\mathcal{D})}$.

Theorem 2.3 ([27], Corollary 2)**.** *Let $\mathcal{C}$ a category and $\mathcal{D}$ a full and pro-reflective subcategory. Then*

(i) $Sh_{(\mathcal{C},\mathcal{D})} = Sh^{\{1\}}_{(\mathcal{C},\mathcal{D})}$;

(ii) $Sh^*_{(\mathcal{C},\mathcal{D})} = Sh^{\mathbb{N}}_{(\mathcal{C},\mathcal{D})}$, where $Sh^*_{(\mathcal{C},\mathcal{D})}$ is the coarse shape category [14]*;*

(iii) If J is a directed partially ordered set having $\max J$, then $Sh^J_{(\mathcal{C},\mathcal{D})} \cong Sh_{(\mathcal{C},\mathcal{D})}$.

3. Movability and uniform movability properties for J-morphisms

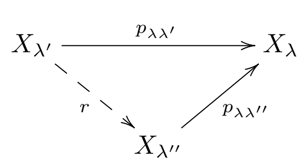
All sets of indices of inverse systems are supposed to be cofinite directed sets. This condition is not restrictive (cf. [17], Ch.I, §1.2).

First we recall from [17] the notions of movable and uniform movable inverse systems.

An object $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of pro-$\mathcal{C}$ is *movable* provided every $\lambda \in \Lambda$ admits a $\lambda' \geq \lambda$ (called a *movability index* of λ) such that each $\lambda'' \geq \lambda$ admits a morphism $r : X_{\lambda'} \to X_{\lambda''}$ of $\mathcal{C}$ which satisfies

$$p_{\lambda\lambda''} \circ r = p_{\lambda\lambda'}, \tag{3.1}$$

i.e., makes the following diagram commutative



An object $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of pro-$\mathcal{C}$ is called a *uniform movable* if every $\lambda \in \Lambda$ admits a $\lambda' \geq \lambda$ (called a *uniform movability index* of λ) such that there is a morphism $\mathbf{r} : X_{\lambda'} \to \mathbf{X}$ in pro-$\mathcal{C}$ satisfying

$$\mathbf{p}_\lambda \circ \mathbf{r} = p_{\lambda\lambda'}, \tag{3.2}$$

where $\mathbf{p}_\lambda : \mathbf{X} \to X_\lambda$ is the morphism of pro-$\mathcal{C}$ given by $1_\lambda : X_\lambda \to X_\lambda$.

Definition 3.1. Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in a category $\mathcal{C}$ and $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ be a J-morphism of inverse systems. We say that the J-morphism (f^j_μ, ϕ) is *movable* (*J-movable*) if every $\mu \in M$ admits $\lambda \in \Lambda$, $\lambda \geq \phi(\mu)$ and $j \in J$, such that each $\mu' \in M$, $\mu' \geq \mu$, and $j' \geq j$ admit a morphism $u^{j'} : X_\lambda \to Y_{\mu'}$ in the category $\mathcal{C}$, which satisfies

$$f^{j'}_\mu \circ p_{\phi(\mu)\lambda} = q_{\mu\mu'} \circ u^{j'}, \tag{3.3}$$

i.e., makes the following diagram commutative

$$\begin{array}{ccc} X_{\phi(\mu)} & \xrightarrow{f^{j'}_\mu} & Y_\mu \\ {\scriptstyle p_{\phi(\mu)\lambda}} \uparrow & & \uparrow {\scriptstyle q_{\mu\mu'}} \\ X_\lambda & \dashrightarrow_{u^{j'}} & Y_{\mu'} \end{array}$$

The pair of indices (λ, j) is called a *J-movability pair* of μ with respect to the J-morphism (f^j_μ, ϕ).

The composition $f^j_\mu \circ p_{\phi(\mu)\lambda}$ for $\lambda \geq \phi(\mu)$ is denoted by $f^j_{\mu\lambda}$ (cf. [17], Ch.II, §2.1). With this notation the relation (3.3) becomes

$$f^{j'}_{\mu\lambda} = q_{\mu\mu'} \circ u^{j'}.$$

Note that if (λ, j) is a J-movability pair of μ with respect to (f^j_μ, ϕ), then so is any pair $(\widetilde{\lambda}, \widetilde{j})$, with $\widetilde{\lambda} \geq \lambda$ and $\widetilde{j} \geq j$.

Example 3.1. Let (X) be a rudimentary system in the category $\mathcal{C}$ and (f^j_μ, ϕ) : $(X) \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$, $\phi(\mu) = 1, \forall \mu \in M$, a J-morphism of inverse systems. It is not hard to verify that (f^j_μ, ϕ) is movable.

More generally, if we consider a morphism $(f^j_\mu, \phi) : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ such that there exists $\lambda_M \in \Lambda$ satisfying $\lambda_M \geq \phi(\mu)$ for any $\mu \in M$, then (f^j_μ, ϕ) is J-movable. Indeed, for an arbitrary index $\mu \in M$ and $\mu' \geq \mu$ there exists $j \in J$ such that for $j' \in J$, $j' \geq j$, we have $f^{j'}_\mu \circ p_{\phi(\mu)\lambda_M} = q_{\mu\mu'} \circ f^{j'}_{\mu'} \circ p_{\phi(\mu')\lambda_M} = q_{\mu\mu'} \circ u^{j'}$, where $u^{j'} = f^{j'}_{\mu'} \circ p_{\phi(\mu')\lambda_M}$ is a morphism from X_{λ_M} to $Y_{\mu'}$. So, (λ_M, j) is a J-movability pair for $\mu \in M$.

Remark 3.1. a) If $J = \{1\}$, that is, inv^J-$\mathcal{C} = inv$-$\mathcal{C}$, then the condition of movability for a morphism of inverse systems $(f_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ is written as $f_\mu \circ p_{\phi(\mu)\lambda} = q_{\mu\mu'} u$, for a $\lambda \in \Lambda$, $\lambda \geq \phi(\mu)$, $\mu' \geq \mu$, and $u : X_\lambda \to Y_{\mu'}$ a morphism in $\mathcal{C}$. And this is the definition of movability for an usual morphism of inverse systems (cf. [10], Definition 2.2).

b) If $(J, \leq) = (\mathbb{N}, \leq)$, i.e., inv^J-$\mathcal{C} = inv^*$-$\mathcal{C}$, the condition of movability for a $*$-morphism $(f^n_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ is the following: every $\mu \in M$ admits $\lambda \in \Lambda$, $\lambda \geq \phi(\mu)$ and $n \in \mathbb{N}$, such that each $\mu' \in M$, $\mu' \geq \mu$, and $m \geq n$, admit a morphism $u^m : X_\lambda \to Y_{\mu'}$ in the category $\mathcal{C}$, which satisfies

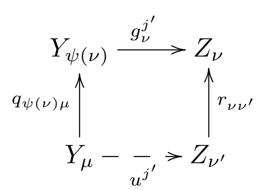
$$f^m_\mu \circ p_{\phi(\mu)\lambda} = q_{\mu\mu'} \circ u^m. \tag{3.4}$$

Proposition 3.1. *An inverse system* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *is movable if and only if the identity J-morphism* $(1^j_{X_\lambda}, 1_\Lambda)$ *is movable.*

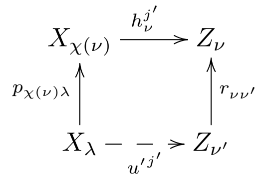
Proof. If λ' is a movability index of λ with respect to $\mathbf{X}$, then a pair (λ', j), $j \in J$, is a J-movability pair for λ with respect to the identity J-morphism, and conversely. □

Theorem 3.1. *Let* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$, $\mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ *be inverse systems in the category $\mathcal{C}$ and let* $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$, $(g^j_\nu, \psi) : \mathbf{Y} \to \mathbf{Z}$ *be J-morphisms of inverse systems. If* (g^j_ν, ψ) *is movable, then the composition* $(h^j_\nu, \chi) = (g^j_\nu, \psi) \circ (f^j_\mu, \phi)$ *is also a movable J-morphism.*

Proof. Recall that by definition of the composition of J-morphisms we have $\chi = \phi \circ \psi$ and $h^j_\nu = g^j_\nu \circ f^j_{\psi(\nu)}$. If (g^j_ν, ψ) is movable, and if (μ, j) is a J-movability pair of an index $\nu \in N$, then for any index $\nu' \in N$, $\nu' \geq \nu$, there is an index $j \in J$ and a morphism $u^{j'} : Y_\mu \to Z_{\nu'}$, $j' \geq j$, such that $g^{j'}_\nu \circ q_{\psi(\nu)\mu} = r_{\nu\nu'} \circ u^{j'}$ or the next diagram is commutative



Now consider $\lambda \in \Lambda$ such that $\lambda \geq \phi(\mu)$, $\lambda \geq \phi(\psi(\nu))$, and $f^{j'}_{\psi(\nu)} \circ p_{\phi(\psi(\nu))\lambda} = q_{\psi(\nu)\mu} \circ f^{j'}_\mu \circ p_{\phi(\mu)\lambda}$. Consider the morphism $u'^{j'} = u^{j'} \circ f^{j'}_\mu \circ p_{\phi(\mu)\lambda} : X_\lambda \to Z_{\nu'}$. For this morphism we obtain: $r_{\nu\nu'} \circ u'^{j'} = (r_{\nu\nu'} \circ u^{j'}) \circ f_\mu \circ p_{\phi(\mu)\lambda} = g_\nu \circ q_{\psi(\nu)} \circ f^{j'}_\mu \circ p_{\phi(\mu)\lambda} = g_\nu \circ f^{j'}_{\psi(\nu)} \circ p_{\phi(\psi(\nu))} = h^{j'}_\nu \circ p_{\chi(\nu)\lambda}$, i.e., the following diagram is commutative



Thus, (h_ν^j, χ) is a movable J-morphism. $\square$

Corollary 3.1. *Let* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *be an arbitrary inverse system and* $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ *be a movable inverse system. Then any* J*-morphism* $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$ *is movable.*

Proof. Since $(f_\mu^j, \phi) = (1_{Y_\mu}^j, 1_M) \circ (f_\mu^j, \phi)$ and $(1_{Y_\mu}^j, 1_M) : \mathbf{Y} \to \mathbf{Y}$ is a movable J-morphism by Proposition 3.1, then (f_μ^j, ϕ) is also J-movable according to Theorem 3.1. $\square$

Theorem 3.2. *Let* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$*,* $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$*,* $\mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ *be inverse systems in the category* $\mathcal{C}$*, and let* $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$*,* $(g_\nu^j, \psi) : \mathbf{Y} \to \mathbf{Z}$ *be* J*-morphisms. If* (f_μ^j, ϕ) *is movable, then the composition* $(h_\nu^j, \chi) = (g_\nu^j, \psi) \circ (f_\mu^j, \phi)$ *is also a movable* J*-morphism.*

Proof. For a given index $\nu \in N$, consider a movability pair (λ, j) of $\psi(\nu)$, $\lambda \geq \phi(\psi(\nu))$, with respect to (f_μ^j, ϕ). Let us prove that (λ, j) is a movability pair of ν with respect to the J-morphism (h_ν^j, χ).

Let $\nu' \in N$, $\nu' \geq \nu$, be any index and let $\mu' \geq \psi(\nu'), \psi(\nu)$ be an index such that for $j' \geq j$

$$r_{\nu\nu'}^{j'} \circ g_{\nu'}^{j'} \circ q_{\psi(\nu')\mu'} = g_\nu^{j'} \circ q_{\psi(\nu)\mu'}.$$

By the J-movability of $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$, for $q_{\psi(\nu)\mu'}$ there exists a morphism $u^{j'} : X_\lambda \to Y_{\mu'}$ such that

$$f_{\psi(\nu)}^{j'} \circ p_{\phi(\psi(\nu))\lambda} = q_{\psi(\nu)\mu'} \circ u^{j'}.$$

Define $U^{j'} : X_\lambda \to Z_{\nu'}$ by

$$U^{j'} = g_{\nu'}^{j'} \circ q_{\psi(\nu')\mu'} \circ u^{j'}.$$

Now we have: $r_{\nu\nu'}^{j'} \circ U^{j'} = r_{\nu\nu'}^{j'} \circ g_{\nu'}^{j'} \circ q_{\psi(\nu')\mu'} \circ u^{j'} = g_\nu^{j'} \circ q_{\psi(\nu)\mu'} \circ u^{j'} = g_\nu^{j'} \circ q_{\psi(\nu)\mu} \circ f_\mu^{j'} \circ p_{\phi(\mu)\lambda} = g_\nu^{j'} \circ f_{\psi(\nu)}^{j'} \circ p_{\chi(\nu)\lambda} = h_\nu^{j'} \circ p_{\chi(\nu)\lambda}$. $\square$

Corollary 3.2. *Let* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *be a movable inverse system and let* $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ *be an arbitrary inverse system. Then any* J*-morphism* $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$ *is movable.*

Proof. Since $(f^j_\mu, \phi) = (f^j_\mu, \phi) \circ (1^j_{X_\lambda}, 1_\Lambda)$ and the identity J-morphism $(1^j_{X_\lambda}, 1_\Lambda)$: $\mathbf{X} \to \mathbf{X}$ is movable by Proposition 3.1, then (f^j_μ, ϕ) is also J-movable according to Theorem 3.2. □

Corollary 3.3. *Let* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *be a movable inverse system in the category* $\mathcal{C}$. *If an inverse system* $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ *is* J*-dominated by* $\mathbf{X}$, *i.e., there exist two* J*-morphisms* $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ *and* $(g^j_\lambda, \psi) : \mathbf{Y} \to \mathbf{X}$ *such that* $(f^j_\mu, \phi) \circ (g^j_\lambda, \psi) = (1^j_{Y_\mu}, 1_M)$, *then* $\mathbf{Y}$ *is movable.*

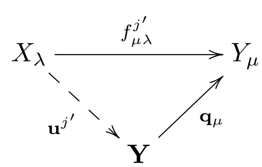
Proof. By hypothesis and Proposition 3.1, $(1^j_{X_\lambda}, 1_\Lambda)$ is J-movable. Then by the equality $(1^j_{X_\lambda}, 1_\Lambda) \circ (g^j_\lambda, \psi) = (g^j_\lambda, \psi)$ and by Theorem 3.1 it follows that (g^j_ν, ψ) is J-movable. Hence, the composition $(f^j_\mu, \phi) \circ (g^j_\lambda, \psi) = (1^j_{Y_\mu}, 1_M)$ is also J-movable by Theorem 3.2. Therefore, Proposition 3.1 implies that $\mathbf{Y}$ is a movable inverse system. □

Remark 3.2. Corollary 3.3 is a generalization of a classical result for the movability of inverse systems [17] (Ch. II, §6.1, Theorem 1) here with a proof based on the J-movability property of J-morphisms.

Definition 3.2. Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in a category $\mathcal{C}$ and let $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ be a J-morphism. We say that the J-morphism (f_μ, ϕ) is *uniformly movable*(*J-uniformly movable*) if every $\mu \in M$ admits $\lambda \in \Lambda$, $\lambda \geq \phi(\mu)$ and $j \in J$ such that for $j' \geq j$ there is a J-morphism of inverse systems $\mathbf{u^{j'}} : X_\lambda \to \mathbf{Y}$ satisfying

$$f^{j'}_{\mu\lambda} = \mathbf{q}_\mu \circ \mathbf{u}^{j'} \tag{3.5}$$

i.e., the following diagram commutes



where $f^{j'}_{\mu\lambda} = f^{j'}_\mu \circ p_{\phi(\mu)\lambda}$ and $\mathbf{q}_\mu : \mathbf{Y} \to Y_\mu$ is the J-morphism of inverse systems given by $1^j_{Y_\mu} : Y_\mu \to Y_\mu$.

The pair (λ, j) is called a *J-uniform movability pair of μ with respect to* (f^j_μ, ϕ).

Remark 3.3. If (λ, j) is a J-uniform movability pair, then so is any pair $(\widetilde{\lambda}, \widetilde{j})$, $\widetilde{\lambda} \geq \lambda$, $\widetilde{j} \geq j$.

Remark 3.4. Note that the J-morphism $\mathbf{u}^{j'} : X_\lambda \to \mathbf{Y}$ determines for every $\mu_1 \in M$ a morphism $u^{j'}_{\mu_1} : X_\lambda \to Y_{\mu_1}$ in $\mathcal{C}$ such that for $\mu_1 \leq \mu_2$ we have $q_{\mu_1\mu_2} \circ u^{j'}_{\mu_2} = u^{j'}_{\mu_1}$

and $u_\mu^{j'} = f_{\mu\lambda}^{j'}$. In particular, for $\mu' \in M$, $\mu' \geq \mu$, we have $q_{\mu\mu'} \circ u_{\mu'}^{j'} = u_\mu^{j'} = f_{\mu\lambda}^{j'}$, so that J-uniform movability of J-morphisms implies J-movability.

Proposition 3.2. *An inverse system* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *is uniformly movable if and only if the identity* J*-morphism* $1_\mathbf{X} = (1_{X_\lambda}^j, 1_\Lambda)$ *is* J*-uniformly movable.*

Proof. Suppose $\mathbf{X}$ is uniformly movable. Let $\lambda \in \Lambda$. Note that a uniform movability index $\lambda' \geq \lambda$ together with $j \in J$ arbitrary constitutes a pair (λ', j) of J-uniform movability of λ with respect to the identity $1_\mathbf{X} = (1_{X_\lambda}^j, 1_\Lambda)$. Conversely, suppose $1_\mathbf{X} : \mathbf{X} \to \mathbf{X}$ is a uniformly movable J-morphism. Note that for any $\lambda \in \Lambda$ if (λ', j) is a J-uniform movability pair of λ with respect to $1_\mathbf{X}$, then λ' is a uniform movability index of λ for $\mathbf{X}$. $\square$

Example 3.2. Let (X) be a rudimentary system in the category $\mathcal{C}$. It is easy to see that any J-morphism of inverse systems $(f_\mu^j) : (X) \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ is J-uniformly movable.

More generally, if we consider a J-morphism $(f_\mu^j, \phi) : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ such that there exists $\lambda_M \in \Lambda$ satisfying $\lambda_M \geq \phi(\mu)$ for any $\mu \in M$, then (f_μ^j, ϕ) is J-uniformly movable. Indeed, it is not difficult to verify that for any index $\mu \in M$, a J-uniformly movable pair is (λ_M, j), for an arbitrary $j \in J$.

Theorem 3.3. *Let* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$*,* $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$*,* $\mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ *be inverse systems in the category* $\mathcal{C}$ *and* $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$*,* $(g_\nu^j, \psi) : \mathbf{Y} \to \mathbf{Z}$ *morphisms of inverse systems. If* (g_ν^j, ψ) *is* J*-uniformly movable, then the composition* $(h_\nu^j, \chi) = (g_\nu^j, \psi) \circ (f_\mu^j, \phi)$ *is also a* J*-uniformly movable morphism.*

Proof. We use the notations from the proof of Theorem 3.1 replacing $r_{\nu\nu'} : Z_{\nu'} \to Z_\nu$ by $\mathbf{r}_\nu : \mathbf{Z} \to Z_\nu$ and $u^{j'} : Y_\mu \to Z_{\nu'}$ by $\mathbf{u}^{j'} : Y_\mu \to \mathbf{Z}$. Then we have $g_\nu^{j'} \circ q_{\psi(\nu)\mu} = \mathbf{r}_\nu^{j'} \circ \mathbf{u}^{j'}$. And by defining $\mathbf{u'}^{j'} = \mathbf{u}^{j'} \circ f_{\mu\lambda}^{j'} : X_\lambda \to \mathbf{Z}$, we obtain $\mathbf{r}_\nu \circ \mathbf{u'}^{j'} = h_{\nu\lambda}^{j'}$. $\square$

Corollary 3.4. *Let* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *be an arbitrary inverse system and let* $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ *be a uniformly movable inverse system. Then any* J*-morphism* $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$ *is* J*-uniformly movable.*

Proof. Since $(f_\mu^j, \phi) = 1_\mathbf{Y}^J \circ (f_\mu^j, \phi)$ and $1_\mathbf{Y}^J : \mathbf{Y} \to \mathbf{Y}$ is J-uniformly movable by Proposition 3.2, then (f_μ^j, ϕ) is also uniformly movable according to Theorem 3.3.

Theorem 3.4. *Let* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$*,* $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$*,* $\mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ *be inverse systems in the category* $\mathcal{C}$ *and let* $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$*,* $(g_\nu^j, \psi) : \mathbf{Y} \to \mathbf{Z}$ *be morphisms of inverse systems. Suppose that* (f_μ^j, ϕ) *is* J*-uniformly movable. Then the composition* $(h_\nu^j, \chi) = (g_\nu^j, \psi) \circ (f_\mu^j, \phi)$ *is also a uniformly movable morphism.*

Proof. Using the notations from the proof of Theorem 3.2, there exists $\mathbf{u}^{j'}: X_\lambda \to \mathbf{Y}$, such that $f^{j'}_{\psi(\nu)\lambda} = \mathbf{q}_{\psi(\nu)} \circ \mathbf{u}^{j'}$. Then for $\mathbf{U}^{j'}: X_\lambda \to \mathbf{Z}$, $\mathbf{U}^{j'} = g^{j'}_\nu \circ \mathbf{u}^{j'}$, we have $h^{j'}_{\nu\lambda} = g^{j'}_\nu \circ f^{j'}_{\psi(\nu)\lambda} = \mathbf{r}_\nu \circ \mathbf{U}^{j'}$. □

Corollary 3.5. *Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be uniformly movable inverse system and let $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be an arbitrary inverse system. Then any J-morphism $(f^j_\mu, \phi): \mathbf{X} \to \mathbf{Y}$ is uniformly movable.*

Proof. Since $(f^j_\mu, \phi) = (f^j_\mu, \phi) \circ (1^j_{X_\lambda}, 1_\Lambda)$ and the identity J-morphism $(1^j_{X_\lambda}, 1_\Lambda)$ is uniformly movable by Proposition 3.2, then (f^j_μ, ϕ) is also J-uniformly movable according to Theorem 3.4. □

Corollary 3.6. *Let $\mathbf{X}$ and $\mathbf{Y}$ be inverse systems in the category $\mathcal{C}$. Suppose that $\mathbf{X}$ is uniformly movable and $\mathbf{Y}$ is J-dominated by $\mathbf{X}$. Then $\mathbf{Y}$ is uniformly movable.*

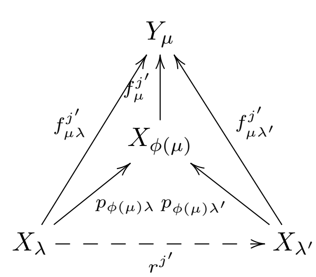
Proof. We use the notations from Corollary 3.3. By hypothesis and Proposition 3.2, $(1^j_{X_\lambda}, 1_\Lambda)$ is J-uniformly movable. Then by the equality $(1^j_{X_\lambda}, 1_\Lambda) \circ (g^j_\nu, \psi) = (g^j_\nu, \psi)$ and by Theorem 3.3 we have that (g^j_ν, ψ) is J-uniformly movable. Hence, by Theorem 3.4 the composition $(f^j_\mu, \phi) \circ (g^j_\lambda, \psi) = (1^j_{Y_\mu}, 1_M)$ is also J-uniformly movable. Finally, using Proposition 3.2 we conclude that $\mathbf{Y}$ is a uniformly movable inverse system. □

4. Co-movability properties for J-morphisms

Definition 4.1. Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in a category $\mathcal{C}$ and let $(f^j_\mu, \phi): \mathbf{X} \to \mathbf{Y}$ be a J-morphism in $\mathcal{C}$. We say that the (f_μ, ϕ) is a *co-movable J-morphism* provided every $\mu \in M$ admits $\lambda \in \Lambda$, $\lambda \geq \phi(\mu)$ and $j \in J$ (the pair (λ, j) being called a *co-movability pair* of μ relative to (f^j_μ, ϕ)) such that each $\lambda' \geq \phi(\mu)$ and $j' \in J$, $j' \geq j$ admit a morphism $r^{j'}: X_\lambda \to X_{\lambda'}$ of $\mathcal{C}$ which satisfies

$$f^{j'}_{\mu\lambda} = f^{j'}_{\mu\lambda'} \circ r^{j'}, \tag{4.1}$$

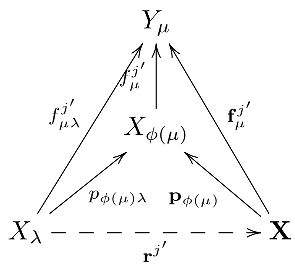
i.e., makes the following outside diagram commutative



Definition 4.2. Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in a category $\mathcal{C}$ and let $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$ be a J-morphism of inverse systems. We say that the (f_μ^j, ϕ) is a *uniformly co-movable J-morphism* provided every $\mu \in M$ admits $\lambda \in \Lambda$, $\lambda \geq \phi(\mu)$ and $j \in J$ (the pair (λ, j) being called a *uniform co-movability pair of* μ relative to (f_μ^j, ϕ)) such that, for $j' \in J$, $j' \geq j$, there is a morphism $\mathbf{r}^{j'} : X_\lambda \to \mathbf{X}$ of inverse systems satisfying

$$f_{\mu\lambda}^{j'} = \mathbf{f}_\mu^{j'} \circ \mathbf{r}^{j'}, \tag{4.2}$$

i.e., makes the following outside diagram commutative



where $\mathbf{f}_\mu^{j'} = f_\mu^{j'} \circ \mathbf{p}_{\phi(\mu)}$.

Remark 4.1. Note that the morphism $\mathbf{r}^{j'} : X_\lambda \to \mathbf{X}$ is given by some morphisms $r_{\lambda'}^{j'} : X_\lambda \to X_{\lambda'}$ such that if $\lambda_1' \leq \lambda_2'$ then $r_{\lambda_1'}^{j'} = p_{\lambda_1'\lambda_2'} \circ r_{\lambda_2'}^{j'}$. The relation $f_{\mu\lambda}^{j'} = \mathbf{f}_\mu^{j'} \circ \mathbf{r}^{j'}$ means $f_{\mu\lambda}^{j'} = f_\mu^{j'} \circ r_{\phi(\mu)}^{j'}$. Therefore, $\lambda' \geq \phi(\mu)$ implies $f_{\mu\lambda}^{j'} = f_\mu^{j'} \circ p_{\phi(\mu)\lambda'} \circ r_{\lambda'}^{j'} = f_{\mu\lambda'}^{j'} \circ r_{\lambda'}^{j'}$. In this way we have that uniform J-co-movability implies J-co-movability.

Remark 4.2. If (λ, j) is a co-movability (uniform co-movability) pair of μ relative to the J-morphism (f_μ^j, ϕ) then so is any pair $(\widetilde{\lambda}, \widetilde{j})$, with $\widetilde{\lambda} \geq \lambda$ and $j \geq j$.

Definition 4.3. A J-morphism $(f_\mu^j, \phi) : ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda) \to ((Y_\mu, *), q_{\mu\mu'}, M)$ of pointed sets is said to have the *Mittag-Leffler property* provided every $\mu \in M$ admits a pair (λ, j), $\lambda \geq \phi(\mu)$, $j \in J$, (an ML pair for μ with respect to (f_μ^j, ϕ)), such that for any $\lambda' \in \Lambda$, with $\lambda' \geq \lambda$, and $j' \geq j$ one has

$$f_{\mu\lambda'}^{j'}(X_{\lambda'}) = f_{\mu\lambda}^{j'}(X_\lambda). \tag{4.3}$$

Note that if $J = \{1\}$ and (f_μ^j, ϕ) is replaced by $1_{(X,*)}$ we obtain the Mittag-Leffler property for an inverse sistems in the category Set_* (cf. [17], Ch. II, §6.2).

Theorem 4.1. *A J-morphism of inverse systems of pointed sets is co-movable if and only if it has the Mittag-Lefler property.*

Proof. Let $(f_\mu^j, \phi) : ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda) \to ((Y_\mu, *), q_{\mu\mu'}, M)$ be a J-morphism with the Mittag-Leffler property. Then for $\mu \in M$ there is a ML pair (λ, j), $\lambda \geq \phi(\mu)$

such that (4.3) holds for each $\lambda' \geq \lambda$ and $j' \geq j$. We can prove that (λ, j) is a co-movability pair of μ with respect to (f^j_μ, ϕ). If $\lambda' \geq \phi(\mu)$ and $\lambda' \geq \lambda$, the relation (4.3) defines a map of pointed sets $r^{j'} : (X_\lambda, *) \to (X_{\lambda'}, *)$ such that $f^{j'}_{\mu\lambda'} \circ r^{j'} = f^{j'}_{\mu\lambda}$. For any other $\lambda'' \geq \phi(\mu)$, one choose $\lambda''' \geq \lambda''$, $\phi(\mu)$ and consider $r'^{j'} : X_\lambda \to X_{\lambda'''}$ such that $f^{j'}_{\mu\lambda'''} \circ r'^{j'} = f^{j'}_{\mu\lambda}$. Then the composition $r^{j'} := p_{\lambda''\lambda'''} \circ r'^{j'}$ satisfies the relation $f^{j'}_{\mu\lambda''} \circ r^{j'} = f^{j'}_{\mu\lambda''} \circ p_{\lambda''\lambda'''} \circ r'^{j'} = f^{j'}_{\mu\lambda'''} \circ r'^{j'} = f^{j'}_{\mu\lambda}$.

Conversely, let (f^j_μ, ϕ) be a co-movable J-morphism. Let $\mu \in M$ and $\lambda \in \Lambda$ with $\lambda \geq \phi(\mu)$ and $j \in J$ a co-movability pair of μ with respect to (f^j_μ, ϕ). Then, for $\lambda' \geq \lambda$ and $j' \geq j$ there exists $r^{j'} : (X_\lambda, *) \to (X_{\lambda'}, *)$ such that $f^{j'}_{\mu\lambda'} \circ r^{j'} = f^{j'}_{\mu\lambda}$. This implies the inclusion $f^{j'}_{\mu\lambda}(X_\lambda) \subseteq f^{j'}_{\mu\lambda'}(X_{\lambda'})$. The converse inclusion follows from the relation $f^{j'}_{\mu\lambda} \circ p_{\lambda\lambda'} = f^{j'}_{\mu\lambda'}$. □

Proposition 4.1. *Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in a the category $\mathcal{C}$ and let $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ be a J-morphism of inverse systems. If $\mathbf{X}$ is a movable (uniformly movable) inverse system and $\mathbf{Y}$ is an arbitrary inverse system, then (f^j_μ, ϕ) is a co-movable (uniformly co-movable) J-morphism.*

Proof. It is easy to prove that if $\mu \in M$ and $\lambda \in \Lambda$ is a movability (uniform movability) index for $\phi(\mu)$, then a pair (λ, j) with an arbitrary $j \in J$ is a co-movability (uniform co-movability) pair for μ with respect to the J-morphism (f^j_μ, ϕ). □

Theorem 4.2. *An inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is movable (uniformly movable) if and only if the identity J-morphism $1^J_{\mathbf{X}}$ is co-movable (uniformly co-movable) for an arbitrary directed partially ordered set J.*

Proof. If X is movable (uniformly movable), then by Proposition 4.1 the morphism $1^J_{\mathbf{X}}$ is co-movable (uniformly co-movable). Conversely, let $1^J_{\mathbf{X}}$ be a co-movable (uniformly co-movable) J-morphism and let (λ', j) be a co-movability (uniform co-movability) pair of a given $\lambda \in \Lambda$ with respect to $1^J_{\mathbf{X}} = (1_{X_\lambda}, 1_\Lambda)$. It is easy to verify that λ' is a movability (uniform movability) index of λ for the inverse system $\mathbf{X}$. □

Using Theorems 4.1, 4.2 and Proposition 4.1, we obtain the following corollary (see [17], Ch. II, §6.2, Corollary 4).

Corollary 4.1. *An inverse system of pointed set $(\mathbf{X}, *)$ is movable if and only if it has the Mittag-Leffler property, in particular, if all bonding functions are surjective.*

The following theorem is a generalization of Proposition 4.1.

Theorem 4.3. *Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$, $\mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ be inverse systems in the category $\mathcal{C}$ and let $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$, $(g^j_\nu, \psi) : \mathbf{Y} \to \mathbf{Z}$ be*

J-morphisms. Suppose that (f_μ^j, ϕ) is co-movable (uniformly co-movable). Then the composition $(h_\nu^j, \chi) = (g_\nu^j, \psi) \circ (f_\mu^j, \phi)$ is also a co-movable (uniformly co-movable) J-morphism.

Proof. At first we note that $h_{\nu\lambda}^j = g_\nu^j \circ f_{\psi(\nu)\lambda}^j$. Then if (λ, j) is a co-movability pair for $\psi(\nu)$, we have $f_{\psi(\nu)\lambda}^{j'} = f_{\psi(\nu)\lambda'}^{j'} \circ r^{j'}$, for $\lambda' \geq \lambda$ and $j' \geq j$. By this we have $h_{\nu\lambda}^{j'} = g_\nu^{j'} \circ f_{\psi(\nu)\lambda'}^{j'} \circ r^{j'} = h_{\nu\lambda'}^{j'} \circ r^{j'}$, which is the condition of co-movability for J-morphism (h_ν^j, χ). For the property of uniform co-movability the proof is similar. □

Remark 4.3. The assertion of Corollary 3.1 in the case of co-movability of J-morphisms is false even if $J = \{1\}$. To show this, consider the following inverse sequences of groups:

$$\mathbf{G} = (G_n, p_{nn'}), \text{where } G_n = \mathbb{Z} \text{ and } p_{nn'}(m) = 2^{n'-n}m;$$

$$\mathbf{H} = (H_n, q_{nn'}), \text{where } H_n = \oplus^n \mathbb{Z} = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \text{ and}$$
$$q_{nn+1}(m_1, \ldots, m_{n'}) = (m_1, \ldots, m_{n'-n})$$

The pro-group $\mathbf{H}$ is movable (see [17], Ch.II, §6.1, Example 2).

Now consider the following morphism $(f_n, 1_\mathbb{N}) : \mathbf{G} \to \mathbf{H}$ with

$$f_n : G_n \to H_n, \quad f_n(m) = (2^{n-1}m, 2^{n-2}m, \ldots, 2m, m).$$

We can verify that in this way we obtain a level morphism of pro-groups. Indeed,

$$(f_n \circ p_{nn'})(m) = f_n(2^{n'-n}m) = (2^{n-1}2^{n'-n}m, 2^{n-2}2^{n'-n}m, \ldots, 2 \cdot 2^{n'-n}m, 2^{n'-n}m) =$$
$$= (2^{n'-1}m, 2^{n'-2}m, \ldots, 2^{n'-n+1}m, 2^{n'-n}m)$$

and

$$(q_{nn'} \circ f_{n'})(m) = q_{nn'}(2^{n'-1}m, 2^{n'-2}m, \ldots, 2m, m) = (2^{n'-1}m, 2^{n'-2}m, \ldots, 2^{n'-n}m).$$

So $f_n \circ p_{nn'} = q_{nn'} \circ f_{n'}$ and hence, $(f_n, 1_\mathbb{N}) : \mathbf{G} \to \mathbf{H}$ is a level morhism of pro-groups.

Now the condition of co-movability (4.1) for the morphism $(f_n, 1_\mathbb{N})$ becomes

$$f_{nn'} = f_{nn''} \circ r \quad \text{or} \quad f_n \circ p_{nn'} = f_n \circ p_{nn''} \circ r.$$

Consider the last relation written for $n = 1$ and $n'' = n' + 1$:

$$(f_1 \circ p_{1n'})(m) = (f_1 \circ p_{1n'+1} \circ r)(m) \Leftrightarrow 2^{n'-1}m = 2^{n'}r(m) \Leftrightarrow m = 2 \cdot r(m) \Leftrightarrow r(m) = \frac{m}{2}$$

for any $m \in \mathbb{Z}$, which is impossible because r is an endomorphism of $\mathbf{Z}$. Thus, the morphism $(f_n, 1_\mathbb{N})$ is not co-movable although $\mathbf{H}$ is movable.

In addition, by Proposition 4.1, we conclude that $\mathbf{G}$ is not movable (the result also proved in [17], Ch.II, §6.1).

Remark 4.4. Since Corollary 3.1 is a consequence of Theorem 3.1, Remark 4.3 suggests that a result for the properties of co-movability and uniform co-movability analogous to that from Theorem 3.1 for movability and uniform movability is false. But imposing for (f_μ^j, ϕ) to be a J-isomorphism, we obtain a positive result.

Theorem 4.4. *Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$, $\mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ be inverse systems in a category $\mathcal{C}$. Let $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$ be a J-isomorphism and let $(g_\nu^j, \psi) : \mathbf{Y} \to \mathbf{Z}$ be a (uniformly) co-movable J-morphism. Then the composition $(h_\nu^j, \chi) = (g_\nu^j, \psi) \circ (f_\mu^j, \phi)$ is also a (uniformly) co-movable J-morphism.*

Proof. Without loss of generality, we can assume that $\phi : \Lambda \to M$ is an increasing function [17] (Ch.I, §1.2, Lemma 2). Since $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$ be a J-isomorphism we can also assume that ϕ is a bijection. Let $({f'}_\lambda^j, \phi') : \mathbf{Y} \to \mathbf{X}$, where $\phi' = \phi^{-1}$, be an inverse J-morphism of (f_μ^j, ϕ).
Now suppose that (g_ν^j, ψ) is a co-movable J-morhiasm. To prove that the composition $(h_\nu^j, \chi) = (g_\nu^j, \psi) \circ (f_\mu^j, \phi)$ is also a co-movable J-morphism, consider an arbitrary $\nu \in N$ and take a co-movability pair (μ, j) of ν with respect to J-morphism (g_ν^j, ψ).

Let's prove that $(\phi(\mu), j)$ is a co-movability pair of ν with respect to J-morphism (h_ν^j, χ). Consider any $\lambda' \geq \chi(\nu)$, $\chi(\nu) = \phi(\psi(\nu))$. Note that $\phi'(\lambda') \geq \psi(\nu)$ because ϕ' is an increasing function. Hence, for any $j' \geq j$ there exists a morphism $r^{j'} : Y_\mu \to Y_{\phi'(\lambda')}$ in the category $\mathcal{C}$ satisfying the relation

$$g_{\nu\mu}^{j'} = g_{\nu\phi'(\lambda')}^{j'} \circ r^{j'}, \text{ i.e., } g_\nu^{j'} \circ q_{\psi(\nu)\mu} = g_\nu^{j'} \circ q_{\psi(\nu)\phi'(\lambda')} \circ r^{j'}. \tag{4.4}$$

Now define the morphism $R^{j'} : X_{\phi(\mu)} \to X_{\lambda'}$ by

$$R^{j'} = {f'}_{\lambda'}^{j'} \circ r^{j'} \circ f_\mu^{j'} \tag{4.5}$$

and prove that $h_{\nu\phi(\mu)}^{j'} = h_{\nu\lambda'}^{j'} \circ R^{j'}$, i.e.,

$$g_\nu^{j'} \circ f_{\psi(\nu)}^{j'} \circ p_{\phi(\psi(\nu))\phi(\mu)} = g_\nu^{j'} \circ f_{\psi(\nu)}^{j'} \circ p_{\phi(\psi(\nu))\lambda'} \circ R^{j'}. \tag{4.6}$$

Indeed, by (4.4) and (4.5), one has

$$g_\nu^{j'} \circ f_{\psi(\nu)}^{j'} \circ p_{\phi(\psi(\nu))\lambda'} \circ R^{j'} = g_\nu^{j'} \circ \left(f_{\psi(\nu)}^{j'} \circ p_{\phi(\psi(\nu))\lambda'}\right) \circ {f'}_{\lambda'}^{j'} \circ r^{j'} \circ f_\mu^{j'} =$$
$$= g_\nu^{j'} \circ \left(q_{\psi(\nu)\phi'(\lambda')} \circ f_{\phi'(\lambda')}^{j'}\right) \circ {f'}_{\lambda'}^{j'} \circ r^{j'} \circ f_\mu^{j'} = g_\nu^{j'} \circ q_{\psi(\nu)\phi'(\lambda')} \circ 1_{Y_{\phi'(\lambda')}} \circ r^{j'} \circ f_\mu^{j'} =$$
$$= g_\nu^{j'} \circ q_{\psi(\nu)\mu} \circ f_\mu^{j'} = g_\nu^{j'} \circ f_{\psi(\nu)}^{j'} \circ p_{\phi(\psi(\nu))\phi(\mu)}.$$

So, the composition $(h_\nu^j, \chi) = (g_\nu^j, \psi) \circ (f_\mu^j, \phi)$ is co-movable. In the same way one can prove the case of uniform co-movability. □

5. Properties of movability and co-movability for J-pro-morphisms

Proposition 5.1. *Let* $(f_\mu^j, \phi), ({f'}_\mu^j, \phi') : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ *be two equivalent* J*-morphisms of inverse systems.*

(i) If the J*-morphism* (f_μ^j, ϕ) *is movable (uniformly movable) then the* J*-morphism* $({f'}_\mu^j, \phi')$ *is also movable (uniformly movable).*

(ii) If the J*-morphism* (f_μ^j, ϕ) *is co-movable (uniformly co-movable) then the* J*-morphism* $({f'}_\mu^j, \phi')$ *is also co-movable (uniformly co-movable).*

Proof. (i) Suppose that (f_μ^j, ϕ) is J-movable and $(f_\mu^j, \phi) \sim ({f'}_\mu^j, \phi')$. We need to prove that $(f_\mu^{\prime j}, \phi')$ is also J-movable.

Let $\mu \in M$ be any index. Consider a movability pair (λ, j) of μ with respect to J-morphism (f_μ^j, ϕ). There is no loss of generality in assuming that $\lambda \geq \phi(\mu), \phi'(\mu)$ and

$$f_\mu^j \circ p_{\phi(\mu)\lambda} = f_\mu^{\prime j} \circ p_{\phi'(\mu)\lambda}. \tag{5.1}$$

Consider any $\mu' \geq \mu$. By assumption for any $j' \geq j$ there is a morphism $u^{j'}$: $X_\lambda \to Y_{\mu'}$ such that

$$f_{\mu\lambda}^{j'} = q_{\mu\mu'} \circ u^{j'}. \tag{5.2}$$

Then by (5.1) and (5.2) we have

$$f_{\mu\lambda}^{\prime j'} = f_\mu^{\prime j'} \circ p_{\phi'(\mu)\lambda} = f_\mu^{j'} \circ p_{\phi(\mu)\lambda} = f_{\mu\lambda}^{j'} = q_{\mu\mu'} \circ u^{j'}$$

which means that (λ, j) is also movability pair of μ with respect to J-morphism $(f_\mu^{\prime j}, \phi')$.

The case of uniform movability is proved similarly.

(ii) Let $\mu \in M$ be any index and let (λ, j) be a co-movability pair for μ with respect to J-morphism (f_μ^j, ϕ). We can assume that $\lambda \geq \phi(\mu), \phi'(\mu)$ and (5.1) holds.

Now we prove that (λ, j) is also a co-movability pair of μ with respect to $(f_\mu^{\prime j}, \phi')$. Let $\lambda' \geq \phi'(\mu)$ be any index and let λ'' be an index with $\lambda'' \geq \lambda', \phi(\mu)$ which satisfies

$$f_\mu^{j'} \circ p_{\phi(\mu)\lambda''} = f_\mu^{\prime j'} \circ p_{\phi'(\mu)\lambda''} \tag{5.3}$$

for given $j' \geq j$. By assumption there is a morphism $r^{j'} : X_\lambda \to X_{\lambda''}$ such that

$$f_{\mu\lambda}^{j'} = f_{\mu\lambda''}^{j'} \circ r^{j'}. \tag{5.4}$$

Define the morphism $R^{j'} : X_\lambda \to X_{\lambda'}$ by

$$R^{j'} = p_{\lambda'\lambda''} \circ r^{j'}. \tag{5.5}$$

By (5.1), (5.3), (5.4), and (5.5) one has

$$f'^{j'}_{\mu\lambda'} \circ R^{j'} = f'^{j'}_{\mu\lambda'} \circ p_{\lambda'\lambda''} \circ r^{j'} = f'^{j'}_{\mu} \circ p_{\phi'(\mu)\lambda'} \circ p_{\lambda'\lambda''} \circ r^{j'} =$$
$$= f^{j'}_{\mu} \circ p_{\phi(\mu)\lambda''} \circ r^{j'} = f^{j'}_{\mu} \circ p_{\phi(\mu)\lambda} = f'^{j'}_{\mu} \circ p_{\phi'(\mu)\lambda} = f'^{j'}_{\mu\lambda},$$

which is the condition for co-movability for the J-morphism $({f'}^{j}_{\mu}, \phi')$. The case of uniform movability can be proved similarly. □

Thanks to Proposition 5.1, we can give the following definition.

Definition 5.1. (i) A J-pro-morphism $\mathbf{f}^J : \mathbf{X} \to \mathbf{Y}$ is called movable (uniformly movable) if $\mathbf{f}^J$ admits a representation (f^j_μ, ϕ) which is J-movable (uniformly J-movable).

(ii) A J-pro-morphism $\mathbf{f}^J : \mathbf{X} \to \mathbf{Y}$ is called co- movable (uniformly co-movable) if $\mathbf{f}^J$ admits a representation (f^j_μ, ϕ) which is J-co-movable (uniformly J-co-movable).

The next theorem follows from Theorems 3.1, 3.2, 3.3, 3.4 and Corollaries 3.1, 3.2, 3.4, 3.5.

Theorem 5.1. *A (pre- or post-) composition of an arbitrary J-pro-morphism with a movable (uniformly movable) J-pro-morphism is a movable (uniformly movable) J-pro-morphism. In particular, if $\mathbf{X}$ or $\mathbf{Y}$ is a movable (uniformly movable) inverse system, then $\mathbf{f}^J : \mathbf{X} \to \mathbf{Y}$ is a movable (uniformly movable) J-pro-morphism.*

Taking into account Corollaries 3.3 and 3.6, we obtain

Proposition 5.2. *Let $\mathbf{X}$ and $\mathbf{Y}$ be J-pro-morphisms. If $\mathbf{Y}$ is a movable (uniformly movable) and $\mathbf{X}$ is dominated by $\mathbf{Y}$ in pro^J-$\mathcal{C}$, then $\mathbf{X}$ is also movable (uniformly movable).*

The following theorem is an immediate consequence of Theorem 4.3

Theorem 5.2. *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ be inverse systems in the category $\mathcal{C}$ and let $\mathbf{f}^J : \mathbf{X} \to \mathbf{Y}$, $\mathbf{g}^J : \mathbf{Y} \to \mathbf{Z}$ be J-pro-morphisms in pro^J-$\mathcal{C}$. If $\mathbf{f}^J$ is a co-movable (uniformly co-movable) J-pro-morphism, then the composition $\mathbf{h}^J = \mathbf{g}^J \circ \mathbf{f}^J$ is also a co-movable (uniformly co-movable) J-pro-morphism.*

Remark 5.1. It follows from the example from Remark 4.3 that if $\mathbf{g}^J$ is a co-movable J-pro-morphism and $\mathbf{f}^J$ is an arbitrary J-pro-morphism, then the composition $\mathbf{h}^J = \mathbf{g}^J \circ \mathbf{f}^J$ is not necessarily a co-movable J-pro-morphism.

However, the following theorem is true (follows from Theorem 4.4).

Theorem 5.3. *Let* $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ *be inverse systems in the category* $\mathcal{C}$ *and let* $\mathbf{f}^J : \mathbf{X} \to \mathbf{Y}$, $\mathbf{g}^J : \mathbf{Y} \to \mathbf{Z}$ *be* J*-pro-morphisms in* pro^J*-*$\mathcal{C}$*. If* $\mathbf{g}^J$ *is a co-movable (uniformly co-movable)* J*-pro-morphism and* $\mathbf{f}^J$ *is a* J*-pro-isomorphism, then the composition* $\mathbf{h}^J = \mathbf{g}^J \circ \mathbf{f}^J$ *is a co-movable (uniformly co-movable)* J*-pro-morphism.*

6. Properties of movability and co-movability for J-shape morphisms

Consider $(\mathcal{C}, \mathcal{D})$ a pair of categories with $\mathcal{D}$ a dense subcategory of $\mathcal{C}$. If $X, Y \in Ob\mathcal{C}$ and $p : X \to \mathbf{X}$, $q : Y \to \mathbf{Y}$ are $\mathcal{D}$-expansions, then by Remark 2.4 and Definitions 2.3 and 2.4, a J-shape morphism from X to Y is an equivalence class $\langle \mathbf{f} \rangle$ of a J-pro-morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$.

Theorem 6.1. *In the above conditions, let* $p' : X \to \mathbf{X}'$ *and* $q' : Y \to \mathbf{Y}'$ *be other* $\mathcal{D}$*-expansions of* X *and* Y*, respectively. If the* J*-pro-morphisms* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, $\mathbf{f}' : \mathbf{X}' \to \mathbf{Y}'$ *define the same* J*-shape morphism* $F : X \to Y$ *and if* $\mathbf{f}$ *is a movable (uniformly movable)* J*-pro-morphism, then* $\mathbf{f}'$ *is the same.*

Proof. By Definition 2.3 there exists a commutative diagram

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\mathbf{i}} & \mathbf{X}' \\ {\scriptstyle \mathbf{f}}\downarrow & & \downarrow{\scriptstyle \mathbf{f}'} \\ \mathbf{Y} & \xrightarrow[\mathbf{j}]{} & \mathbf{Y}' \end{array}$$

where $\mathbf{i}$ and $\mathbf{j}$ are J-pro-isomorphisms. If $\mathbf{f}$ is a movable (uniformly movable), then by Theorem 5.1 the composition $\mathbf{j} \circ \mathbf{f}$ is J-movable (uniformly J-movable). Therefore, by the same theorem, $\mathbf{f}' = (\mathbf{j} \circ \mathbf{f}) \circ \mathbf{i}'^{-1}$ is J-movable (uniformly J-movable). □

Definition 6.1. A J-shape morphism $F : X \to Y$ is called movable (uniformly movable) if it can be represented by a movable (uniformly movable) J-pro-morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, $F = \langle \mathbf{f} \rangle$.

Theorem 6.2. *With the notation from Theorem 6.1, if* $\mathbf{f}$ *is a co-movable (uniformly co-movable)* J*-pro-morphism, then* $\mathbf{f}'$ *is the same.*

Proof. As above we have $\mathbf{j} \circ \mathbf{f} = \mathbf{f}' \circ \mathbf{i}$. If $\mathbf{f}$ is co-movable (uniformly co-movable), then by Theorem 5.2, the composition $\mathbf{j} \circ \mathbf{f}$ is co-movable (uniformly co-movable). Then $\mathbf{f}' = (\mathbf{j} \circ \mathbf{f}) \circ \mathbf{i}^{-1}$ is co-movable (uniformly co-movable) by Theorem 5.3. □

Definition 6.2. A J-shape morphism $F : X \to Y$ is called co-movable (uniformly co-movable) if it can be represented by a co-movable (uniformly co-movable) J-pro-morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, $F = \langle \mathbf{f} \rangle$.

Remark 6.1. All properties of movability (uniform movability) and co-movability (uniform co-movability) of J-morphisms and J-pro-morphisms of inverse systems can be transferred to appropriate properties for J-shape morphisms and for morphisms in the category $\mathcal{C}$ of a shape theory $Sh^J_{(\mathcal{C},\mathcal{D})}$. For example, by Theorems 5.1, 5.2, and 5.3 we obtain the following theorem.

Theorem 6.3. *(i) A (pre-or post-) composition of an arbitrary J-shape morphism with a movable (uniformly movable) J-shape morphism is a movable (uniformly movable) J-shape morphism. In particular, if X or Y is a movable (uniformly movable) object, then any J-shape morphism $F : X \to Y$ is movable (uniformly movable);*

(ii) Let $F : X \to Y$, $G : Y \to Z$ be J-shape morphisms in the J-shape category $Sh^J_{(\mathcal{C},\mathcal{D})}$. If F is co-movable (uniformly co-movable), then the composition $H = G \circ F$ also is co-moavble (uniformly co-movable).

(iii) If $F : X \to Y$ is a J-shape isomorphism and $G : Y \to Z$ is a co-movable (uniformly co-movable) J-shape morphism, then $H = G \circ F$ is a co-movable (uniformly co-movable) J-shape morphism.

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ON THE EUCLIDEAN DISTANCE BETWEEN TWO GAUSSIAN POINTS AND THE NORMAL COVARIOGRAM OF $\mathbb{R}^d$

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Abstract. The concept of covariogram is extended from bounded convex bodies in $\mathbb{R}^d$ to the entire space $\mathbb{R}^d$ by obtaining integral representations for the distribution and density functions of the Euclidean distance between two d-dimensional Gaussian points that have correlated coordinates governed by a covariance matrix. When $d = 2$, a closed-form expression for the density function is obtained. Precise bounds for the moments of the considered distance are found in terms of the extreme eigenvalues of the covariance matrix.

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1. Introduction

Consider two fundamental characteristics of a bounded body $\mathbb{D} \subset \mathbb{R}^d$. Let the first be the covariogram of $\mathbb{D}$ which has a geometric nature: for any vector $\boldsymbol{t} \in \mathbb{R}^d$, it represents the d-dimensional Lebesgue measure of the region shared between $\mathbb{D}$ and its translated copy by vector $\boldsymbol{t}$. We denote the covariogram of $\mathbb{D}$ by $C_{\mathbb{D}}(\boldsymbol{t})$.

Let the second characteristic be the Euclidean distance between two random points chosen independently and uniformly from $\mathbb{D} \subset \mathbb{R}^d$. This is a well-known random variable studied in geometric probability (see, for example [1]). We denote it by $D_d(\mathbb{D})$. Extensive research has been conducted on this random variable for various bounded bodies $\mathbb{D}$, including computation of the average distance within a cube [2], on the surface of a cube [3], within a hyperball [4], as well as bounding the average distance within a hypercube [5] or furthermore, within compact subsets of $\mathbb{R}^d$ with unit diameter [4]. In dimensions $d \le 3$, closed-form expressions are obtained for the density function of $D_d(\mathbb{D})$ in [6]-[11] for numerous geometric shapes of $\mathbb{D}$. A unified approach for determining the density function of $D_d(\mathbb{D})$ for typical compact sets is suggested in [12]. It also provides a good list of references for related results of theoretical and applied character.

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When $\mathbb{D}$ is a bounded convex body with a non-empty interior in $\mathbb{R}^d$, then the two considered characteristics of $\mathbb{D}$ are interrelated as follows:

$$f_{D_d(\mathbb{D})}(h) = \frac{h^{d-1}}{L_d^2(\mathbb{D})} \int_{S_{d-1}} C_{\mathbb{D}}(h\boldsymbol{u})d\boldsymbol{u}, \ h > 0, \tag{1.1}$$

where S_{d-1} is the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$, centered at the origin, and $L_d(\mathbb{D})$ is Lebesgue d-measure of $\mathbb{D}$.

In this paper, we aim to extend the concepts of covariogram $C_{\mathbb{D}}(t)$ and interpoint distance $D_d(\mathbb{D})$ from bounded convex bodies to the entire space $\mathbb{R}^d$ and establish a relation between them.

The first problem that arises in our way is the nature of randomness of choosing a point from $\mathbb{D} = \mathbb{R}^d$. The uniform distribution is no longer applicable to this case and therefore we will naturally replace it with a multivariate normal distribution.

The second obstacle lies in the challenge of applying the language and sense of geometry to define the covariogram of $\mathbb{R}^d$. We will define it analytically based on the following observation. If $\mathbb{D}$ is a convex body and $\mathcal{P}_1$, $\mathcal{P}_2$ are chosen uniformly and independently from $\mathbb{D}$, then it is easy to check (see, for example, [11]) that

$$f_{\mathcal{P}_1-\mathcal{P}_2}(\boldsymbol{t}) = \frac{C_{\mathbb{D}}(\boldsymbol{t})}{L_d^2(\mathbb{D})},$$

which can be equivalently written as

$$f_{\mathcal{P}_1-\mathcal{P}_2}(\boldsymbol{t}) = \frac{C_{\mathbb{D}}(\boldsymbol{t})}{C_{\mathbb{D}}^2(\boldsymbol{0})}. \tag{1.2}$$

Thus, the covariogram should be a positive function defined on the entire space that satisfies (1.2).

We have defined the normal covariogram of $\mathbb{R}^d$ and established an analogous relationship to (1.1) in section 4, with the foundational basis of the proof presented in the preceding section. Notably, section 3 unveils novel findings, including integral representations for the distribution and density functions of the Euclidean distance between two d-dimensional Gaussian points, characterized by correlated coordinates through a covariance matrix. Precise bounds for the moments of the considered distance in terms of the extreme eigenvalues of the covariance matrix are found. When $d = 2$, an expression for the density function in terms of a modified Bessel function is obtained. In section 2, we independently address the scenario of uncorrelated coordinates and deduce the density and moments of the interpoint distance, drawing upon the results by Mathai and Provost [13].

In the upcoming text, a d-dimensional vector $\mathbf{v} \in \mathbb{R}^d$ will be assumed to be a column vector, or, equivalently, a $d \times 1$ matrix. The transpose of matrix $\mathbf{A}$ will be denoted by $\mathbf{A}^T$. $\mathbf{0}$ will stand for the vector with all zero coordinates, $\mathbf{1}$ for the vector whose all coordinates are equal to 1. $\mathbf{I}_d$ will represent the identity $d \times d$ matrix, $\| \cdot \|_d$ the Euclidean norm in $\mathbb{R}^d$, and $|\mathbf{A}|$ the determinant of matrix $\mathbf{A}$.

If $\mathbf{X}$ is a d-variate normal random vector having mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$ then we denote this condition by $\boldsymbol{X} \sim N_d(\boldsymbol{\mu}, \mathbf{\Sigma})$. We denote $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, ..., \lambda_d]^T$, where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d > 0$ are the eigenvalues of $\mathbf{\Sigma}$.

From now onwards, we assume $\boldsymbol{\mu} = \mathbf{0}$ and the diagonal of $\mathbf{\Sigma}$ consisting of 1s. If $\boldsymbol{X_1}, \boldsymbol{X_2} \sim N_d(\mathbf{0}, \mathbf{\Sigma})$ are independent, we denote

$$D_d = \|\boldsymbol{X_1} - \boldsymbol{X_2}\|_d.$$

2. The density of D_d

Let $\boldsymbol{U} = \boldsymbol{X_1} - \boldsymbol{X_2}$. Since $\boldsymbol{U} \sim N_d(\mathbf{0}, 2\mathbf{\Sigma})$ and $D_d^2 = \boldsymbol{U}^T\boldsymbol{U}$, then the distribution function of D_d^2 can be written in the following form (see [13], page 95):

Theorem 2.1.

$$\mathbb{P}(D_d^2 \leq y) \stackrel{def}{=} F_{D_d^2}(\mathbf{\Sigma}, y) = \sum_{k=0}^{\infty} (-1)^k c_k \frac{y^{\frac{d}{2}+k}}{\Gamma\left(\frac{d}{2}+k+1\right)}, \; y > 0, \tag{2.1}$$

where

$$c_0 = \frac{1}{2^d \sqrt{|\mathbf{\Sigma}|}}, \; c_k = \frac{1}{k} \sum_{r=0}^{k-1} \delta_{k-r} c_r, \; k \geq 1, \tag{2.2}$$

$$\delta_k = \frac{1}{2^{2k+1}} \sum_{i=1}^{d} \frac{1}{\lambda_i^k}, \tag{2.3}$$

and Γ *is the Gamma function*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0.$$

When the coordinates of the Gaussian points are uncorrelated univariate standard normal variables, then $\mathbf{\Sigma} = \boldsymbol{I}_d$ is the identity $d \times d$ matrix and, consequently, $\boldsymbol{\lambda} = \mathbf{1}$. In this case, one can obtain from Theorem 2.1 that D_d follows $GG(a, d, p)$, a generalized Gamma distribution, introduced by E. W. Stacy [14], which has the probability density function

$$f(x; a, d, p) = \frac{\left(p/a^d\right) x^{d-1} e^{-(x/a)^p}}{\Gamma(d/p)}, \; x > 0,$$

where $d > 0$ and $p > 0$ are the shape parameters, and a is a scale parameter. The result is formulated below.

Let $f_{D_d}(\mathbf{\Sigma}, \cdot)$ be the density function of D_d.

Theorem 2.2.

$$f_{D_d}(\boldsymbol{I}_d,\, R) = \frac{R^{d-1}e^{-\frac{R^2}{4}}}{2^{d-1}\Gamma\left(\frac{d}{2}\right)},\ R > 0, \tag{2.4}$$

that is, if $\boldsymbol{\Sigma} = \boldsymbol{I}_d$ *then* $D_d \sim GG(2, d, 2)$.

Proof. Since $\boldsymbol{\lambda} = \mathbf{1}$, (2.3) and (2.2) imply $c_0 = 2^{-d}$ and

$$c_k = \frac{d}{k2^{2k+1}} \sum_{r=0}^{k-1} 4^r c_r,\ k \geq 1.$$

It is easy to verify by mathematical induction that

$$c_k = \frac{1}{2^d k! 4^k} \prod_{j=0}^{k-1} \left(\frac{d}{2} + j\right),\ k \geq 1,$$

which, based on the identity $x\Gamma(x) = \Gamma(x+1)$, $x > 0$, can be rewritten as

$$c_k = \frac{\Gamma(\frac{d}{2} + k)}{2^d k! 4^k \Gamma(\frac{d}{2})},\ k \geq 1. \tag{2.5}$$

By substituting (2.5) in (2.1) and using $f_{D_d}(\boldsymbol{I}_d,\, R) = 2R\frac{\partial}{\partial R} F_{D_d^2}(\boldsymbol{I}_d,\, R^2)$, we obtain

$$f_{D_d}(\boldsymbol{I}_d,\, R) = 2R \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\frac{d}{2}+k)(R^2)^{\frac{d}{2}+k-1}(\frac{d}{2}+k)}{2^d k! 4^k \Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+k+1)} =$$

$$= 2R \cdot \frac{R^{d-2}}{2^d \Gamma(\frac{d}{2})} \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{R^2}{4}\right)^k = \frac{R^{d-1}e^{-\frac{R^2}{4}}}{2^{d-1}\Gamma\left(\frac{d}{2}\right)}.$$

□

The moments of the generalized Gamma distribution are well known. If $X \sim GG(a, d, p)$, then (see, for example [15], section 17.8.7)

$$\mathbb{E}\left(X^r\right) = a^r \frac{\Gamma\left(\frac{d+r}{p}\right)}{\Gamma\left(\frac{d}{p}\right)},\ r = 0, 1, 2, \ldots\ .$$

As a result, from Theorem 2.2 we immediately obtain the corresponding formula for the moments of D_d.

Corollary 2.1. *If* $\boldsymbol{\Sigma} = \boldsymbol{I}_d$*, then*

$$\mathbb{E}\left(D_d^r\right) = 2^r \frac{\Gamma\left(\frac{d+r}{2}\right)}{\Gamma\left(\frac{d}{2}\right)},\ r = 0, 1, 2, \ldots\ . \tag{2.6}$$

In general, when $\boldsymbol{\Sigma} \neq \boldsymbol{I}_d$, even when $d = 2$, it is hard to compute the coefficients c_k from the recursive formulas (2.2) and evaluate the infinite sum (2.1).

3. An integral representation of the distribution function of D_d

As usual, we denote by $F_{D_d}(\boldsymbol{\Sigma}, \cdot)$ the distribution function of D_d.

Theorem 3.1. *Let $\mathcal{E}_d(\boldsymbol{\lambda}, R)$ be the ellipsoid*

$$\{\boldsymbol{y} = [y_1, y_2, ..., y_d]^T : \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_d y_d^2 \le R^2\}.$$

Then

$$F_{D_d}(\boldsymbol{\Sigma}, R) = \frac{1}{(2\sqrt{\pi})^d} \int_{\mathcal{E}(\boldsymbol{\lambda},R)} \exp\left(-\frac{1}{4}\boldsymbol{y}^T\boldsymbol{y}\right) d\boldsymbol{y}, \; R > 0. \tag{3.1}$$

Proof. Consider the probability density function of $\boldsymbol{U} = \boldsymbol{X_1} - \boldsymbol{X_2}$:

$$f_{\boldsymbol{U}}(\boldsymbol{u}) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \exp\Big(-\frac{1}{4}\boldsymbol{u}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{u}\Big), \; \boldsymbol{u} \in \mathbb{R}^d. \tag{3.2}$$

We denote

$$\mathbf{diag}(\boldsymbol{\lambda}) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix}, \; \mathbf{diag}(\boldsymbol{\lambda}^{-1}) = \begin{bmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_d^{-1} \end{bmatrix}.$$

Due to orthogonal diagonalization theorem for symmetric matrices, there exists an orthogonal matrix $\boldsymbol{Q} = [q_{ij}]_{d\times d}$ such that $\boldsymbol{\Sigma} = \boldsymbol{Q}\,\mathbf{diag}(\boldsymbol{\lambda})\boldsymbol{Q}^T$, and therefore, $\boldsymbol{\Sigma}^{-1} = \boldsymbol{Q}\,\mathbf{diag}(\boldsymbol{\lambda}^{-1})\boldsymbol{Q}^T$. Denoting the i-th column of $\boldsymbol{Q}$ by $\boldsymbol{q}_i$, we obtain

$$\boldsymbol{u}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{u} = [\boldsymbol{u}^T\boldsymbol{q}_1, \, \boldsymbol{u}^T\boldsymbol{q}_2, \, ..., \, \boldsymbol{u}^T\boldsymbol{q}_d\,]\,\mathbf{diag}(\boldsymbol{\lambda}^{-1}) \begin{bmatrix} \boldsymbol{q}_1^T\boldsymbol{u} \\ \boldsymbol{q}_2^T\boldsymbol{u} \\ \vdots \\ \boldsymbol{q}_d^T\boldsymbol{u} \end{bmatrix} = \sum_{i=1}^d \frac{(\boldsymbol{q}_i^T\boldsymbol{u})^2}{\lambda_i},$$

and therefore,

$$f_{\boldsymbol{U}}(\boldsymbol{u}) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \exp\Big(-\frac{1}{4}\sum_{i=1}^d \frac{(\boldsymbol{q}_i^T\boldsymbol{u})^2}{\lambda_i}\Big), \; \boldsymbol{u} \in \mathbb{R}^d. \tag{3.3}$$

Let $x_1, \, x_2, \, ..., \, x_d > 0$, $\boldsymbol{U} = [U_1, \, U_2, \, ..., \, U_d]^T$, $\boldsymbol{U}^* = [|U_1|, \, |U_2|, \, ..., \, |U_d|]^T$ and $\boldsymbol{u} = [u_1, \, u_2, \, ..., \, u_d]^T$. Then, due to (3.3),

$$\mathbb{P}(|U_1| \le x_1, |U_2| \le x_2, ..., |U_d| \le x_d)) \overset{\text{def}}{=} F_{\boldsymbol{U}^*}(x_1, x_2, ..., x_d) =$$
$$= \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \int_{-x_1}^{x_1} du_1 \int_{-x_2}^{x_2} du_2 ... \int_{-x_d}^{x_d} \exp\Big(-\frac{1}{4}\sum_{i=1}^d \frac{(\boldsymbol{q}_i^T\boldsymbol{u})^2}{\lambda_i}\Big) du_d.$$

The joint density function of the random variables $|U_1|$, $|U_2|$, ..., $|U_d|$ can be reached by partial differentiation of the last iterated integral, i.e.

$$f_{\boldsymbol{U}^*}(x_1, x_2, ..., x_d) = \frac{\partial^d}{\partial x_d \partial x_{d-1} ... \partial x_1} F_{\boldsymbol{U}^*}(x_1, x_2, ..., x_d).$$

Applying the Leibnitz's rule of differentiation d times, we conclude

$$f_{\boldsymbol{U}^*}(x_1, x_2, ..., x_d) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \sum_{\boldsymbol{w} \in \Omega(x_1, x_2, ..., x_d)} \exp\Big(-\frac{1}{4}\sum_{i=1}^d \frac{(\boldsymbol{q}_i^T\boldsymbol{w})^2}{\lambda_i}\Big), \tag{3.4}$$

where $\Omega(x_1, x_2, ..., x_d) = \{\boldsymbol{w} = [w_1, w_2, ..., w_d]^T \,:\, |w_i| = x_i,\ i = 1, 2, ..., d\}$.

We now aim to replace the summing index in (3.4) and run it over all the binary strings of length d. For any $\boldsymbol{s} = (s_1, s_2, ..., s_d) \in \{0,\,1\}^d$ consider the unique vector $\boldsymbol{w}_{\boldsymbol{s}} = [w_1^{(\boldsymbol{s})}, w_2^{(\boldsymbol{s})}, ..., w_d^{(\boldsymbol{s})}]^T \in \Omega(x_1, x_2, ..., x_d)$ such that

$$w_i^{(\boldsymbol{s})} = \begin{cases} x_i & \text{if } s_i = 0 \\ -x_i & \text{if } s_i = 1 \end{cases}.$$

Formula (3.4) can be equivalently written in the following form:

$$f_{\boldsymbol{U}^*}(x_1, x_2, ..., x_d) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \sum_{\boldsymbol{s}\in\{0,\,1\}^d} \exp\bigg(-\frac{1}{4}\sum_{i=1}^{d} \frac{(\boldsymbol{q}_i^T \boldsymbol{w}_{\boldsymbol{s}})^2}{\lambda_i} \bigg). \tag{3.5}$$

Since $F_{D_d}(\boldsymbol{\Sigma}, R) = \mathbb{P}(\|U^*\| \leq R)$, $R > 0$, the formula (3.4) implies

$$F_{D_d}(\boldsymbol{\Sigma}, R) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \int_{B_d^{\boldsymbol{0}}(R)} \sum_{\boldsymbol{s}\in\{0,\,1\}^d} \exp\bigg(-\frac{1}{4}\sum_{i=1}^{d} \frac{(\boldsymbol{q}_i^T \boldsymbol{w}_{\boldsymbol{s}})^2}{\lambda_i} \bigg) d\boldsymbol{x}, \tag{3.6}$$

where $B_d^{\boldsymbol{0}}(R) = \{(x_1, x_2, ..., x_d) \,:\, x_1^2 + x_2^2 + ... + x_d^2 \leq R^2,\ x_i > 0,\ i = 1, 2, ..., d\}$ is an 2^d-quadrant of the d-dimensional ball $B_d(R)$ of radius R centered at the origin, and $d\boldsymbol{x} = dx_1 dx_2 ... dx_d$. Hereinafter, for any $\boldsymbol{s} = (s_1, s_2, ..., s_d) \in \{0,\,1\}^d$, the symbol $B_d^{\boldsymbol{s}}(R)$ will stand for the 2^d-quadrant of $B_d(R)$ consisting of the points $(x_1, x_2, ..., x_d)$ such that $x_i > 0$, if $s_i = 0$ and $x_i < 0$, if $s_i = 1$.

By interchanging the sum with the integral in (3.6) and denoting

$$g_{\boldsymbol{s}}(x_1, x_2, ..., x_n) = \exp\bigg(-\frac{1}{4}\sum_{i=1}^{d} \frac{(\boldsymbol{q}_i^T \boldsymbol{w}_{\boldsymbol{s}})^2}{\lambda_i} \bigg),$$

we receive

$$F_{D_d}(\boldsymbol{\Sigma}, R) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \sum_{\boldsymbol{s}\in\{0,\,1\}^d} \int_{B_d^{\boldsymbol{0}}(R)} g_{\boldsymbol{s}}(x_1, x_2, ..., x_d) dx_1 dx_2 ... dx_d. \tag{3.7}$$

Let us perform the following change of variable in the integral of $g_{\boldsymbol{s}}$ over $B_d^{\boldsymbol{0}}(R)$:

$$t_i = x_i, \text{ if } s_i = 0,$$

$$t_i = -x_i, \text{ if } s_i = 1.$$

The Jacobian $\frac{D(x_1, x_2, ..., x_d)}{D(t_1, t_2, ..., t_d)}$ is either 1 or -1, therefore after this change of variable we obtain

$$\int_{B_d^{\boldsymbol{0}}(R)} g_{\boldsymbol{s}}(x_1, x_2, ..., x_d) dx_1 dx_2 ... dx_d = \int_{B_d^{\boldsymbol{s}}(R)} g_{\boldsymbol{0}}(x_1, x_2, ..., x_d) dx_1 dx_2 ... dx_d. \tag{3.8}$$

Since the sets $B_d^{\boldsymbol{s}}(R)$, $\boldsymbol{s} \in \{0,\,1\}^d$ are pairwise disjoint and the union of their closures is exactly equal to $B_d(R)$, from (3.7) and (3.8) we establish

$$F_{D_d}(\boldsymbol{\Sigma}, R) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \int_{B_d(R)} g_{\boldsymbol{0}}(x_1, x_2, ..., x_d) dx_1 dx_2 ... dx_d. \tag{3.9}$$

We have $\boldsymbol{w_0} = \boldsymbol{x} = [x_1, x_2, ..., x_d]^T$, which means that

$$g_{\boldsymbol{0}}(x_1, x_2, ..., x_d) = \exp\left(-\frac{1}{4}\sum_{i=1}^{d} \frac{(\boldsymbol{q}_i^T \boldsymbol{x})^2}{\lambda_i} \right). \tag{3.10}$$

To finish the proof, we make one more change of variable in the integral of $g_{\boldsymbol{0}}$ over the ball $B_d(R)$. Consider a new variable $\boldsymbol{y} = [y_1, y_2, ..., y_d]$, where

$$y_i = \frac{\boldsymbol{q}_i^T \boldsymbol{x}}{\sqrt{\lambda_i}}, \;\; i = 1, 2, ..., d. \tag{3.11}$$

Using orthogonality of $\boldsymbol{Q}$, we will have

$$\frac{D(x_1, x_2, ..., x_d)}{D(y_1, y_2, ..., y_d)} = \sqrt{\lambda_1}\sqrt{\lambda_2}...\sqrt{\lambda_d}\,|\boldsymbol{Q}| = |\boldsymbol{\Sigma}|^{1/2} \tag{3.12}$$

and

$$\sum_{i=1}^{d} x_i^2 = \sum_{i=1}^{d} (\sqrt{\lambda_1} q_{i1} y_1 + \sqrt{\lambda_2} q_{i2} y_2 + ... + \sqrt{\lambda_d} q_{id} y_d)^2 = \sum_{i=1}^{d} \lambda_i y_i^2. \tag{3.13}$$

Now (3.1) follows from (3.9)-(3.13).

Corollary 3.1. *The probability density function of D_d is representable as follows:*

$$f_{D_d}(\boldsymbol{\Sigma}, R) = \frac{R^{d-1}}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \int_{S_{d-1}} \exp\left(-\frac{R^2}{4} \boldsymbol{u}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{u} \right) d\boldsymbol{u}. \tag{3.14}$$

Proof. As we saw in the last part of the proof of Theorem 3.1, the formula (3.1) is equivalent to

$$F_{D_d}(\boldsymbol{\Sigma}, R) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \int_{B_d(R)} \exp\left(-\frac{1}{4} \boldsymbol{x}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{x} \right) d\boldsymbol{x}. \tag{3.15}$$

The change of variable $\boldsymbol{x} = r\boldsymbol{u}$, $\boldsymbol{u} \in S_{d-1}$, $d\boldsymbol{x} = r^{d-1} dr d\boldsymbol{u}$ in (3.15) produces

$$F_{D_d}(\boldsymbol{\Sigma}, R) = \frac{1}{(2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2}} \int_{S_{d-1}} d\boldsymbol{u} \int_0^R \exp\left(-\frac{r^2}{4} \boldsymbol{u}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{u} \right) r^{d-1} dr.$$

By taking the derivatives of both sides in the last equation, we establish (3.14). □

As an application of the obtained integral representations, we easily found the probability density function of the Euclidean distance between two bivariate Gaussian points in the case when there is an intercoordinate correlation ρ.

Theorem 3.2. *If* $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, *then*

$$f_{D_2}(\boldsymbol{\Sigma}, R) = \frac{R e^{-\frac{R^2}{4|\boldsymbol{\Sigma}|}}}{2\sqrt{|\boldsymbol{\Sigma}|}} I_0\left(\frac{\rho R^2}{4|\boldsymbol{\Sigma}|} \right),$$

where

$$I_0(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{((2k)!!)^2}$$

is the modified Bessel function of the first kind of order zero.

Proof. It is easy to see that $\lambda_1 = 1+\rho$, $\lambda_2 = 1-\rho$. By (3.14), we have

$$f_{D_2}(\mathbf{\Sigma}, R) = \frac{R}{4\pi\sqrt{1-\rho^2}} \int_0^{2\pi} \exp\left(-\frac{R^2\cos^2\varphi}{4+4\rho} - \frac{R^2\sin^2\varphi}{4-4\rho} \right) d\varphi =$$

$$= \frac{Re^{-\frac{R^2}{4(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} \int_0^{\pi} e^{a\cos 2\varphi} d\varphi,$$

where

$$a = \frac{\rho R^2}{4(1-\rho^2)}.$$

Since $|\mathbf{\Sigma}| = 1-\rho^2$, to complete the proof it remains to show that

$$\frac{1}{\pi}\int_0^{\pi} e^{a\cos 2\varphi} d\varphi = I_0(a).$$

Indeed, Taylor's expansion for e^x solves this problem:

$$\frac{1}{\pi}\int_0^{\pi} e^{a\cos 2\varphi} d\varphi = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{a^k}{k!}\int_0^{\pi}\cos^k 2\varphi d\varphi = \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{a^k}{k!}\int_0^{2\pi}\cos^k\psi d\psi =$$

$$= \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{a^{2k}}{(2k)!}\int_0^{\pi/2}\cos^{2k}\psi d\psi = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{a^{2k}}{(2k)!}\frac{(2k-1)!!}{(2k)!!}\frac{\pi}{2} = \sum_{k=0}^{\infty}\frac{a^{2k}}{((2k)!!)^2} = I_0(a).$$

□

As another application, we established lower and upper bounds for the moments of D_d in terms of the largest and the smallest eigenvalues of the covariance matrix.

Theorem 3.3. *Let* $\mathbb{E}(D_d^r)$ *be the* r*-th moment of* D_d*. Then*

$$\frac{2^r\Gamma\left(\frac{d+r}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\frac{\lambda_d^{\frac{d+r}{2}}}{|\mathbf{\Sigma}|^{1/2}} \le \mathbb{E}(D_d^r) \le \frac{2^r\Gamma\left(\frac{d+r}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\frac{\lambda_1^{\frac{d+r}{2}}}{|\mathbf{\Sigma}|^{1/2}},\ r = 0,1,2,\dots . \tag{3.16}$$

Proof. As $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_d > 0$ then we have

$$\frac{1}{\lambda_1}\sum_{i=1}^{d}(\boldsymbol{q}_i^T\boldsymbol{u})^2 \le \boldsymbol{u}^T\mathbf{\Sigma}^{-1}\boldsymbol{u} \le \frac{1}{\lambda_d}\sum_{i=1}^{d}(\boldsymbol{q}_i^T\boldsymbol{u})^2. \tag{3.17}$$

Due to orthogonality of $\boldsymbol{Q}$,

$$\sum_{i=1}^{d}(\boldsymbol{q}_i^T\boldsymbol{u})^2 = \|\boldsymbol{u}\|_d^2 = 1,\ \text{if } \boldsymbol{u}\in S_{d-1},$$

therefore, the integral representation (3.14) and inequalities (3.17) yield

$$\frac{e^{-\frac{R^2}{4\lambda_d}}R^{d-1}}{(2\sqrt{\pi})^d|\mathbf{\Sigma}|^{1/2}}\int_{S_{d-1}} d\boldsymbol{u} \le f_{D_d}(\mathbf{\Sigma}_d, R) \le \frac{e^{-\frac{R^2}{4\lambda_1}}R^{d-1}}{(2\sqrt{\pi})^d|\mathbf{\Sigma}|^{1/2}}\int_{S_{d-1}} d\boldsymbol{u}.$$

The surface area of S_{d-1} is well-known and equal to $\frac{2(\sqrt{\pi})^d}{\Gamma(\frac{d}{2})}$, so we obtain

$$\frac{R^{d-1}e^{-\frac{R^2}{4\lambda_d}}}{2^{d-1}\Gamma(\frac{d}{2})|\mathbf{\Sigma}|^{1/2}} \le f_{D_d}(\mathbf{\Sigma}_d, R) \le \frac{R^{d-1}e^{-\frac{R^2}{4\lambda_1}}}{2^{d-1}\Gamma(\frac{d}{2})|\mathbf{\Sigma}|^{1/2}}. \tag{3.18}$$

Multiplying all sides of (3.18) by R^r and applying integral over $(0, +\infty)$ to all sides leads to

$$\frac{\lambda_d^{\frac{d+r}{2}}}{|\boldsymbol{\Sigma}|^{1/2}} I(d) \leq \mathbb{E}\left(D_d^r\right) \leq \frac{\lambda_1^{\frac{d+r}{2}}}{|\boldsymbol{\Sigma}|^{1/2}} I(d),$$

where

$$I(d) = \frac{1}{2^{d-1}\Gamma(\frac{d}{2})} \int_0^{+\infty} R^{d+r-1} e^{-\frac{R^2}{4}} dR.$$

It remains to apply (2.4) and (2.6) to see that

$$I(d) = 2^r \frac{\Gamma\left(\frac{d+r}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}.$$

4. The normal covariogram of $\mathbb{R}^d$

The covariogram of a bounded domain $\mathbb{D} \subset \mathbb{R}^d$ is known to be the function

$$C_{\mathbb{D}}(\boldsymbol{t}) = L_d(\mathbb{D} \cap \{\mathbb{D} + \boldsymbol{t}\}), \ \ \boldsymbol{t} \in \mathbb{R}^d,$$

where $\mathbb{D} + \boldsymbol{t} = \{\mathcal{P} + \boldsymbol{t} \, : \, \mathcal{P} \in \mathbb{D}\}$ and $L_d(\cdot)$ is the d-dimensional Lebesgue measure in $\mathbb{R}^d$. If $\mathbb{D}$ is a convex body and $\mathcal{P}_1$, $\mathcal{P}_2$ are chosen uniformly and independently from $\mathbb{D}$, then the probability density function of $\mathcal{P}_1 - \mathcal{P}_2$ can be expressed by the covariogaram of $\mathbb{D}$ as shown in (1.2). This motivates us to extend the concept of the covariogram for $\mathbb{D} = \mathbb{R}^d$.

Definition 4.1. *Let $\mathcal{P}_1$, $\mathcal{P}_2 \sim N_d(\mathbf{0}, \boldsymbol{\Sigma})$ be independent and $f_{\mathcal{P}_1 - \mathcal{P}_2}$ be the probability density function of $\mathcal{P}_1 - \mathcal{P}_2$. The function $C_{\boldsymbol{\Sigma}} : \mathbb{R}^d \to (0, +\infty)$ that satisfies*

$$f_{\mathcal{P}_1 - \mathcal{P}_2}(\boldsymbol{t}) = \frac{C_{\boldsymbol{\Sigma}}(\boldsymbol{t})}{C^2_{\boldsymbol{\Sigma}}(\mathbf{0})},$$

is called the normal covariogram of $\mathbb{R}^d$ associated with $\boldsymbol{\Sigma}$.

By taking $\boldsymbol{t} = \mathbf{0}$ in this definition and using (3.2) we immediately obtain

$$C_{\boldsymbol{\Sigma}}(\mathbf{0}) = (2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2},$$

and then

$$C_{\boldsymbol{\Sigma}}(\boldsymbol{t}) = (2\sqrt{\pi})^d |\boldsymbol{\Sigma}|^{1/2} \exp\left(-\frac{1}{4} \boldsymbol{t}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{t}\right), \, \boldsymbol{t} \in \mathbb{R}^d. \tag{4.1}$$

It is remarkable that $C_{\boldsymbol{I}_d}(\boldsymbol{t}) = (2\sqrt{\pi})^d \exp\left(-\frac{1}{4}\|\boldsymbol{t}\|_d^2\right)$. It illustrates that if $\mathbb{R}^d$ is considered as a space of points with uncorrelated coordinates then the covariogram of the space is naturally independent on the direction of translation.

Taking into account (1.1), the following identity provides a further argument to ensure that the normal covariogram naturally extends the concept of covariogram.

Theorem 4.1.

$$f_{D_d}(\mathbf{\Sigma}, R) = \frac{R^{d-1}}{C^2_{\mathbf{\Sigma}}(\mathbf{0})} \int_{S_{d-1}} C_{\mathbf{\Sigma}}(R\boldsymbol{u}) d\boldsymbol{u}, \;\; R > 0. \tag{4.2}$$

Proof. By (4.1),

$$\frac{R^{d-1}}{C^2_{\mathbf{\Sigma}}(\mathbf{0})} \int_{S_{d-1}} C_{\mathbf{\Sigma}}(R\boldsymbol{u}) d\boldsymbol{u} = \frac{R^{d-1}}{(2\sqrt{\pi})^d |\mathbf{\Sigma}|^{1/2}} \int_{S_{d-1}} \exp\left(- \frac{R^2}{4} \boldsymbol{u}^T \mathbf{\Sigma}^{-1} \boldsymbol{u} \right) d\boldsymbol{u}.$$

Now due to (3.14), the right-hand-side of the above equality coincides with the left-hand-side of (4.2).

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UNIQUENESS OF L-FUNCTIONS AND GENERAL MEROMORPHIC FUNCTIONS IN LIGHT OF TWO SHARED SETS

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Abstract. In this paper, we have dealt with the uniqueness problem of a general meromorphic function with $\mathcal{L}$ function in terms of two shared sets. In our main theorem, we deal with general meromorphic functions instead of meromorphic functions having finitely many poles. As a corollary of our main theorem, we have shown that our result not only fills the gap of some theorems of [3] and [1] for $m = n-1$ but also reduces the cardinality of the main range set and hence our result significantly improves all the results in this direction.

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Keywords: L-function; meromorphic function; shared set.

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Throughout the paper by meromorphic functions we shall mean it is meromorphic in the complex plane and by L-functions we mean it is L-functions in the Selberg class which is defined [7, 8] to be a Dirichelet series

$$\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{-s}} \tag{1.1}$$

satisfying the following axioms:

- (i) *Ramanujan hypothesis :* $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$;
- (ii) *Analytic continuation :* There is a non-negative integer m such that $(s-1)^m \mathcal{L}(s)$ is an entire function of finite order;
- (iii) *Functional equation:* $\mathcal{L}$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega \overline{\Lambda_{\mathcal{L}}(1-\overline{s})}, \tag{1.2}$$

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where

$$\Lambda_{\mathcal{L}}(s)=\mathcal{L}(s)Q^{s}\prod_{j=1}^{k}\Gamma(\lambda_j s+\nu_j), \tag{1.3}$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω with $\text{Re}\nu_j \geq 0$ and $|\omega| = 1$;

- (iv) *Euler product hypothesis :* $\log \mathcal{L}(s) = \sum\limits_{n=1}^{\infty} \frac{b(n)}{n^s}$, where $b(n) = 0$ unless n is a positive power of a prime and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

This class includes many of the known entire Dirichlet series with Euler product, including the Riemann zeta function and the Dirichlet L-functions. Since an L-function can be analytically continued to a meromorphic function, the study of uniquely determining an L-function, gradually moved towards uniquely determining the L-functions with respect to the meromorphic functions having finitely many poles. A lot of research has already been pursued by various researchers [7, 8, 6, 9, 3] in this direction. Below we recall some of these results and the gradual development. But before that, we recall some basic definitions. For standard notations of Nevanlinna theory, we suggest our reader to follow [2].

Definition 1.1. [4, 5] *Let k be a non-negative integer or infinity. For $a \in \overline{\mathbb{C}}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [4] *For $A \subset \overline{\mathbb{C}}$ we define $E_f(A, k) = \cup_{a\in A} E_k(a; f)$, where k is a non-negative integer or infinity. If $E_f(A, k) = E_g(A, k)$, then we say that f and g share the set A with weight k.*

We write f, g share (A, k) to mean that f, g share the set A with weight k. We say that f, g share a set A IM or CM if and only if f, g share $(A, 0)$ or (A, ∞) respectively.

2. Gradual development and motivation

In 2010, B. Q. Li proved the following theorem.

Theorem A. [6] *Let a and b be two distinct finite values, and let f be a meromorphic function in the complex plane such that f has finitely many poles in the complex plane. If f and a non-constant L-function $\mathcal{L}$ share (a,∞) and $(b,0)$, then $\mathcal{L}=f$.*

In 2018, Yuan, Li and Yi [9] considered the uniqueness of L-functions with meromorphic functions having finitely many poles under set sharing and proved the following theorem.

Theorem B. [9] *Let $S=\{\omega_1,\omega_2,...,\omega_l\}$, where $\omega_1,\omega_2,...,\omega_l$ are all distinct zeros of the polynomial $P(\omega)=\omega^n+a\omega^m+b$. Here l is a positive integer satisfying $1\leq l\leq n$, n and m are relatively prime positive integers with $n\geq 5$ and $n>m$, and a,b,c are nonzero finite constants, where $c\neq\omega_j$ for $1\leq j\leq l$. Let f be a non-constant meromorphic function such that f has finitely many poles in the complex plane, and let $\mathcal{L}$ be a non-constant L-function. If f and $\mathcal{L}$ share S CM and c IM, then $\mathcal{L}=f$.*

In 2020, Kundu and Banerjee [3] considered the case $c=0$ of Theorem B and provided the following theorem.

Theorem C. [3] *Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles and S be as defined in* Theorem B. *Here a,b are two non-zero constants and n,m are relatively prime positive integers such that $n>2m$. If f and a non-constant L -function $\mathcal{L}$ share (S,∞) and $(0,0)$, then $\mathcal{L}=f$.*

Very recently, Banerjee and Kundu [1] proved the following result.

Theorem D. [1] *Let S be defined as in* Theorem B*, f be a meromorphic function having finitely many poles in $\mathbb{C}$ and let $\mathcal{L}$ be a non-constant L -function. Suppose $E_f(S,s)=E_{\mathcal{L}}(S,s)$ and for some finite $c\notin S, f$ and $\mathcal{L}$ share $(c,0)$. Also let $a_i(i=1,2,\ldots,n-m)$ be the zeros of nz^k+ma, where $k=n-m(\geq 1)$ and denote $S'=\{a_1,a_2,\ldots,a_{n-m}\}$.*

I. *Suppose $c=0$.*

When (i) $s\geq 2, n>2m+2$ or (ii) $s=1, n>2m+3$ or (iii) $s=0, n>2m+8$; then $f\equiv\mathcal{L}$.

II. *Suppose $c\neq 0$.*

(A) *Let $c\in S'$. When $l=n$ and (i) $s=1, n>2k+2$ or (ii) $s=0, n>2k+5$; or when $l=n-1$ and (i) $s\geq 2, n>2k+2$ or (ii) $s=1, n>2k+3$ or (iii) $s=0, n>2k+8$; then $f\equiv\mathcal{L}$.*

(B) *Next let $c\notin S'$. When (i) $s\geq 2, n>2k+4$ or (ii) $s=1, n>2k+5$ or (iii) $s=0, n>2k+10$; then $f\equiv\mathcal{L}$.*

From the above discussion, one would naturally observe that in *Theorem B-D* all the authors always considered the set S to be the zeros of the polynomial $P(z) = z^n + az^m + b$. Now if we take $m = n - 1$, then the condition of *Theorem C* and condition **I** of *Theorem D* become absurd; i.e., for $m = n - 1$ *Theorem C* and **I** of *Theorem D* is not applicable. Also one can notice that all the authors always considered a special class of meromorphic functions; i.e., they always considered meromorphic functions having finitely many poles. Therefore the uniqueness of L-functions with general meromorphic functions is yet to be dealt with. At this moment naturally, the following two questions come into mind.

Question 2.1. *For $m = n - 1$, if a non-constant meromorphic function having finitely many poles and an L-function share (S, t) and $(c, 0)$, are they equal?*

Question 2.2. *If a general non-constant meromorphic function and an L-function share (S, t) and $(c, 0)$, then are they equal?*

In this paper, we have answered the above two questions affirmatively. Not only that by considering the polynomial $P(z) = z^n + az^{n-1} + b$, we have shown the uniqueness of a general non-constant meromorphic function with a non-constant L-function when they share the set (S, t) and $(\eta, 0)$, where η is the zero of $P'(z)$. As a corollary of our main theorem, we have shown that our result not only fills the gap of *Theorem C* and I of *Theorem D* for $m = n-1$ but also significantly improves *Theorem B-C* and **I** of *Theorem D*.

3. Main result

Now we state the following theorem which is the main result of the paper.

Theorem 3.1. *Let $P(z) = z^n + az^{n-1} + b$, with $n \geq 3$ and a, b are non-zero constants such that the polynomial has no multiple zero. Suppose that f, $\mathcal{L}$ share (S, t) and $(\eta, 0)$, where $t \in \mathbb{N} \cup \{0\}$, S be the set of zeros of $P(z)$, η be the zero of $P'(z)$, f be a non-constant meromorphic function and $\mathcal{L}$ be a non-constant L-function.*

- (I) *Suppose $\eta = 0$. If*
  - (i) *$t \geq 5$, with*
    - $n > 2 + (2 + \frac{3}{n-2})(1 - \Theta(\infty; f))$;
  - (ii) *$t = 4$, with*
    - $n > \max\{2 + (2 + \frac{3}{n-2})(1 - \Theta(\infty; f)), 3\}$;
    - $n = 3$ *and* $\Theta(\infty; f) > \frac{5}{6}$;

(iii) $t = 3$, *with*
- $n > \max\{2 + (2 + \frac{3}{n-2})\,(1 - \Theta(\infty; f))\,,\ 4\}$;
- $n = 4$ *and* $\Theta(\infty; f) > \frac{11}{23}$;
- $n = 3$ *and* $\Theta(\infty; f) > \frac{7}{8}$;

(iv) $t = 2$, *with*
- $n > 2 + \left(2 + \frac{12}{3n-8}\right)(1 - \Theta(\infty; f))$;

(v) $t = 1$, *with*
- $n > \max\{2 + \frac{5}{2} + (\frac{6}{n-3})(1 - \Theta(\infty; f)),\ 4\}$;
- $n = 4$ *and* $\Theta(\infty; f) > \frac{13}{17}$.

(vi) $t = 0$, *with*
- $n > \max\{2 + \left(4 + \frac{14}{n-4}\right)(1 - \Theta(\infty; f)),\ 4\}$;

then we get $f \equiv \mathcal{L}$.

(II) *Suppose* $\eta \neq 0$. *If*

(i) $t \geq 2$, *with*
- $n > \max\{4 + 2(1 - \Theta(\infty; f)),\ 4\}$;

(ii) $t = 1$, *with*
- $n > \max\{5 + \frac{5}{2}(1 - \Theta(\infty; f)),\ 4\}$;

(iii) $t = 0$, *with*
- $n > \max\{8 + 4(1 - \Theta(\infty; f)),\ 4\}$;

then we get $f \equiv \mathcal{L}$.

Corollary 3.1. *Let* $P(z) = z^n + az^{n-1} + b$, *with* $n \geq 3$ *and* a, b *are non-zero constants such that the polynomial has no multiple zero. Suppose that* f, $\mathcal{L}$ *share* (S, t) *and* $(\eta, 0)$, *where* $t \in \mathbb{N} \cup \{0\}$, S *be the set of zeros of* $P(z)$, η *be the zero of* $P'(z)$, f *be a non-constant meromorphic function having finitely many poles and* $\mathcal{L}$ *be a non-constant* L*-function.*

(I) *Suppose* $\eta = 0$. *If (i)* $n \geq 3$ *when* $t \geq 2$, *(ii)* $n \geq 4$ *when* $t = 1$, *(iii)* $n \geq 5$ *when* $t = 0$; *then we get* $f \equiv \mathcal{L}$.

(II) *Suppose* $\eta \neq 0$. *If (i)* $n \geq 5$ *when* $t \geq 2$, *(ii)* $n \geq 6$ *when* $t = 1$, *(iii)* $n \geq 9$ *when* $t = 0$; *then we get* $f \equiv \mathcal{L}$.

Remark 3.1. *In Corollary 3.1 one can observe that for* $\eta = 0$ *the least cardinality of the set* S *is* $3,\ 4$ *and* 5 *when* $t \geq 2,\ t = 1$ *and* $t = 0$ *respectively, whereas in Theorem D it was* $5,\ 6$ *and* 11. *Again in Theorem C the cardinality* 3 *was achieved in the case of CM sharing but from Corollary 3.1 the same is achieved for weight* 2 *only. Also in Theorem 3.1 we deal with general meromorphic functions instead of meromorphic functions having finitely many poles. Therefore our result is not only improved but also an extended version of Theorem B-C and I of Theorem D.*

4. Lemmas

In this section, we discuss some lemmas which will play key role to prove our main result. For the convenience of the reader, let us shortly recall some definitions and notations which will be required to prove the lemmas.

Definition 4.1. *Let f be a meromorphic function. We denote the order of f by $\rho(f)$, where*

$$\rho(f) = \limsup_{r\to\infty} \frac{\log(T(r,f))}{\log r}. \tag{4.1}$$

By $S(r,f)$ we mean any quantity satisfying $S(r,f) = O(\log(rT(r,f)))$ for all r possibly outside a set of finite linear measure. If f is a function of finite order, then $S(r,f) = O(\log r)$ for all r.

Definition 4.2. *Let f and g be two non-constant meromorphic functions such that f and g share $(a,0)$, where $a \in \overline{\mathbb{C}}$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. Then*

- *$\overline{N}_d(r,a;f)$ denotes the reduced counting function of those a-points of f and g where $p > q$.*
- *$N_E^{1)}(r,a;f)$ denotes the counting function of those a-points of f and g where $p = q = 1$.*

  In the same way we can define $\overline{N}_d(r,a;g)$ and $N_E^{1)}(r,a;g)$.
- *$\overline{N}(r,a;f|=1)$ denotes the reduced counting function of simple a-points of f.*
- *$\overline{N}_*(r,a;f,g)$ denotes the reduced counting function of those a-points of f and g where $p \neq q$. Clearly $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f) = \overline{N}_d(r,a;f) + \overline{N}_d(r,a;g)$.*
- *$\overline{N}(r,a;f\,|\geq m)$ denotes the reduced counting function of those a points of f whose multiplicities are not less than m.*
- *$N(r,a;f \mid \; g \neq b_1,b_2,\ldots,b_q)$ denotes the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1,2,\ldots,q$; where $a, b_1, b_2, \ldots, b_q \in \overline{\mathbb{C}}$.*

Definition 4.3. *Let $f(z)$ be a non-constant meromorphic function in the complex plane and $a \in \overline{\mathbb{C}}$. Then*

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

Observe that $0 \leq \Theta(a,f) \leq 1$.

For two non-constant meromorphic functions F and G, set

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{4.2}$$

and

$$\Phi = \frac{F'}{F-1} - \frac{G'}{G-1}. \tag{4.3}$$

Lemma 4.1. [10] *Let F, G share $(1,0)$ and $H \not\equiv 0$. Then*

$$N_E^{1)}(r,1;F) = N_E^{1)}(r,1;G) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 4.2. *Let f be a non-constant meromorphic function and $\mathcal{L}$ be an non-constant L-function sharing a set S IM, where $|S| \geq 3$. Then $\rho(f) = \rho(\mathcal{L}) = 1$. Furthermore, $S(r,f) = O(\log r) = S(r,\mathcal{L})$.*

Proof. Proceeding in a similar method as done in the proof of Theorem 5, [9, see p. 6], we can obtain $\rho(f) = \rho(\mathcal{L}) = 1$. So we omit it.
Since $\rho(f) = \rho(\mathcal{L}) = 1$, so from the definition of $S(r,f)$ we get $S(r,f) = O(\log r) = S(r,\mathcal{L})$. □

Lemma 4.3. *Let f, g be two non-constant meromorphic functions sharing (1,t), where $t \in \mathbb{N} \cup \{0\}$. Then*

$$\overline{N}(r,1;f) + \overline{N}(r,1;g) \leq N_E^{1)}(r,1;f) + (1-t)\,\overline{N}_*(r,1;f,g) + N(r,1;f).$$

Proof. Since f and g share $(1,t)$, we observe that

$$\overline{N}(r,1;f) + \overline{N}(r,1;g) = 2\overline{N}(r,1;f).$$

Case-I : Suppose $t \geq 2$.
Let z_0 be a 1 point of f with multiplicity p and a 1 point of g of multiplicity q. Since f and g share $(1,t)$, therefore $p \leq t$ implies $p = q$.

Subcase - I : When $p \leq t$. If $p = 1$, then z_0 is counted once in both $N_E^{1)}(r,1;f)$ and $N(r,1;f)$. On the other hand z_0 is not counted in $\overline{N}_*(r,1;f,g)$. Again if $p \neq 1$, then z_0 is counted p times (i.e., at least 2 times) in $N(r,1;f)$ and in this case z_0 is not counted in $N_E^{1)}(r,1;f)$ and $\overline{N}_*(r,1;f,g)$. Therefore z_0 is counted at least 2 times in $N_E^{1)}(r,1;f) + (1-t)\,\overline{N}_*(r,1;f,g) + N(r,1;f)$.

Subcase - II : When $p \geq (t+1)$. If $p = q$, then z_0 is counted p time (i.e., at least 3 times) in $N(r,1;f)$ and z_0 is not counted in $N_E^{1)}(r,1;f)$ and $\overline{N}_*(r,1;f,g)$. When $p \neq q$, then z_0 is counted $(1-t)$ times in $(1-t)\overline{N}_*(r,1;f,g)$ and counted p times (i.e., at least $t+1$ times) in $N(r,1;f)$ and z_0 is not counted in $N_E^{1)}(r,1;f)$; i.e., z_0 is counted at least $(1-t)+(t+1)=2$ times in $N_E^{1)}(r,1;f)+(1-t)\,\overline{N}_*(r,1;f,g)+N(r,1;f)$. Now since z_0 is counted two times in $\overline{N}(r,1;f)+\overline{N}(r,1;g)$. Therefore in any sub case we have

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)\leq N_E^{1)}(r,1;f)+(1-t)\,\overline{N}_*(r,1;f,g)+N(r,1;f).$$

Case-II : Suppose $t=1$.
Then clearly

$$\overline{N}(r,1;f)\leq N(r,1;f|=1)+N(r,1;f)=N_E^{1)}(r,1;f)+N(r,1;f).$$

Therefore, for $t=1$

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)\leq N_E^{1)}(r,1;f)+(1-t)\,\overline{N}_*(r,1;f,g)+N(r,1;f).$$

Case-III : Suppose that $t=0$. Let z_0 be a 1 point of f with multiplicity p and a 1 point of g of multiplicity q. If $p=q=1$, then z_0 is counted 2 times in both $2\overline{N}(r,1;f)$ and $N_E^{1)}(r,1;f)+(1-t)\,\overline{N}_*(r,1;f,g)+N(r,1;f)$, as $\overline{N}_*(r,1;f,g)$ does not count z_0. If $p=1,\ q\neq 1$, then also z_0 is counted 2 times in both $2\overline{N}(r,1;f)$ and $N_E^{1)}(r,1;f)+(1-t)\,\overline{N}_*(r,1;f,g)+N(r,1;f)$, as in this case $N_E^{1)}(r,1;f)$ does not count z_0. Finally if $p\neq 1$, then z_0 is counted at least 2 times in $(1-t)\,\overline{N}_*(r,1;f,g)+N(r,1;f)$ and z_0 is not counted in $N_E^{1)}(r,1;f)$.
Therefore, for $t=0$,

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)\leq N_E^{1)}(r,1;f)+(1-t)\,\overline{N}_*(r,1;f,g)+N(r,1;f).$$

And hence in any case,

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)\leq N_E^{1)}(r,1;f)+(1-t)\,\overline{N}_*(r,1;f,g)+N(r,1;f).\qquad\square$$

Lemma 4.4. *Let $P(z)=z^n+az^{n-1}+b$, with $n\geq 3$ and a, b are non-zero constants such that the polynomial has no multiple zero and S be the set of all zeros of $P(z)$. Define*

$$F=\frac{f^n+af^{n-1}}{-b}\ \textit{and}\ G=\frac{\mathcal{L}+a\mathcal{L}^{n-1}}{-b}\tag{4.4}$$

and

$$P^{'}(z)=n\prod_{i=1}^{2}(z-\eta_i)^{q_i},\tag{4.5}$$

where $\eta_1=0$, $\eta_2=\frac{a(n-1)}{n}$, $q_1=n-2$ and $q_2=1$.

Let f be a non-constant meromorphic functions and $\mathcal{L}$ be a non-constant L-function such that $f,\mathcal{L}$ share (S,t) and η_j IM and if $\Phi\not\equiv 0$ then

$$\overline{N}(r,\eta_j;f)\leq\frac{1}{q_j}[\overline{N}_*(r,1;F,G)+\overline{N}(r,\infty;f)]+O(\log r).$$

Proof. By the given condition clearly F and G share $(1,t)$. Also by (4.5) we have

$$F' = -\frac{n}{b}\prod_{i=1}^{2}(f-\eta_i)^{q_i}\, f' \quad \text{and} \quad G' = -\frac{n}{b}\prod_{i=1}^{2}(\mathcal{L}-\eta_i)^{q_i}\, \mathcal{L}'.$$

Thus we see that

$$\Phi = \frac{-n\prod_{i=1}^{2}(f-\eta_i)^{q_i}\, f'}{b(F-1)} - \frac{-n\prod_{i=1}^{2}(\mathcal{L}-\eta_i)^{q_i}\, \mathcal{L}'}{b(G-1)}. \tag{4.6}$$

Let z_0 be a zero of $f-\eta_j$ with multiplicity r and a zero of $\mathcal{L}-\eta_j$ with multiplicity v. Then that would be a zero of Φ of multiplicity $\mu = \min\{q_j r + r - 1, q_j v + v - 1\} \geq q_j$. So by a simple calculation we can write

$$\begin{aligned}
&\overline{N}\,(r,\eta_j;f) = \overline{N}\,(r,\eta_j;\mathcal{L}) \leq \frac{1}{\mu}N(r,0;\Phi) \leq \frac{1}{\mu}T(r,\Phi)\\
&\leq \frac{1}{\mu}[N(r,\Phi) + S(r,F) + S(r,G)]\\
&\leq \frac{1}{\mu}\left[\overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G)\right] + S(r,F) + S(r,G)\\
&\leq \frac{1}{q_j}\left[\overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;\mathcal{L})\right] + S(r,f) + S(r,\mathcal{L}).
\end{aligned}$$

Now using Lemma (4.2) and the fact that $\overline{N}(r,\infty;\mathcal{L}) = O(\log r)$, we have

$$\overline{N}\,(r,\eta_j;f) = \overline{N}\,(r,\eta_j;\mathcal{L}) \leq \frac{1}{q_j}\left[\overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;f)\right] + O(\log r). \qquad \square$$

Lemma 4.5. *Let* $F^* - 1 = a_n\prod_{i=1}^{n}(f-w_i)$ *and* $G^* - 1 = a_n\prod_{i=1}^{n}(\mathcal{L}-w_i)$, *where* f *be a non-constant meromorphic function,* $\mathcal{L}$ *be an non-constant L-function,* $a_n, w_i \in \mathbb{C}-\{0\}$ *for all* $i \in \{1,2,\ldots,n\}$. *Further suppose that* F^* *and* G^* *share* $(1,t)$, *where* $t \in \mathbb{N}\cup\{0\}$ *and* $\eta_j \neq w_i$ *for* $i = 1,2,\cdots,n$. *Then*

$$\overline{N}_d(r,1;F^*) \leq \frac{1}{t+1}\left[\overline{N}(r,\eta_j;f) + \overline{N}(r,\infty;f) - N_1(r,0;f')\right] + O(\log r),$$

where $N_1(r,0;f') = N(r,0;f'|f \neq 0,\eta_1,w_1,w_2,...,w_n)$. *Similar expression also holds for* $\overline{N}_d(r,1;G^*)$.

Proof. Since F^* and G^* share $(1,t)$, in view of *Lemma (4.2)* and $\overline{N}(r,\infty;\mathcal{L}) = O(\log r)$, we find by using first fundamental theorem that

$$\begin{aligned}
\overline{N}_d(r,1;F^*) &\le \overline{N}(r,1;F^*|\ge t+2) \le \frac{1}{t+1}\left[N(r,1;F^*) - \overline{N}(r,1;F^*)\right]\\
&\le \frac{1}{t+1}\left[\sum_{i=1}^{n}\left(N(r,w_i;f) - \overline{N}(r,w_i;f)\right)\right]\\
&\le \frac{1}{t+1}\left[N(r,0;f'|f-\eta_j \neq 0) - N_1(r,0;f')\right]\\
&\le \frac{1}{t+1}\left[N(r,0;\frac{f'}{f-\eta_j}) - N_1(r,0;f')\right]\\
&\le \frac{1}{t+1}\left[T(r,\frac{f'}{f-\eta_j}) - N_1(r,0;f')\right] + O(1)\\
&\le \frac{1}{t+1}\left[N(r,\infty;\frac{f'}{f-\eta_j}) - N_1(r,0;f')\right] + S(r,f)\\
&\le \frac{1}{t+1}\left[\overline{N}(r,\infty;f) + \overline{N}(r,\eta_j;f) - N_1(r,0;f')\right] + O(\log r).
\end{aligned}$$

This proves the lemma. □

Remark 4.1. *Let F and G be defined by* (4.4). *If F, G share $(1,t)$ and f, $\mathcal{L}$ share η_j IM, then using* Lemma (4.4) *and* Lemma (4.5), *in view of $\overline{N}(r,\infty;\mathcal{L}) = O(\log r)$, we get*

$$\begin{aligned}
\overline{N}_*(r,1;F,G) &= \overline{N}_d(r,1;F) + \overline{N}_d(r,1;G)\\
&\le \frac{1}{t+1}\left[\overline{N}(r,\eta_j;f) + \overline{N}(r,\infty;f) + \overline{N}(r,\eta_j;\mathcal{L})\right] + O(\log r)\\
&\le \frac{2}{t+1}\overline{N}(r,\eta_j;f) + \frac{1}{t+1}\overline{N}(r,\infty;f) + O(\log r) \qquad (4.7)\\
&\le \frac{2}{q_j(t+1)}\left[\overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;f)\right] + \frac{1}{t+1}\overline{N}(r,\infty;f) + O(\log r).
\end{aligned}$$

This implies that

$$\begin{aligned}
\left(1 - \frac{2}{q_j(t+1)}\right)\overline{N}_*(r,1;F,G) &\le \frac{(2+q_j)}{q_j(t+1)}\overline{N}(r,\infty;f) + O(\log r)\\
((t+1)q_j - 2)\,\overline{N}_*(r,1;F,G) &\le (2+q_j)\overline{N}(r,\infty;f) + O(\log r). \qquad (4.8)
\end{aligned}$$

Lemma 4.6. *Let $P(z) = z^n + az^{n-1} + b$, with $n \ge 3$ and a, b are non-zero constants such that the polynomial has no multiple zero. Suppose that f, $\mathcal{L}$ share (S,t) and $(\eta,0)$, where $t \in \mathbb{N}\cup\{0\}$, η be the zero of $P'(z)$, f be a non-constant meromorphic function and $\mathcal{L}$ be a non-constant L-function. Further suppose that*

(4.9)
$$\mathcal{F} = \frac{f^n + af^{n-1}}{-b} = -\frac{1}{b}f^{n-1}(f+a) \;\; \textit{and} \;\; \mathcal{G} = \frac{\mathcal{L}^n + a\mathcal{L}^{n-1}}{-b} = -\frac{1}{b}\mathcal{L}^{n-1}(\mathcal{L}+a).$$

When $\eta = 0$ and

(i) $t \geq 5$, *with*
- $n > 2 + (2 + \frac{3}{n-2})(1 - \Theta(\infty; f))$;

(ii) $t = 4$, *with*
- $n > \max\{2 + (2 + \frac{3}{n-2})(1 - \Theta(\infty; f)), 3\}$;
- $n = 3$ *and* $\Theta(\infty; f) > \frac{5}{6}$;

(iii) $t = 3$, *with*
- $n > \max\{2 + (2 + \frac{3}{n-2})\,(1 - \Theta(\infty; f))\,,\ 4\}$;
- $n = 4$ *and* $\Theta(\infty; f) > \frac{11}{23}$;
- $n = 3$ *and* $\Theta(\infty; f) > \frac{7}{8}$;

(iv) $t = 2$, *with*
- $n > 2 + \left(2 + \frac{12}{3n-8}\right)(1 - \Theta(\infty; f))$;

(v) $t = 1$, *with*
- $n > \max\{2 + \frac{5}{2} + (\frac{6}{n-3})(1 - \Theta(\infty; f)),\ 4\}$;
- $n = 4$ *and* $\Theta(\infty; f) > \frac{13}{17}$.

(vi) $t = 0$, *with*
- $n > \max\{2 + \left(4 + \frac{14}{n-4}\right)(1 - \Theta(\infty; f)),\ 4\}$;

or, $\eta \neq 0$ *and*

(i) $t \geq 2$, *with*
- $n > 4 + 2(1 - \Theta(\infty; f))$;

(ii) $t = 1$, *with*
- $n > 5 + \frac{5}{2}(1 - \Theta(\infty; f))$;

(iii) $t = 0$, *with*
- $n > 8 + 4(1 - \Theta(\infty; f))$;

we get $\frac{1}{\mathcal{F}-1} = \frac{A}{\mathcal{G}-1} + B$, *where* $A(\neq 0),\ B \in \mathbb{C}$.

Proof. According to the assumptions of the lemma, we clearly have $\mathcal{F}$, $\mathcal{G}$ share $(1, t)$ and f, $\mathcal{L}$ share $(\eta, 0)$. Here

$$\mathcal{F}' = -\frac{1}{b} f^{n-2}(nf + a(n-1))f' = -\frac{n}{b} f^{n-2}\left(f - \frac{a(1-n)}{n}\right) f'$$

and

$$\mathcal{G}' = -\frac{1}{b} \mathcal{L}^{n-2}(n\mathcal{L} + a(n-1))\mathcal{L}' = -\frac{n}{b} \mathcal{L}^{n-2}\left(\mathcal{L} - \frac{a(1-n)}{n}\right) \mathcal{L}'.$$

Now consider H as given by (4.2) for $\mathcal{F}$ and $\mathcal{G}$. Firstly we suppose that $H \not\equiv 0$. Now we distinguish the following cases.

Case 1. $\Phi \equiv 0$.

Then by integrating we get, $(F - 1) = A(G - 1)$, where $A(\neq 0) \in \mathbb{C}$. Therefore, $F' = AG'$ and $F'' = AG''$. Which implies that $H \equiv 0$. Which is a contradiction.

Case 2. $\Phi \not\equiv 0$.

First let us assume that $\eta=\eta_1=0$. Then clearly $q_1=n-2$. Also let $\eta_2=\frac{a(1-n)}{n}$. Since $H\not\equiv 0$, it can be easily verified that H has only simple poles and these poles come from the following points.

(i) η_2 -points of f and $\mathcal{L}$.

(ii) η_1-points of f and $\mathcal{L}$ having different multiplicity.

(iii) Poles of f and $\mathcal{L}$.

(iv) 1 -points of $\mathcal{F}$ and $\mathcal{G}$ having different multiplicities.

(v) Those zeros of f' and $\mathcal{L}'$, which are not zeros of $\prod_{i=1}^{2}(f-\eta_i)(\mathcal{F}-1)$ and $\prod_{i=1}^{2}(\mathcal{L}-\eta_i)(\mathcal{G}-1)$ respectively. Therefore we obtain

$$\begin{aligned}N(r,H)\leq{}&\overline{N}(r,\eta_2;f)+\overline{N}(r,\eta_2;\mathcal{L})+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})\\&+\overline{N}_*(r,\eta_1;f,\mathcal{L})+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;\mathcal{L}'),\end{aligned}\tag{4.10}$$

where $\overline{N}_0(r,0;f')$ and $\overline{N}_0(r,0;\mathcal{L}')$ denotes the reduced counting functions of those zeros of f' and $\mathcal{L}'$, which are not zeros of $\prod_{i=1}^{2}(f-\eta_i)(\mathcal{F}-1)$ and $\prod_{i=1}^{2}(\mathcal{L}-\eta_i)(\mathcal{G}-1)$ respectively.

Using the second fundamental theorem, we get,

$$\begin{aligned}(n+1)T(r,f)\leq{}&\overline{N}(r,1;\mathcal{F})+\textstyle\sum_{i=1}^{2}\overline{N}(r,\eta_i;f)+\\&+\overline{N}(r,\infty;f)-\overline{N}_0(r,0;f')+S(r,f),\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}(n+1)T(r,\mathcal{L})\leq{}&\overline{N}(r,1;\mathcal{G})+\textstyle\sum_{i=1}^{2}\overline{N}(r,\eta_i;\mathcal{L})+\\&+\overline{N}(r,\infty;\mathcal{L})-\overline{N}_0(r,0;\mathcal{L}')+S(r,\mathcal{L}).\end{aligned}\tag{4.12}$$

Now combining (4.11) and (4.12) with the help of Lemmas (4.1) – (4.4) and then using $\overline{N}(r,\infty;\mathcal{L})=O(\log r)$, and (4.10) we get,

$$\begin{aligned}&(n+1)\{T(r,f)+T(r,\mathcal{L})\}\leq\overline{N}(r,1;\mathcal{F})+\overline{N}(r,1;\mathcal{G})+\sum_{i=1}^{2}\left[\overline{N}(r,\eta_i;f)+\overline{N}(r,\eta_i;\mathcal{L})\right]\\&+[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})]-\overline{N}_0(r,0;f')-\overline{N}_0(r,0;\mathcal{L}')+S(r,f)+S(r,\mathcal{L})\\\leq\ &N_E^{1)}(r,1;\mathcal{F})+(1-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+N(r,1;F)+\\&\sum_{i=1}^{2}[\overline{N}(r,n_i;f)+\overline{N}(r,\eta_i;\mathcal{L})]+\overline{N}(r,\infty;f)-\overline{N}_0(r,0;f')-\overline{N}_0(r,0;\mathcal{L})+\ O(\log r)\end{aligned}$$

$$
\begin{aligned}
(4.13)\leq\quad & \left[\overline{N}\left(r,\eta_2;f\right)+\overline{N}\left(r,\eta_2;\mathcal{L}\right)\right]+\overline{N}(r,\infty;f)+\overline{N}_*(r,\eta_1;f,\mathcal{L})\\
& +\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;\mathcal{L}')+(1-t)\\
& \overline{N}_*(r,1;\mathcal{F},\mathcal{G})+N(r,1;\mathcal{F})+\sum_{i=1}^{2}[\overline{N}(r,n_i;f)+\overline{N}(r,\eta_i;\mathcal{L})]\\
& +\overline{N}(r,\infty;f)-\overline{N}_0(r,0;f')-\overline{N}_0(r,0;\mathcal{L}')+O(\log r)\\
\leq\quad & 2\left[\overline{N}\left(r,\eta_2;f\right)+\overline{N}\left(r,\eta_2;\mathcal{L}\right)\right]+2\overline{N}(r,\infty;f)+3\overline{N}(r,\eta_1;f)\\
& +N(r,1;\mathcal{F})+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r)\\
& \leq 2\{T(r,f)+T(r,\mathcal{L})\}+2\overline{N}(r,\infty;f)+\frac{3}{n-2}[\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
& +\overline{N}(r,\infty;f)]+nT(r,f)+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r)\\
& \leq 2\{T(r,f)+T(r,\mathcal{L})\}+(2+\frac{3}{n-2})\overline{N}(r,\infty;f)+\frac{3}{n-2}\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
& +nT(r,f)+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r)\\
& \leq (2+n)T(r,f)+(2+\frac{3}{n-2})\overline{N}(r,\infty;f)+2T(r,\mathcal{L})\\
& +(2+\frac{3}{n-2}-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& nT(r,\mathcal{L})\leq T(r,f)+T(r,\mathcal{L})+(2+\tfrac{3}{n-2})\overline{N}(r,\infty;f)+\\
& \quad +(2+\tfrac{3}{n-2}-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\ O(\log r).
\end{aligned}
\tag{4.14}
$$

In a similar manner, we get

$$
\begin{aligned}
& nT(r,f)\leq T(r,f)+T(r,\mathcal{L})+(2+\tfrac{3}{n-2})\overline{N}(r,\infty;f)+\\
& \quad +(2+\tfrac{3}{n-2}-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\ O(\log r).
\end{aligned}
\tag{4.15}
$$

Let $T(r)=\max\{T(r,f)\ ,\ T(r,\mathcal{L})\}$. Then from (4.14) and (4.15) we get,

$$
\begin{aligned}
nT(r)\ &\leq\ 2T(r)+(2+\tfrac{3}{n-2})\overline{N}(r,\infty;f)+\\
& +(2+\tfrac{3}{n-2}-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r).
\end{aligned}
\tag{4.16}
$$

Again from (4.13) we get

$$(n+1)\{T(r,f)+T(r,\mathcal{L})\}\leq\left[\overline{N}\left(r,\eta_2;f\right)+\overline{N}\left(r,\eta_2;\mathcal{L}\right)\right]+\overline{N}(r,\infty;f)$$

$$
\begin{aligned}
&+\overline{N}_*(r,\eta_1;f,\mathcal{L})+\ \overline{N}_*(r,1;\mathcal{F},\mathcal{G})+(1-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+N(r,1;\mathcal{F})\\
&+\sum_{i=1}^{2}[\overline{N}(r,n_i;f)+\overline{N}(r,\eta_i;\mathcal{L})]+\overline{N}(r,\infty;f)+O(\log r)\\
\leq\ &\left[\overline{N}\,(r,\eta_2;f)+\overline{N}\,(r,\eta_2;\mathcal{L})\right]+\overline{N}(r,\infty;f)+\overline{N}_d(r,\eta_1;f)\\
&+\overline{N}_d(r,\eta_1;\mathcal{L})+\ \overline{N}_*(r,1;\mathcal{F},\mathcal{G})+(1-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
&+N(r,1;\mathcal{F})+[\overline{N}(r,n_2;f)+\overline{N}(r,\eta_2;\mathcal{L})]+\overline{N}(r,n_1;f)\\
&+\overline{N}(r,\eta_1;\mathcal{L})+\overline{N}(r,\infty;f)+O(\log r)\\
\leq\ &2\{T(r,f)+T(r,\mathcal{L})\}+2\overline{N}(r,\infty;f)+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
&+nT(r,f)+N_2(r,\eta_1;f)+N_2(r,\eta_1;\mathcal{L})+O(\log r)\\
\leq\ &(n+3)T(r,f)+3T(r,\mathcal{L})+2\overline{N}(r,\infty;f)\\
&+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r)
\end{aligned}
$$

$$
\begin{aligned}
nT(r,\mathcal{L})\leq 2T(r,f)+2T(r,\mathcal{L})+2\overline{N}(r,\infty;f)+\\
+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r).
\end{aligned}
\tag{4.17}
$$

In a similar manner, we get

$$
\begin{aligned}
nT(r,f)\leq 2T(r,f)+2T(r,\mathcal{L})+2\overline{N}(r,\infty;f)+\\
+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r).
\end{aligned}
\tag{4.18}
$$

Then combining (4.17) and (4.18) we get

$$
nT(r)\ \leq\ 4T(r)+2\overline{N}(r,\infty;f)+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r). \tag{4.19}
$$

Subcase 2.1. When $t\geq 5$ or $t=4$ with $n\geq 4$, we get from (4.16) that

$$
nT(r)\ \leq\ 2T(r)+(2+\frac{3}{n-2})\overline{N}(r,\infty;f)+O(\log r);
$$

i.e.,

$$
nT(r)\ \leq\ (2+(2+\frac{3}{n-2})(1-\Theta(\infty;f))+\epsilon)T(r)+O(\log r),
$$

which is a contradiction for $n>2+(2+\frac{3}{n-2})(1-\Theta(\infty;f))$.

When $t=4$ and $n=3$, we get from (4.16) that

$$
3T(r)\ \leq\ 2T(r)+5\overline{N}(r,\infty;f)+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r). \tag{4.20}
$$

Now from (4.8) we get, $\overline{N}_*(r,1;F,G)\leq\overline{N}(r,\infty;f)+O(\log r)$. Hence (4.20) gives $T(r)\ \leq\ 6(1-\Theta(\infty;f)+\epsilon)T(r)+O(\log r)$, which is a contradiction as $\Theta(\infty;f)>\frac{5}{6}$.

Subcase 2.2. When $t=3$ and $n\geq 5$, we get from (4.16) that

$$
nT(r)\ \leq\ 2T(r)+(2+\frac{3}{n-2})\overline{N}(r,\infty;f)+O(\log r);
$$

i.e.,

$$nT(r) \;\le\; (2+(2+\frac{3}{n-2})\,(1-\Theta(\infty;f)+\epsilon))T(r)+O(\log r),$$

which is a contradiction for $n>2+(2+\frac{3}{n-2})(1-\Theta(\infty;f))$.

When $n=4$, we get from (4.16) that

$$4T(r) \;\le\; 2T(r)+\frac{7}{2}\overline{N}(r,\infty;f)+\frac{1}{2}\overline{N}_{*}(r,1;\mathcal{F},\mathcal{G})+O(\log r). \tag{4.21}$$

Now using (4.8) in (4.21) we get,

$$4T(r) \;\le\; 2T(r)+\left(\frac{7}{2}+\frac{1}{3}\right)\overline{N}(r,\infty;f)+O(\log r);$$

i.e.,

$$4(r) \;\le\; 2T(r)+\frac{23}{6}(1-\Theta(\infty;f)+\epsilon)T(r)+O(\log r),$$

which is a contradiction for $\Theta(\infty;f)>\frac{11}{23}$.
When $t=3$ and $n=3$ from (4.16) we get

$$3T(r) \;\le\; 2T(r)+5\overline{N}(r,\infty;f)+2\overline{N}_{*}(r,1;\mathcal{F},\mathcal{G})+O(\log r).$$

Now using (4.8) we get $3T(r)\le 2T(r)+8\overline{N}(r,\infty;f)+O(\log r)$; i.e.,

$$3T(r)\le 2T(r)+8(1-\Theta(\infty;f)+\epsilon)T(r)+O(\log r);$$

which is a contradiction for $\Theta(\infty;f)>\frac{7}{8}$.

Subcase 2.3. When $t=2$, from (4.8) we get

$$(3n-8)\overline{N}_{*}(r,1;\mathcal{F},\mathcal{G})\le n\overline{N}(r,\infty;f)+O(\log r).$$

Since $(3n-8)>0$, we get

$$\overline{N}_{*}(r,1;\mathcal{F},\mathcal{G})\le \frac{n}{3n-8}\overline{N}(r,\infty;f)+O(\log r). \tag{4.22}$$

Now from (4.16) using (4.22) we get

$$\begin{aligned} nT(r) &\le 2T(r)+(2+\frac{3}{n-2})\overline{N}(r,\infty;f)+\frac{3}{n-2}\overline{N}_{*}(r,1;\mathcal{F},\mathcal{G})+O(\log r)\\ &\le \left(2+\left(2+\frac{12}{3n-8}\right)(1-\Theta(\infty;f)+\epsilon)\right)T(r)+O(\log r), \end{aligned}$$

which is a contradiction for $n>2+\left(2+\frac{12}{3n-8}\right)(1-\Theta(\infty;f)$.

Subcase 2.4. When $t=1$, $n\ge 5$ then $(2n-6)>0$.
Therefore from (4.8) we get

$$(2n-6)\overline{N}_{*}(r,1;\mathcal{F},\mathcal{G})\le n\overline{N}(r,\infty;f)+O(\log r). \tag{4.23}$$

Now from (4.16) using (4.23), we get

$$\begin{aligned} nT(r) &\leq 2T(r)+(2+\frac{3}{n-2})\overline{N}(r,\infty;f)+(1+\frac{3}{n-2})\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r), \\ &\leq 2T(r)+\left(2+\frac{1}{2}+\frac{6}{n-3}\right)(1-\Theta(\infty;f)+\epsilon)\,T(r)+O(\log r) \end{aligned}$$

which is a contradiction for $n>2+\frac{5}{2}+(\frac{6}{n-3})(1-\Theta(\infty;f))$.

When $t=1$ and $n=4$, we get from (4.8) that

$$\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\leq 2\overline{N}(r,\infty;f)+O(\log r). \tag{4.24}$$

Now from (4.16) using (4.24), we get

$$\begin{aligned} 4T(r) &\leq 2T(r)+\frac{7}{2}\overline{N}(r,\infty;f)+\frac{5}{2}\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r) \\ &\leq 2T(r)+\frac{17}{2}(1-\Theta(\infty;f)+\epsilon)T(r)+O(\log r), \end{aligned}$$

which is a contradiction for $\Theta(\infty;f)>\frac{13}{17}$.

Subcase 2.5. When $t=0$, $n\geq 5$ then $(n-4)>0$.
Therefore from (4.8), we get

$$(n-4)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\leq n\overline{N}(r,\infty;f)+O(\log r). \tag{4.25}$$

Now from (4.16) using (4.25), we get

$$\begin{aligned} nT(r) &\leq 2T(r)+(2+\frac{3}{n-2})\overline{N}(r,\infty;f)+(2+\frac{3}{n-2})\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r) \\ &\leq 2T(r)+\left(4+\frac{14}{n-4}\right)(1-\Theta(\infty;f)+\epsilon)T(r)+O(\log r), \end{aligned}$$

which is a contradiction for $n>2+\left(4+\frac{14}{n-4}\right)(1-\Theta(\infty;f))$.

Next suppose that $\eta=\eta_2(\neq 0)$. Hence $q_2=1$. Then proceeding similarly as we have done above for (4.16) and (4.19) we can easily get the following

$$nT(r)\ \leq\ 2T(r)+5\overline{N}(r,\infty;f)+(5-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r) \tag{4.26}$$

and

$$nT(r)\ \leq\ 4T(r)+2\overline{N}(r,\infty;f)+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r). \tag{4.27}$$

Subcase 2.6. When $t\geq 2$, we get from (4.27) that

$$nT(r)\ \leq\ 4T(r)+2\overline{N}(r,\infty;f)+O(\log r);$$

i.e.,

$$nT(r)\ \leq\ (4+2(1-\Theta(\infty;f))+\epsilon)T(r)+O(\log r),$$

which is a contradiction for $n > 4 + 2(1 - \Theta(\infty; f))$.

Subcase 2.7. When $t = 1$, then from (4.7) we get

$$\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) \le \overline{N}(r, \eta; f) + \frac{1}{2}\overline{N}(r, \infty; f) + O(\log r). \tag{4.28}$$

Now from (4.27) using (4.28), we get

$$\begin{aligned} nT(r) &\le 4T(r) + 2\overline{N}(r, \infty; f) + \overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + O(\log r) \\ &\le 5T(r) + \frac{5}{2}\left(1 - \Theta(\infty; f) + \epsilon\right) T(r) + O(\log r), \end{aligned}$$

which is a contradiction for $n > 5 + \frac{5}{2}(1 - \Theta(\infty; f))$.

Subcase 2.8. When $t = 0$, then from (4.7) we get

$$\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) \le 2\overline{N}(r, \eta; f) + \overline{N}(r, \infty; f) + O(\log r). \tag{4.29}$$

Now from (4.27) using (4.29), we get

$$\begin{aligned} nT(r) &\le 4T(r) + 2\overline{N}(r, \infty; f) + 2\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + O(\log r) \\ &\le 8T(r) + 4(1 - \Theta(\infty; f) + \epsilon)T(r) + O(\log r), \end{aligned}$$

which is a contradiction for $n > 8 + 4(1 - \Theta(\infty; f))$.
Thus we see from above that $H \equiv 0$. Hence on integration, we obtain

$$\frac{1}{\mathcal{F} - 1} = \frac{A}{\mathcal{G} - 1} + B,$$

where $A(\neq 0), B \in \mathbb{C}$. □

Lemma 4.7. *Let $\mathcal{F}$ and $\mathcal{G}$ be defined by* (4.9)*, then $\mathcal{F}\mathcal{G} \neq a$, where a is non-zero complex constant.*

Proof. On the contrary, suppose that $\mathcal{F}\mathcal{G} = \zeta \neq 0$. Then

$$f^{n-1}(f + a)\mathcal{L}^{n-1}(\mathcal{L} + a) = \zeta b^2 = \zeta_1(say) \neq 0. \tag{4.30}$$

Let $\alpha_1 = 0$ and $\alpha_2 = -a$. Then it is clear from (4.30) that each α_i-point of f is a pole of $\mathcal{L}$ and vice-versa.

Let z_0 be a α_2 point of $\mathcal{L}$ of multiplicity r, then it will be a pole of f of multiplicity ν, such that $r = \nu n$. Since $\nu \ge 1$, so $r \ge n$; i.e., $\frac{1}{r} \le \frac{1}{n}$. Similar argument can be made for α_1 point of $\mathcal{L}$. Now using the second fundamental theorem in view of

$\overline{N}(r,\infty;\mathcal{L})=O(\log r)$ we get

$$\begin{aligned}T(r,\mathcal{L}) &\leq \overline{N}(r,\alpha_1;\mathcal{L})+\overline{N}(r,\alpha_2;\mathcal{L})+\overline{N}(r,\infty;\mathcal{L})+S(r,\mathcal{L})\\ &\leq \frac{2}{n}T(r,\mathcal{L})+O(\log r),\end{aligned}$$

which is a contradiction as $n\geq 3$. □

5. Proof Of the theorems

Proof of Theorem 3.1 Let f be a non-constant meromorphic function and $\mathcal{L}$ be a non-constant L-function. Suppose $E_f(S,t)=E_{\mathcal{L}}(S,t)$ and $E_f(\eta,0)=E_{\mathcal{L}}(\eta,0)$ where $t\in\mathbb{N}\cup\{0\}$ and η is the zero of $P'(z)$. Consider $\mathcal{F}$ and $\mathcal{G}$ as defined by (4.9).

Therefore in view of the Lemma 4.6 we get

$$\frac{1}{\mathcal{F}-1}=\frac{A}{\mathcal{G}-1}+B, \tag{5.1}$$

where $A(\neq 0), B\in\mathbb{C}$. Hence we have

$$T(r,\mathcal{F})=T(r,\mathcal{G})+O(1). \tag{5.2}$$

Since

$$T(r,\mathcal{F})=nT(r,f)+O(1)\ \ \textit{and}\ \ T(r,\mathcal{G})=nT(r,\mathcal{L})+O(1). \tag{5.3}$$

So (5.2) implies that

$$T(r,f)=T(r,\mathcal{L})+O(1). \tag{5.4}$$

Case 1. If $B\neq 0$. Then from (5.1) we get,

$$\mathcal{F}=\frac{(B+1)\mathcal{G}+(A-B-1)}{B\mathcal{G}+(A-B)}. \tag{5.5}$$

Subcase 1.1. If $B\neq -1$. Then from (5.5) we get,

$$\mathcal{F}=\frac{(B+1)\left(\mathcal{G}-\frac{B-A+1}{B+1}\right)}{B\left(\mathcal{G}-\frac{B-A}{B}\right)}. \tag{5.6}$$

Now clearly, $\frac{B-A+1}{B+1}\neq\frac{B-A}{B}$, as if $\frac{B-A+1}{B+1}=\frac{B-A}{B}$ then $A=0$, which is absurd.

Subcase 1.1.1. If $B-A\neq 0$. Then from (5.6) it is clear that $\overline{N}\left(r,\frac{B-A}{B};\mathcal{G}\right)=\overline{N}(r,\infty;\mathcal{F})$.

Now using second fundamental theorem, (5.4) and $\overline{N}(r,\infty;\mathcal{L}) = O(\log r)$ we get,

$$\begin{aligned} T(r,\mathcal{G}) &\leq \overline{N}(r,0;\mathcal{G}) + \overline{N}\left(r,\frac{B-A}{B};\mathcal{G}\right) + \overline{N}(r,\infty;\mathcal{G}) + S(r,\mathcal{G}) \\ &\leq \overline{N}(r,0;\mathcal{L}) + \overline{N}(r,-a;\mathcal{L}) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;\mathcal{L}) + S(r,\mathcal{G}) \\ &\leq 2T(r,\mathcal{L}) + (1-\Theta(\infty;f)+\epsilon)T(r,f) + S(r,\mathcal{G}) \\ &\leq (3-\Theta(\infty;f)+\epsilon)T(r,\mathcal{L}) + S(r,\mathcal{G}) \\ &\leq \left(\frac{3-\Theta(\infty;f)+\epsilon}{n}\right)T(r,\mathcal{G}) + S(r,\mathcal{G}), \end{aligned}$$

which is a contradiction.

Subcase 1.1.2. If $B - A = 0$. Then from (5.6) we get,

$$\mathcal{F} = \frac{(B+1)\left(\mathcal{G} - \frac{1}{B+1}\right)}{B\mathcal{G}}. \tag{5.7}$$

Now it is clear from (5.7) that $\overline{N}\left(r,\frac{1}{B+1};\mathcal{G}\right) = \overline{N}(r,0;\mathcal{F})$ and $\overline{N}(r,0;\mathcal{G}) = \overline{N}(r,\infty;\mathcal{F})$.
Now using second fundamental theorem, (5.4) and $\overline{N}(r,\infty;\mathcal{L}) = O(\log r)$ we get,

$$\begin{aligned} T(r,\mathcal{G}) &\leq \overline{N}(r,0;\mathcal{G}) + \overline{N}\left(r,\frac{1}{B+1};\mathcal{G}\right) + \overline{N}(r,\infty;\mathcal{G}) + S(r,\mathcal{G}) \\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,-a;f) + \overline{N}(r,\infty;\mathcal{L}) + S(r,\mathcal{G}) \\ &\leq 2T(r,f) + (1-\Theta(\infty;f)+\epsilon)T(r,f) + S(r,\mathcal{G}) \\ &\leq (3-\Theta(\infty;f)+\epsilon)T(r,\mathcal{L}) + S(r,\mathcal{G}) \\ &\leq \left(\frac{3-\Theta(\infty;f)+\epsilon}{n}\right)T(r,\mathcal{G}) + S(r,\mathcal{G}), \end{aligned}$$

which is a contradiction.

Subcase 1.2. If $B = -1$. Then from (5.5) we get,

$$\mathcal{F} = \frac{A}{-\mathcal{G} + A + 1}. \tag{5.8}$$

Subcase 1.2.1. If $A \neq -1$. Then from (5.8) it is clear that $\overline{N}\left(r,(A+1);\mathcal{G}\right) = \overline{N}(r,\infty;\mathcal{F})$.
Now using second fundamental theorem, (5.4) and $\overline{N}(r,\infty;\mathcal{L}) = O(\log r)$ we get,

$$\begin{aligned} T(r,\mathcal{G}) &\leq \overline{N}(r,0;\mathcal{G}) + \overline{N}\left(r,(A+1);\mathcal{G}\right) + \overline{N}(r,\infty;\mathcal{G}) + S(r,\mathcal{G}) \\ &\leq \overline{N}(r,0;\mathcal{L}) + \overline{N}(r,-a;\mathcal{L}) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;\mathcal{L}) + S(r,\mathcal{G}) \\ &\leq (3-\Theta(\infty;f)+\epsilon)T(r,\mathcal{L}) + S(r,\mathcal{G}) \\ &\leq \left(\frac{3-\Theta(\infty;f)+\epsilon}{n}\right)T(r,\mathcal{G}) + S(r,\mathcal{G}), \end{aligned}$$

which is a contradiction.

Subcase 1.2.2. If $A=-1$. Then from (5.8) we get,

$$\mathcal{F}\mathcal{G}=1,$$

which is a contradiction in view of *Lemma (4.7)*.

Case 2. If $B=0$. Then from (5.1) we get,

$$\mathcal{G}-1=A(\mathcal{F}-1). \tag{5.9}$$

Subcase 2.1. $A\neq 1$. Then from (5.9) we get,

$$A\mathcal{F}=\mathcal{G}-(1-A). \tag{5.10}$$

Suppose η is not an e.v.P. of f and $\mathcal{L}$. Then there exists z_0 such that $f(z_0)=\mathcal{L}(z_0)=\eta$.
Let $\xi=-\frac{1}{b}\eta^{n-1}(\eta+a)$. Then clearly $F(z_0)=G(z_0)=\xi$ and since $P(z)$ has only simple zeros, we have $\xi\neq 1$.
Now from (5.10) we get,

$$(\xi-1)(A-1)=0,$$

which is a contradiction.
Now let η be an e.v.P of $\mathcal{L}$ and hence it will be an e.v.P of f also.

Subcase 2.1.1. Suppose that $\eta=0$. Then 0 is an e.v.P of f and $\mathcal{L}$.
Again, it is clear from (5.10) that $\overline{N}\left(r,(1-A);\mathcal{G}\right)=\overline{N}(r,0;\mathcal{F})$.
Now using second fundamental theorem, (5.4) and $\overline{N}(r,\infty;\mathcal{L})=O(\log r)$ we get,

$$\begin{aligned}
T(r,\mathcal{G}) &\leq \overline{N}(r,0;\mathcal{G})+\overline{N}\left(r,(1-A);\mathcal{G}\right)+\overline{N}(r,\infty;\mathcal{G})+S(r,\mathcal{G})\\
&\leq \overline{N}(r,0;\mathcal{L})+\overline{N}(r,-a;\mathcal{L})+\overline{N}(r,0;f)+\overline{N}(r,-a;f)+\overline{N}(r,\infty;\mathcal{L})+S(r,\mathcal{G})\\
&\leq T(r,\mathcal{L})+T(r,f)+S(r,\mathcal{G})\leq\left(\frac{2}{n}\right)T(r,\mathcal{G})+S(r,\mathcal{G}),
\end{aligned}$$

which is a contradiction as $n\geq 3$.

Subcase 2.1.2. let $\eta\neq 0$, then using second fundamental theorem, (5.4) and $\overline{N}(r,\infty;\mathcal{L})=O(\log r)$ we get,

$$\begin{aligned}
T(r,\mathcal{G}) &\leq \overline{N}(r,0;\mathcal{G})+\overline{N}\left(r,(1-A);\mathcal{G}\right)+\overline{N}(r,\infty;\mathcal{G})+S(r,\mathcal{G})\\
&\leq \overline{N}(r,0;\mathcal{L})+\overline{N}(r,-a;\mathcal{L})+\overline{N}(r,0;f)+\overline{N}(r,-a;f)+\overline{N}(r,\infty;\mathcal{L})+S(r,\mathcal{G})\\
&\leq 2T(r,\mathcal{L})+2T(r,f)+S(r,\mathcal{G})\leq\left(\frac{4}{n}\right)T(r,\mathcal{G})+S(r,\mathcal{G}),
\end{aligned}$$

which is a contradiction as $n\geq 5$.

Subcase 2.2. $A = 1$ and hence $\mathcal{F} = \mathcal{G}$. That is, we get

$$-\frac{1}{b}f^{n-1}(f+a) = -\frac{1}{b}\mathcal{L}^{n-1}(\mathcal{L}+a) \tag{5.11}$$

$$\implies (f^n - \mathcal{L}^n) + a(f^{n-1} - \mathcal{L}^{n-1}) = 0. \tag{5.12}$$

Let $h = \frac{f}{\mathcal{L}}$. Then from (5.11) we get,

$$\mathcal{L}(h^n - 1) + a(h^{n-1} - 1) = 0. \tag{5.13}$$

If $h \neq 1$, then we can write (5.13) as

$$\mathcal{L} = -a\frac{(h-v)(h-v^2)...(h-v^{n-2})}{(h-u)(h-u^2)...(h-u^{n-1})}, \tag{5.14}$$

where $u = \exp(2\pi i/n)$ and $v = \exp(2\pi i/(n-1))$. Noting that n and $(n-1)$ are relatively prime positive integers, then the numerator and denominator of (5.14) have no common factors. Since $\mathcal{L}$ can have atmost one pole in the complex plane, hence whenever $n \geq 3$ we can see that there exists at least one distinct roots of $h^n = 1$ such that they are Picard exceptional values of h.

Subcase 2.2.1. When $\eta = 0$; i.e., f and $\mathcal{L}$ share $(0, 0)$, then from (5.11) it is clear that f and $\mathcal{L}$ have same zeros and poles with counting multiplicity. Therefore, h is an entire function with no zeros; i.e., when $n \geq 3$ there are at least two Picard exceptional value of h, and so it follows by (5.14) that h and thus $\mathcal{L}$ are constants, which is impossible.

Therefore, we must have $h = 1$; i.e., $f = \mathcal{L}$.

Subcase 2.2.2. When $\eta \neq 0$, for $n \geq 5$ there are at least three Picard exceptional value of h, and so it follows by (5.14) that h and thus $\mathcal{L}$ are constants, which is impossible.

Therefore, we must have $h = 1$; i.e., $f = \mathcal{L}$.

Proof of Corollary 3.1 If f be a meromorphic function having finitely many poles, then we have

$$\Theta(\infty; f) = 1. \tag{5.15}$$

Therefore using (5.15), the desired results follow from the proofs of Theorem 3.1.

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UNICITY OF MEROMORPHIC FUNCTIONS CONCERNING DIFFERENTIAL-DIFFERENCE POLYNOMIALS

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Abstract. In this paper, we study unicity of meromorphic functions concerning differential-difference polynomials and mainly prove: Let $k_1, k_2, \cdots, k_n$ be non-negative integers and $k = \max\{k_1, k_2, \cdots, k_n\}$, let l be the number of distinct values of $\{0, c_1, c_2, \cdots, c_n\}$, let s be the number of distinct values of $\{c_1, c_2, \cdots, c_n\}$, let $f(z)$ be a non-constant meromorphic function of finite order satisfying $N(r,f) \le \frac{1}{8(lk+l+2s-1)+1} T(r,f)$, let $m_1(z), m_2(z), \cdots, m_n(z)$, $a(z), b(z)$ be small functions of $f(z)$ such that $a(z) \not\equiv b(z)$, let $(c_1,k_1), (c_2,k_2), \cdots, (c_n,k_n)$ be distinct and let $F(z) = m_1(z)f^{(k_1)}(z+c_1) + m_2(z)f^{(k_2)}(z+c_2) + \cdots + m_n(z)f^{(k_n)}(z+c_n)$. If $f(z)$ and $F(z)$ share $a(z), b(z)$ CM, then $f(z) \equiv F(z)$. Our results improve and extend some results due to [1, 18, 20].

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1. Introduction and main results

In this paper, a meromorphic function always means meromorphic in the whole complex plane. We use the following standard notations in value distribution theory, see [7, 15, 16]: $T(r,f), N(r,f), m(r,f), \cdots$.

We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ possible outside of an exceptional set E with finite logarithmic measure $\int_E dr/r < \infty$. A meromorphic function $\alpha(z)$ is said to be a small function of $f(z)$ if it satisfies $T(r,\alpha) = S(r,f)$.

Let $\alpha(z)$ be a small function of both $f(z)$ and $g(z)$. If $f(z)-\alpha(z)$ and $g(z)-\alpha(z)$ have the same zeros counting multiplicities (ignoring multiplicities), then we call that $f(z)$ and $g(z)$ share $\alpha(z)$ CM (IM). Let $N(r,\alpha)$ be the counting function of common zeros of both $f(z)-\alpha(z)$ and $g(z)-\alpha(z)$ with counting multiplicities. If

$$N\left(r, \frac{1}{f-\alpha}\right) + N\left(r, \frac{1}{g-\alpha}\right) - 2N(r,\alpha) \le S(r,f) + S(r,g),$$

then we call that $f(z)$ and $g(z)$ share $\alpha(z)$ CM almost.

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Let $f(z)$ be a non-constant meromorphic function. Define

$$\rho(f)=\overline{\lim_{r\to\infty}}\frac{\log^{+}T(r,f)}{\log r}$$

by the order of $f(z)$.

For a nonzero complex constant $\eta\in\mathbb{C}$, we define the difference operators of $f(z)$ as $\triangle_\eta f(z)=f(z+\eta)-f(z)$ and $\triangle_\eta^k f(z)=\triangle_\eta(\triangle_\eta^{k-1}f(z)), k\in\mathbb{N}, k\geq 2$.

Let $f(z)$ be a non-constant meromorphic function, let $n_0,n_1,\cdots,n_k$ be non-negative integers, let $c_0,c_1,\cdots,c_k$ be finite values, we call that $M(f)=f^{n_0}(z+c_0)(f')^{n_1}(z+c_1)\cdots(f^{(k)})^{n_k}(z+c_k)$ is a differential-difference monomial, and its degree $\gamma_M=n_0+n_1+\cdots+n_k$. Let $H=a_1M_1(f)+a_2M_2(f)+\cdots+a_nM_n(f)$ be a homogeneous differential-difference polynomial, where $a_1(z),a_2(z),\cdots,a_n(z)$ are small functions of $f(z)$ and $\gamma_{M_1}=\gamma_{M_2}=\cdots=\gamma_{M_n}$.

Let $N_{k)}(r,f)$ be the counting function for poles of $f(z)$ with multiplicity $\leq k$ and let $N_{(k}(r,f)$ be the counting function for poles of $f(z)$ with multiplicity $\geq k$.

Nevanlinna [7, 15, 16] proved the famous five-value theorem.

Theorem A. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let $a_j(j=1,2,\cdots,5)$ be five distinct values on extend complex plane. If $f(z)$ and $g(z)$ share $a_j(j=1,2,\cdots,5)$ IM, then $f(z)\equiv g(z)$.*

Li and Qiao[11] proved the five-small function theorem.

Theorem B. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let $a_j(z)(j=1,2,\cdots,5)$ be five distinct small functions of both $f(z)$ and $g(z)$ (one may be ∞). If $f(z)$ and $g(z)$ share $a_j(z)(j=1,2,\cdots,5)$ IM, then $f(z)\equiv g(z)$.*

In 1976, Rubel and Yang[14] proved the following result.

Theorem C. *Let $f(z)$ be a non-constant entire function, and let a,b be two distinct finite values. If $f(z)$ and $f'(z)$ share a,b CM, then $f(z)\equiv f'(z)$.*

In 1992, Zheng and Wang[19] proved:

Theorem D. *Let $f(z)$ be a non-constant entire function, and let $a(z),b(z)$ be two distinct small functions of $f(z)$. If $f(z)$ and $f'(z)$ share $a(z),b(z)$ CM, then $f(z)\equiv f'(z)$.*

In 1995, Fang[5] proved the following theorem.

Theorem E. *Let $f(z)$ be a non-constant meromorphic function such that $N(r,f)=S(r,f)$, let n be a positive integer, let a,b be two distinct finite complex values, and let $F(z)=f^{(n)}(z)+a_1(z)f^{(n-1)}(z)+\cdots+a_n(z)f(z)$, where $a_1(z),a_2(z),\cdots,a_n(z)$ are small functions of $f(z)$. If $f(z)$ and $F(z)$ share a,b CM almost, then $f(z)\equiv F(z)$.*

In 2006, Chen[1] studied the case of meromorphic function satisfying $N(r,f)\leq\frac{1}{8n+17}T(r,f)$, and proved the following result.

Theorem F. *Let n be a positive integer, let $f(z)$ be a non-constant meromorphic function satisfying $N(r,f) \leq \frac{1}{8n+17}T(r,f)$, and let $a(z), b(z)$ be two distinct small functions of $f(z)$, and let $F(z) = f^{(n)}(z) + a_1(z)f^{(n-1)}(z) + \cdots + a_n(z)f(z)$, where $a_1(z), a_2(z), \cdots, a_n(z)$ are small functions of $f(z)$. If $f(z)$ and $F(z)$ share $a(z), b(z)$ CM, then $f(z) \equiv F(z)$.*

Recently, a number of articles focused on value distribution in shifts or difference operators of meromorphic functions. In particular, some papers studied the unicity of meromorphic functions sharing values with their shifts or difference operators (see [3, 4, 6, 9, 12, 13, 20]).

In 2011, Heittokangas et al.[9] proved the following result.

Theorem G. *Let $f(z)$ be a non-constant entire function of finite order, let η be a nonzero constant, and let a, b be two distinct finite values. If $f(z)$ and $f(z+\eta)$ share a, b CM, then $f(z) \equiv f(z+\eta)$.*

In 2014, Zhang and Liao[20] proved the following result.

Theorem H. *Let $f(z)$ be an entire function of finite order, let η be a nonzero constant, and let a, b be two distinct finite values. If $f(z)$ and $\triangle_\eta f(z)$ share a, b CM, then $f(z) \equiv \triangle_\eta f(z)$.*

Liu et al.[12] replaced $\triangle_\eta f(z)$ by the general difference polynomial and proved the following result:

Theorem I. *Let $f(z)$ be a non-constant entire function of finite order, let n be a positive integer, let $a(z)$, $b(z)$ be be two distinct small functions of $f(z)$, and let $F(z) = m_1 f(z+c_1) + m_2 f(z+c_2) + \cdots + m_n f(z+c_n)$, where $m_1, m_2, \cdots, m_n$ are nonzero complex numbers and $c_1, c_2, \cdots, c_n$ are distinct finite values. If $f(z)$ and $F(z)$ share $a(z), b(z)$ CM, then $f(z) \equiv F(z)$.*

In 2017, Yang and Liu[18] extended Theorem J and proved the following theorem.

Theorem J. *Let $f(z)$ be a non-constant meromorphic function of finite order, let n be a positive integer, let $a(z), b(z)$ be two distinct small functions of $f(z)$, let $m_1, m_2, \cdots, m_n$ be nonzero complex numbers, let $c_1, c_2, \cdots, c_n$ be distinct finite complex numbers, and let*

$$F(z) = m_1 f(z+c_1) + m_2 f(z+c_2) + \cdots + m_n f(z+c_n).$$

If $f(z)$ and $F(z)$ share $a(z)$, $b(z)$ CM almost and $N(r,f) \leq \frac{1}{27n}T(r,f)$, then $f(z) \equiv F(z)$.

In this paper, we extend and improve the above results.

Theorem 1.1. *Let $k_1, k_2, \cdots, k_n$ be non-negative integers and $k = \max\{k_1, k_2, \cdots, k_n\}$, let l be the number of distinct values of $\{0, c_1, c_2, \cdots, c_n\}$, let s be the number of distinct values of $\{c_1, c_2, \cdots, c_n\}$, let $f(z)$ be a non-constant meromorphic function* *of finite order satisfying* $N(r,f) \leq \frac{1}{8(lk+l+2s-1)+1}T(r,f)$, *let* $m_1(z)$, $m_2(z)$, $\cdots$, $m_n(z)$, $a(z), b(z)$ *be small functions of* $f(z)$ *such that* $a(z) \not\equiv b(z)$, *let* (c_1,k_1),(c_2,k_2), $\cdots$, (c_n,k_n) *be distinct and let*

$$F(z) = m_1(z)f^{(k_1)}(z+c_1) + m_2(z)f^{(k_2)}(z+c_2) + \cdots + m_n(z)f^{(k_n)}(z+c_n). \tag{1.1}$$

If $f(z)$ *and* $F(z)$ *share* $a(z), b(z)$ *CM, then* $f(z) \equiv F(z)$.

Remark 1.1. Let $l = s = 1, k = n, 0 = c_1 = c_2 = \cdots = c_n$, then Theorem 1 is also valid. If $F(z) = f^{(n)}(z)+a_1(z)f^{(n-1)}(z)+\cdots+a_n(z)f(z)$, where $a_1(z), a_2(z) \cdots, a_n(z)$ are small functions of $f(z)$. Then by Theorem 1.1 we get Theorem F.

Corollary 1.1. *Let* $k = 0$, *let* $f(z)$ *be a non-constant meromorphic function of finite order, let* n *be a positive integer, let* $a(z)$, $b(z)$ *be be two distinct small functions of* $f(z)$, *let* $m_1, m_2, \cdots, m_n$ *be nonzero complex numbers, let* $c_1, c_2, \cdots, c_n$ *be distinct finite complex numbers, and let*

$$F(z) = m_1 f(z+c_1) + m_2 f(z+c_2) + \cdots + m_n f(z+c_n).$$

If $f(z)$ *and* $F(z)$ *share* $a(z)$, $b(z)$ *CM almost and* $N(r,f) \leq \frac{1}{24n+1}T(r,f)$, *then* $f(z) \equiv F(z)$.

By Corollary 1.1, we get Theorem J.

The following example illustrates that the condition

$$N(r,f) \leq \frac{1}{8(lk+l+2s-1)+1}T(r,f)$$

is necessary in Theorem 1.1.

Example 1.1. Let $f(z) = \frac{e^z+1}{e^z-1}$ and $F(z) = f(z) - f(z+c) - f(z+2c) = -\frac{e^{z+c}+1}{e^{z+c}-1}$, where $c = \pi i$. It is easy to see that $f(z)$ and $F(z)$ share $1, -1$ CM. But $f(z) \not\equiv F(z)$.

Theorem 1.2. *Let* $F(z), l, k, s$ *be the same as Theorem 1, let* $f(z)$ *be a non-constant meromorphic function of finite order satisfying* $N_{1)}(r,f) \leq \frac{1}{5(lk+l+2s-1)}T(r,f)$, *and let* $a(z), b(z)$ *be distinct small functions of* $f(z)$. *If* $f(z)$ *and* $F(z)$ *share* $a(z), b(z), \infty$ *CM, then* $f(z) \equiv F(z)$.

The following example illustrates that the condition

$$N_{1)}(r,f) \leq \frac{1}{5(lk+l+2s-1)}T(r,f)$$

is necessary in Theorem 1.2.

Example 1.2. Let $f(z) = \frac{e^z+1}{e^z-1}$ and $F(z) = f(z) + f(z+c) - f(z+2c) - f(z+3c) - f(z+4c) = -\frac{e^z+1}{e^z-1}$, where $c = 2\pi i$. It is easy to see that $f(z)$ and $F(z)$ share $1, -1, \infty$ CM. But $f(z) \not\equiv F(z)$.

Theorem 1.3. *Let $f(z)$ be a non-constant meromorphic function of finite order, let $a(z), b(z)$ be two distinct small functions of $f(z)$, and let $H(f)$ be a homogeneous differential-difference polynomial of f with $\deg H = m$. If $f^m(z)(m \geq 2)$ and $H(f)$ share $a(z), b(z), \infty$ CM, then $f^m(z) \equiv H(f)$.*

2. Some Lemmas

For the proof of our results, we need the following lemmas.

Lemma 2.1. [7, 15, 16]. *Let $f(z)$ be a non-constant meromorphic function, and let $a_i(z)(i = 1, 2)$ be two distinct small functions of $f(z)$. Then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f).$$

Lemma 2.2. [17]. *Let $f(z)$ be a non-constant meromorphic function, and let $a_i(z)(i = 1, 2, 3)$ be three distinct small functions of $f(z)$. Then for any $0 < \varepsilon < 1$,*

$$2T(r, f) \leq \overline{N}(r, f) + \sum_{i=1}^{3} \overline{N}\left(r, \frac{1}{f - a_i}\right) + \varepsilon T(r, f) + S(r, f).$$

Lemma 2.3. [2]. *Let $f(z)$ be a non-constant meromorphic function of finite order, and let η be a non-zero finite complex number. Then*

$$N(r, f(z + \eta)) = N(r, f(z)) + S(r, f).$$

Lemma 2.4. [2, 8, 10]. *Let $f(z)$ be a non-constant meromorphic function of finite order, let k be a positive integer and let η be a non-zero finite complex number. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f), \quad m\left(r, \frac{f(z + \eta)}{f(z)}\right) = S(r, f).$$

Lemma 2.5. [5]. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions satisfying*

$$N(r, f) + N\left(r, \frac{1}{f}\right) = S(r, f), \quad N(r, g) + N\left(r, \frac{1}{g}\right) = S(r, g).$$

If $f(z)$ and $g(z)$ share 1 CM almost, then either $f(z)g(z) \equiv 1$ or $f(z) \equiv g(z)$.

Lemma 2.6. [7, 15, 16]. *Let $f(z)$ be a non-constant meromorphic function, let $n(\geq 2)$ be a positive integer, and let $a_1(z), a_2(z) \cdots a_n(z)$ be distinct small functions of $f(z)$. Then*

$$m\left(r, \frac{1}{f - a_1}\right) + \cdots + m\left(r, \frac{1}{f - a_n}\right) \leq m\left(r, \frac{1}{f - a_1} + \cdots + \frac{1}{f - a_n}\right) + S(r, f).$$

Lemma 2.7. [16]. *Let k be a positive integer and let $f(z)$ be a meromorphic function such that $f^{(k)}(z)\not\equiv 0$. Then*

$$\begin{aligned} T(r,f^{(k)}) \leq& T(r,f)+k\overline{N}(r,f)+S(r,f),\\ N\left(r,\frac{1}{f^{(k)}}\right) \leq& N\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f). \end{aligned}$$

Lemma 2.8. [1]. *Let $0\leq\lambda\leq\frac{1}{4}$ and let $f(z)$ and $g(z)$ be two meromorphic functions satisfying*

$$\overline{N}(r,f)\leq\lambda T(r,f),\ \overline{N}(r,g)\leq\lambda T(r,g).$$

If $f(z)$ and $g(z)$ share 0, 1 CM almost, and

$$\overline{\lim_{\substack{r\to\infty\\ r\in I}}}\frac{N(r,0)+N(r,1)}{T(r,f)+T(r,g)}<\frac{2-8\lambda}{3},$$

where $I\subset[0,\infty)$ is a set of infinite linear measure, then $\frac{1}{f-1}-\frac{c}{g-1}=d$, where $c(\neq 0), d$ are two constants.

By imitating the proof of Lemma 2.8, we can prove the following lemma.

Lemma 2.9. *Let $0\leq\lambda<1$ and let $f(z)$ and $g(z)$ be two meromorphic functions satisfying*

$$\overline{N}(r,f)\leq\lambda T(r,f),\ \overline{N}(r,g)\leq\lambda T(r,g).$$

If $f(z)$ and $g(z)$ share 0, 1, ∞ CM almost, and

$$\overline{\lim_{\substack{r\to\infty\\ r\in I}}}\frac{N(r,0)+N(r,1)}{T(r,f)+T(r,g)}<\frac{2-2\lambda}{3},$$

where $I\subset[0,\infty)$ is a set of infinite linear measure, then $\frac{1}{f-1}-\frac{c}{g-1}=d$, where $c(\neq 0), d$ are two constants.

Lemma 2.10. *Let k, l be non-negative integers. Then*

$$\begin{aligned} &(1)\quad \frac{9lk+25l-8}{15lk+45l-13}<\frac{16lk+48l-22}{24lk+72l-21}(k\geq 1,l\geq 1),\\ &(2)\quad \frac{9lk+25l-24}{15lk+45l-43}<\frac{16lk+48l-54}{24lk+72l-69}(k\geq 0,l\geq 2),\\ &(3)\quad \frac{6lk+16l-6}{9lk+27l-9}<\frac{10lk+30l-12}{15lk+45l-15}(k\geq 1,l\geq 1),\\ &(4)\quad \frac{6lk+16l-16}{9lk+27l-27}<\frac{10lk+30l-32}{15lk+45l-45}(k\geq 0,l\geq 2). \end{aligned}$$

3. Proof of Theorems

Proof of Theorem 1.1. Set

$$g(z) = \frac{f(z) - a(z)}{b(z) - a(z)}, \tag{3.1}$$

$$G(z) = \frac{F(z) - a(z)}{b(z) - a(z)}. \tag{3.2}$$

Since $f(z)$ and $F(z)$ share $a(z), b(z)$ CM, we know that $g(z)$ and $G(z)$ share 0, 1 CM almost.

It follows from (3.1) and (3.2) that

$$T(r, g) = T(r, f) + S(r, f), \tag{3.3}$$

$$T(r, G) = T(r, F) + S(r, f), \tag{3.4}$$

$$N(r, g) = N(r, f) + S(r, f). \tag{3.5}$$

Hence, by (1.1), (3.2) and Lemma 2.3, we get

$$N(r, G) = N(r, F) + S(r, f) \le s\left(N(r, f) + k\overline{N}(r, f)\right) + S(r, f). \tag{3.6}$$

It follows that

$$T(r, F) \le (s + sk)T(r, f) + S(r, f). \tag{3.7}$$

Hence, we obtain

$$S(r, g) = S(r, f), S(r, f) = S(r, g),$$

$$S(r, F) = S(r, f), S(r, G) = S(r, f).$$

Since $g(z)$ and $G(z)$ share 0, 1 CM almost, we have

$$\begin{aligned} N(r, 0) + N(r, 1) &\le N\left(r, \frac{1}{G - g}\right) + S(r, f) \\ &\le N\left(r, \frac{1}{F - f}\right) + S(r, f) \\ &\le T(r, F - f) + S(r, f) \\ &= m(r, F - f) + N(r, F - f) + S(r, f). \end{aligned} \tag{3.8}$$

It follows from Lemmas 2.3 and 2.4 that

$$m(r, F - f) \le m\left(r, \frac{F - f}{f}\right) + m(r, f) + S(r, f) \le m(r, f) + S(r, f), \tag{3.9}$$

$$N(r, F - f) \le lN(r, f^{(k)}) + S(r, f) \le l\left(N(r, f) + k\overline{N}(r, f)\right) + S(r, f). \tag{3.10}$$

By (3.8)-(3.10) and $N(r,f) \leq \frac{1}{8(lk+l+2s-1)+1}T(r,f)$, we obtain

$$\begin{aligned} N(r,0)+N(r,1) \leq & m(r,f)+l\left(N(r,f)+k\overline{N}(r,f)\right)+S(r,f) \\ \leq & T(r,f)+(l-1+lk)N(r,f)+S(r,f) \\ \leq & \frac{9lk+9l+16s-8}{8(lk+l+2s-1)+1}T(r,f)+S(r,f). \end{aligned} \tag{3.11}$$

By Nevanlinna's first fundamental theorem and (3.11), we have

$$\begin{aligned} & 2T(r,f)=2T(r,g)+S(r,f)=T\left(r,\frac{1}{g}\right)+T\left(r,\frac{1}{g-1}\right)+S(r,f) \\ \leq & N(r,0)+N(r,1)+m\left(r,\frac{1}{g}\right)+m\left(r,\frac{1}{g-1}\right)+S(r,f) \\ \leq & \frac{9lk+9l+16s-8}{8(lk+l+2s-1)+1}T(r,f)+m\left(r,\frac{1}{f-a}\right)+m\left(r,\frac{1}{f-b}\right)+S(r,f). \end{aligned} \tag{3.12}$$

Set

$$\begin{aligned} a_1(z)= & m_1a^{(k_1)}(z+c_1)+m_2a^{(k_2)}(z+c_2)+\cdots+m_na^{(k_n)}(z+c_n), \\ b_1(z)= & m_1b^{(k_1)}(z+c_1)+m_2b^{(k_2)}(z+c_2)+\cdots+m_nb^{(k_n)}(z+c_n). \end{aligned} \tag{3.13}$$

By Lemma 2.4, we obtain

$$m\left(r,\frac{F-a_1}{f-a}\right)=S(r,f),\ m\left(r,\frac{F-b_1}{f-b}\right)=S(r,f).$$

Set

$$W(F,a_1,b_1)=\begin{vmatrix} F & a_1 & b_1 \\ F' & a_1' & b_1' \\ F'' & a_1'' & b_1'' \end{vmatrix}.$$

By Lemma 2.4, we have

$$m\left(r,\frac{W(F,a_1,b_1)}{f-a_1}\right)=S(r,f),\ m\left(r,\frac{W(F,a_1,b_1)}{f-b_1}\right)=S(r,f). \tag{3.14}$$

If $W(F,a_1,b_1)\equiv 0$, then $b_1\equiv ka_1$, where k is a nonzero constant. Obviously, $W(F,a_1)\not\equiv 0$, where

$$W(F,a_1)=\begin{vmatrix} F & a_1 \\ F' & a_1' \end{vmatrix}.$$

Then by Lemma 2.4, we have

$$m\left(r,\frac{W(F,a_1)}{f-a_1}\right)=S(r,f),\ m\left(r,\frac{W(F,a_1)}{f-ka_1}\right)=S(r,f). \tag{3.15}$$

By (3.3), (3.15), Lemmas 2.3, 2.4 and 2.6, we obtain

$$
\begin{aligned}
& m\left(r, \frac{1}{f-a}\right)+m\left(r, \frac{1}{f-b}\right) \\
\leq & m\left(r, \frac{F-a_1}{f-a}\right)+m\left(r, \frac{F-b_1}{f-b}\right)+m\left(r, \frac{1}{F-a_1}\right)+m\left(r, \frac{1}{F-b_1}\right)+S(r, f) \\
\leq & m\left(r, \frac{1}{F-a_1}+\frac{1}{F-k a_1}\right)+S(r, f) \\
\leq & m\left(r, \frac{W(F, a_1)}{F-a_1}+\frac{W(F, a_1)}{F-k a_1}\right)+m\left(r, \frac{1}{W(F, a_1)}\right)+S(r, f) \\
\leq & T\left(r, W(F, a_1)\right)+S(r, f) \\
\leq & T(r, c_1 F'+c_2 F)+S(r, f) \leq T(r, F)+\overline{N}(r, F)+S(r, f) \\
\leq & T(r, F)+\frac{s}{8(lk+l+2s-1)+1} T(r, f)+S(r, f),
\end{aligned} \tag{3.16}
$$

where c_1 and c_2 are small functions of f.

If $W(F, a_1, b_1) \not\equiv 0$, then by (3.14), Lemmas 2.4 and 2.6, we have

$$
\begin{aligned}
& m\left(r, \frac{1}{f-a}\right)+m\left(r, \frac{1}{f-b}\right) \\
\leq & m\left(r, \frac{F-a_1}{f-a}\right)+m\left(r, \frac{F-b_1}{f-b}\right)+m\left(r, \frac{1}{F-a_1}\right)+m\left(r, \frac{1}{F-b_1}\right)+S(r, f) \\
\leq & m\left(r, \frac{1}{F-a_1}+\frac{1}{F-b_1}\right)+S(r, f) \\
\leq & m\left(r, \frac{W(F, a_1, b_1)}{F-a_1}+\frac{W(F, a_1, b_1)}{F-b_1}\right)+m\left(r, \frac{1}{W(F, a_1, b_1)}\right)+S(r, f) \\
\leq & T\left(r, W(F, a_1, b_1)\right)+S(r, f) \\
\leq & T(r, d_1 F''+d_2 F'+d_3 F)+S(r, f) \leq T(r, F)+2\overline{N}(r, F)+S(r, f) \\
\leq & T(r, F)+\frac{2s}{8(lk+l+2s-1)+1} T(r, f)+S(r, f),
\end{aligned} \tag{3.17}
$$

where d_1, d_2 and d_3 are small functions of f.

It follows from (3.16) and (3.17), we deduce that

$$
\begin{aligned}
& m\left(r, \frac{1}{f-a}\right)+m\left(r, \frac{1}{f-b}\right) \\
\leq & T(r, F)+\frac{2s}{8(lk+l+2s-1)+1} T(r, f)+S(r, f).
\end{aligned} \tag{3.18}
$$

By (3.12) and (3.18), we have

$$
\begin{aligned}
2T(r, f) \leq & \frac{9lk+9l+16s-8}{8(lk+l+2s-1)+1} T(r, f)+T(r, F) \\
& +\frac{2s}{8(lk+l+2s-1)+1} T(r, f)+S(r, f),
\end{aligned}
$$

that is

$$\frac{7lk+7l+14s-6}{8(lk+l+2s-1)+1}T(r,f) \le T(r,F)+S(r,f). \tag{3.19}$$

Taking $\lambda = \frac{1}{8(lk+l+2s-1)+1}$. Then by (1), (2) of Lemma 2.10, (3.11), (3.19) and $N(r,f) \le \frac{1}{8(lk+l+2s-1)+1}T(r,f)$, we get

$$\begin{aligned}\overline{\lim_{\substack{r\to\infty\\ r\in I}}}\frac{N(r,0)+N(r,1)}{T(r,g)+T(r,G)} &= \overline{\lim_{\substack{r\to\infty\\ r\in I}}}\frac{N(r,0)+N(r,1)}{T(r,f)+T(r,F)+S(r,f)}\\ &\le \overline{\lim_{\substack{r\to\infty\\ r\in I}}}\frac{\frac{9lk+9l+16s-8}{8(lk+l+2s-1)+1}T(r,f)+S(r,f)}{T(r,f)+\frac{7lk+7l+14s-6}{8(lk+l+2s-1)+1}T(r,f)+S(r,f)}\\ &\le \frac{9lk+9l+16s-8}{15lk+15l+30s-13} < \frac{16lk+16l+32s-22}{24(lk+l+2s-1)+3} = \frac{2-8\lambda}{3}.\end{aligned} \tag{3.20}$$

Hence, by Lemma 2.8, we have

$$\frac{1}{G-1} - \frac{c}{g-1} = d, \tag{3.21}$$

where $c(\neq 0), d$ are two constants. Now we consider two cases.

Case 1. $d=0$. Hence

$$G = \frac{g-1}{c}+1. \tag{3.22}$$

Next, we consider three subcases.

Case 1.1. $N(r,0) \neq S(r,f)$.

Thus there exists z_0 such that $g(z_0) = G(z_0) = 0$. It follows from (3.22) that $g(z) \equiv G(z)$.

Case 1.2. $N(r,0) = S(r,f)$, $N(r,1) \neq S(r,f)$.

Obviously,

$$N\left(r,\frac{1}{G}\right) = N\left(r,\frac{1}{g-1+c}\right). \tag{3.23}$$

Suppose that $c \neq 1$. Then by (3.3), (3.5) and (3.23), we obtain

$$\begin{aligned}T(r,f) =& T(r,g)+S(r,f)\\ \le& N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-1+c}\right)+N(r,g)+S(r,f)\\ \le& N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{G}\right)+N(r,f)+S(r,f)\\ \le& \frac{1}{8(lk+l+2s-1)+1}T(r,f)+S(r,f).\end{aligned}$$

It follows $T(r,f) \le S(r,f)$, a contradiction. So $c=1$, that is $g(z) \equiv G(z)$.

Case 1.3. $N(r,0) = S(r,f)$, $N(r,1) = S(r,f)$.

By (3.3), (3.5) and Lemma 2.1, we have

$$\begin{aligned}T(r,f) =&T(r,g)+S(r,f)\\ \le&N\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g}\right)+N(r,g)+S(r,f)\\ \le&N(r,1)+N(r,0)+N(r,f)+S(r,f)\\ \le&\frac{1}{8(lk+l+2s-1)+1}T(r,f)+S(r,f).\end{aligned}$$

It follows $T(r,f)\le S(r,f)$, a contradiction.

Case 2. $d\neq 0$. In the following, we consider two subcases.

Case 2.1. $\frac{c}{d}\neq 1,0$.

By (3.3), (3.5), (3.6), (3.11) and Lemma 2.2, we have

$$\begin{aligned}&2T(r,f)=2T(r,g)+S(r,f)\\ \le&N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g-\left(1-\frac{c}{d}\right)}\right)+N(r,g)+S(r,f)\\ \le&N(r,0)+N(r,1)+N(r,g)+N(r,G)+S(r,f)\\ \le&\frac{9lk+9l+16s-8}{8(lk+l+2s-1)+1}T(r,f)+(s+1)N(r,f)+sk\overline{N}(r,f)+S(r,f)\\ \le&\frac{9lk+9l+17s+sk-7}{8(lk+l+2s-1)+1}T(r,f)+S(r,f).\end{aligned}$$

It follows $T(r,f)\le S(r,f)$, a contradiction.

Case 2.2. $\frac{c}{d}=1$. Hence $c=d(d\neq 0)$,

$$\frac{1}{G(z)-1}=\frac{dg(z)}{g(z)-1}. \tag{3.24}$$

Obviously $N(r,0)=S(r,f)$. Otherwise, there exists z_0 such that $g(z_0)=G(z_0)=0$. Thus by (3.24) $G(z_0)=\infty$, a contradiction. If $d\neq -1$, then we have

$$\begin{aligned}&T(r,f)=T(r,g)+S(r,f)\\ \le&N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-\frac{1}{d+1}}\right)+N(r,g)+S(r,f)\le N(r,f)+S(r,f)\\ \le&\frac{1}{8(lk+l+2s-1)+1}T(r,f)+S(r,f).\end{aligned}$$

It follows $T(r,f)\le S(r,f)$, a contradiction.

If $d=-1$, then by (3.24), we obtain $g(z)G(z)\equiv 1$. Thus, we have

$$(f-a)^2=\frac{(b-a)^2(f-a)}{F-a}. \tag{3.25}$$

By Nevanlinna's first fundamental theorem and (3.25), we have

$$\begin{aligned}
2T(r,f) \leq & T\left(r,(f-a)^2\right)+S(r,f)\\
= & T\left(r,\frac{(b-a)^2(f-a)}{F-a}\right)+S(r,f)\\
\leq & T\left(r,\frac{F-a}{f-a}\right)+S(r,f)\\
= & N\left(r,\frac{F-a}{f-a}\right)+m\left(\frac{F-a}{f-a}\right)+S(r,f)\\
\leq & m\left(r,\frac{F-a_1}{f-a}\right)+m\left(r,\frac{a_1-a}{f-a}\right)+N(r,F)+S(r,f)\\
\leq & m\left(r,\frac{1}{f-a}\right)+N(r,F)+S(r,f)\\
\leq & T(r,f)+N(r,F)+S(r,f).
\end{aligned}$$

It follows that

$$\begin{aligned}
T(r,f) \leq & N(r,F)+S(r,f)\\
\leq & s\left(N(r,f)+k\overline{N}(r,f)\right)+S(r,f)\\
\leq & \frac{s+sk}{8(lk+l+2s-1)+1}T(r,f)+S(r,f),
\end{aligned}$$

that is $T(r,f)\leq S(r,f)$, a contradiction.

Combining Case 1 with Case 2, we deduce that $g(z)\equiv G(z)$. It follows that $f(z)\equiv F(z)$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Set

$$g(z)=\frac{f(z)-a(z)}{b(z)-a(z)},\ G(z)=\frac{F(z)-a(z)}{b(z)-a(z)}.$$

By $f(z)$ and $F(z)$ share $a(z), b(z), \infty$ CM, we know that $g(z)$ and $G(z)$ share 0, 1, ∞ CM almost.

We prove Theorem 1.2 by contradiction, suppose that $f(z)\not\equiv F(z)$, that is $g(z)\not\equiv G(z)$. Let

$$\phi=\frac{G(g-1)}{g(G-1)}. \tag{3.26}$$

Obviously, we know that $\phi(z)\not\equiv 0,\infty$, and

$$N(r,\phi)+N\left(r,\frac{1}{\phi}\right)=S(r,g)+S(r,G)=S(r,f).$$

By (3.26), we have

$$g-G=(\phi-1)g(G-1). \tag{3.27}$$

Let z_0 be a common pole of both $g(z)$ and $G(z)$ with multiplicity $m \geq 2$. Since $g(z)$ and $G(z)$ share ∞ CM almost, then by (3.27), we know that z_0 is the zero of $\phi(z) - 1$ with multiplicity at least m.

Next, we consider two cases.

Case 1. $\phi'(z) \not\equiv 0$, by (3.27), Lemma 2.7 and $N_{1)}(r, f) \leq \frac{1}{5(lk+l+2s-1)} T(r, f)$, we obtain

$$\begin{aligned} N(r, f) = N(r, g) &= N_{1)}(r, g) + N_{(2}(r, g) \\ &\leq \frac{1}{5(lk + l + 2s - 1)} T(r, f) + 2N\left(r, \frac{1}{\phi'}\right) + S(r, g) \\ &\leq \frac{1}{5(lk + l + 2s - 1)} T(r, f) + 2N\left(r, \frac{1}{\phi}\right) + 2\overline{N}(r, \phi) + S(r, g) \\ &\leq \frac{1}{5(lk + l + 2s - 1)} T(r, f) + S(r, g). \end{aligned}$$

Thus, we have

$$N(r, f) \leq \frac{1}{5(lk + l + 2s - 1)} T(r, f) + S(r, f). \tag{3.28}$$

Case 2. $\phi'(z) \equiv 0$, that is $\phi(z) \equiv c$. If $c = 1$, then by (3.26), we get $g(z) \equiv G(z)$, a contradiction. If $c \neq 1$, then by $\frac{G(g-1)}{g(G-1)} \equiv c$, we know that (3.28) is valid also.

By means of (3), (4) of Lemma 2.10 and Lemma 2.9, it is easy to prove Theorem 1.2 by imitating the proof of Theorem 1.1 and replacing (3.11), (3.19) and (3.20) respectively with the following three formulas:

$$\begin{aligned} N(r, 0) + N(r, 1) &\leq m(r, f) + l\left(N(r, f) + k\overline{N}(r, f)\right) + S(r, f) \\ &\leq T(r, f) + (l - 1 + lk)N(r, f) + S(r, f) \\ &\leq \frac{6lk + 6l + 10s - 6}{5(lk + l + 2s - 1)} T(r, f) + S(r, f). \end{aligned}$$

$$\frac{4lk + 4l + 8s - 4}{5(lk + l + 2s - 1)} T(r, f) \leq T(r, F) + S(r, f).$$

$$\begin{aligned} \overline{\lim_{\substack{r\to\infty \\ r\in I}}} \frac{N(r, 0) + N(r, 1)}{T(r, g) + T(r, G)} &= \overline{\lim_{\substack{r\to\infty \\ r\in I}}} \frac{N(r, 0) + N(r, 1)}{T(r, f) + T(r, F) + S(r, f)} \\ &\leq \overline{\lim_{\substack{r\to\infty \\ r\in I}}} \frac{\frac{6lk+6l+10s-6}{5(lk+l+2s-1)} T(r, f) + S(r, f)}{T(r, f) + \frac{4lk+4l+8s-4}{5(lk+l+2s-1)} T(r, f) + S(r, f)} \\ &\leq \frac{6lk + 6l + 10s - 6}{9lk + 19l + 18s - 9} < \frac{10lk + 10l + 20s - 12}{15lk + 15l + 30s - 15} = \frac{2 - 2\lambda}{3}. \end{aligned}$$

Proof of Theorem 1.3. Set

$$g(z) = \frac{f^m - a(z)}{b(z) - a(z)},$$

$$G(z) = \frac{H - a(z)}{b(z) - a(z)}.$$

By f^m and H share $a(z), b(z), \infty$ CM, we know that $g(z)$ and $G(z)$ share 0, 1, ∞ CM almost. Next, we consider two cases.

Case 1. $a(z) \equiv 0$, $b(z) \not\equiv 0$. In the following, we consider two subcases.

Case 1.1. $N(r, \frac{1}{G}) \neq S(r, G)$. From the conditions of Theorem 1.3, we have

$$N(r, 0) - \overline{N}(r, 0) \neq S(r, G) + S(r, g). \tag{3.29}$$

Set $\psi(z) = \frac{G'(z)}{1-G(z)} - \frac{g'(z)}{1-g(z)}$. If $\psi(z) \not\equiv 0$, then by Nevanlinna's first fundamental theorem, we get

$$\begin{aligned} N(r, 0) - \overline{N}(r, 0) \leq & N(r, \frac{1}{\psi}) \leq T(r, \psi) + O(1) \\ = & m(r, \psi) + N(r, \psi) + O(1) = S(r, G) + S(r, g), \end{aligned}$$

which contradicts with(3.29). Hence $\psi(z) \equiv 0$, we get $G(z) - 1 = c(g(z) - 1)$. By (3.29), we know that there exists z_0 satisfying $g(z_0) = G(z_0) = 0$. Hence $c = 1$, that is $g(z) \equiv G(z)$. It follows $f^m(z) \equiv F(z)$.

Similarly, $N(r, G) = S(r, G)$ and $N(r, g) = S(r, g)$.

Case 1.2. $N(r, \frac{1}{G}) = S(r, G)$.

Obviously, $N(r, \frac{1}{g}) = S(r, g)$, by Lemma 2.5, $N(r, \frac{1}{G}) + N(r, G) = S(r, G)$, $N(r, \frac{1}{g}) + N(r, g) = S(r, g)$. It follows from $g(z)$ and $G(z)$ share 1 CM almost, that $g(z)G(z) \equiv 1$, we have $f^m F \equiv b^2$, that is $\frac{F}{f^m} = \frac{b^2}{f^{2m}}$. Hence, we get

$$m\left(r, \frac{F}{f^m}\right) = m\left(r, \frac{b^2}{f^{2m}}\right) = 2mT(r, f) + S(r, f), \tag{3.30}$$

it follows from $m\left(r, \frac{F}{f^m}\right) \leq S(r, f)$ and (3.30) that $T(r, f) = S(r, f)$, a contradiction.

Case 2. $a(z) \not\equiv 0$.

In the following, we consider two subcases.

Case 2.1. $a(z) \not\equiv 0, b(z) \not\equiv 0$.

Let $f^m(z) \not\equiv F(z)$, by Lemma 2.2, we have

$$\begin{aligned} 2T(r, f^m) \leq & \overline{N}\left(r, \frac{1}{f^m}\right) + \overline{N}(r, f^m) + \overline{N}\left(r, \frac{1}{f^m - a}\right) + \overline{N}\left(r, \frac{1}{f^m - b}\right) \\ & + \varepsilon T(r, f^m) + S(r, f^m) \\ \leq & \frac{1}{m} N\left(r, \frac{1}{f^m}\right) + N\left(r, \frac{1}{f^m - F}\right) + \varepsilon T(r, f^m) + S(r, f^m) \\ \leq & \frac{1}{m} T(r, f^m) + T(r, f^m - F) + \varepsilon T(r, f^m) + S(r, f^m) \\ \leq & \frac{1}{m} T(r, f^m) + T(r, f^m) + \varepsilon T(r, f^m) + S(r, f^m) \\ \leq & \left(\frac{1}{m} + 1 + \varepsilon\right) T(r, f^m) + S(r, f^m). \end{aligned}$$

Let $\varepsilon = \frac{1}{4} < \frac{1}{2}$, and $m \geq 2$. It follows that $T(r, f^m) \leq S(r, f^m)$, a contradiction.

Case 2.2. $a(z) \not\equiv 0, b(z) \equiv 0$.

By using the same argument as used in Case 1, we obtain a contradiction. So $f^m(z) \equiv H(f(z))$. This completes the proof of Theorem 1.3.

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